A tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields

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· tropical geometry … "combinatorial shadow" of algebraic geometry

X: sm. proj. var. $/\mathbb{C}$.

the Hodge conj. — surjective.

$$CH^p(X)_{\mathbb{Q}} \xrightarrow{} H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$$

$$H^{p,p}_{trop}(X) \quad \text{tropical cohomology.}$$

Today — isom.

Future

Main Theorem 1.

X: sm. alg. var / a field K w/ the trivial valuation

$K \to \{0, \infty\}$

$0 \mapsto \infty$

$a \neq 0 \mapsto 0$.

Then Liu's tropical cycle class map

$$CH^p(X)_{\mathbb{Q}} \to H^{p,p}_{trop}(X)$$

is an isomorphism.

tropical cohomology

$X^{Ber} := \{$valuations $\mathcal{O}_X \to \mathbb{R} \cup \{\infty\}$ trivial on $K\}$

the Berkovich space

(" an infinite union of cones ")

$\mathcal{F}^p$: a sheet on X^{Ber}

$:=$ "$\wedge^p$ contangent space" generated by $f_1 \wedge \cdots \wedge f_p$ $f_i \in \mathcal{O}_X^\times$

$H^{p,q}_{\mathrm{Trop}}(X) := H^q(X^{\mathrm{Ber}}, \mathcal{F}^p)$.

tropical K-group.

$K_T^p(K(X)/K) := \mathcal{F}^p_{\cdot}$ (the trivial valuation of the generic pt. $\in X$) ($\leftarrow$ stalk)

$\left(\overset{!!}{K_T^p(K(X))}\right)$ $\left(\overset{ch=0}{\simeq} \mathbb{Q}\left\langle \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_p}{f_p} \right\rangle_{f_i \in K(X)^\times} \subset \Omega^p(K(X)/K) \right)$

K_T^p : sheaf'n on X_{zar}

$\rightsquigarrow$ $\exists$ Gersten resolution of K_T^p. (Rost's cycle module)

$\rightsquigarrow H^p(X_{\mathrm{zar}}, K_T^p) \simeq CH^p(X)_{\mathbb{Q}}$

Main Theorem 2. ($\Rightarrow$ Main Theorem 1)

$H^{p,q}_{\mathrm{Trop}}(X) \simeq H^q(X_{\mathrm{zar}}, K_T^p)$

Sketch.

Key Prop.

(1) étale excision

(2) $\mathbb{A}^1$-homotopy invariance

(3) exsitence of corestriction map.

$\rightsquigarrow$ By a theorem on "cohomology theories"
(Quillen, Bloch-Ogus, Gabber, Rost, Colliot-Télène-Hoobler-Kahn)

$\forall X$, $r>0$. $\exists\ E_2^{p,q} = H^p(X_{Zar}, \mathcal{H}^{r,q}_{trop}) \Rightarrow H^{r,p+q}_{trop}(X)$

$\uparrow$ sheaf'n

Lem.
- $E_2^{p,q} = 0\ (q \neq 0)$
- $\mathcal{H}^{r,0}_{trop} \simeq K_T^r$.

(☹ easy)

Lem. $\Rightarrow$ Main Theorem 2 ☒

<u>Sketch ($\mathbb{A}^1$-homotopy invariance).</u>

- Leray spectral seq. for $\pi : (X \times \mathbb{A}^1)^{Ber} \to X^{Ber}$ (+ more)

$\rightsquigarrow$ it suffices to compute for $v \in \mathbb{A}^{1,ad}_L$ (Huber's adic sp.)

$$K_T^p(\kappa(v)) \xrightarrow{d} \bigoplus_{w \in \overline{\{v\}} \setminus \{v\}} K_T^{p-1}(\kappa(w))$$

$\uparrow$ res. fld

L : complete valuation field ht ≤ 1.

Rem.
$\kappa(v) \simeq k'(s)$ (s indeterminant) $\exists k'/\kappa(L)$: fin. ext.
(when $\overline{\{v\}} \neq \{v\}$)

$\overline{\{v\}} \overset{bij}{\simeq} \mathbb{P}^1_{k'}$.

$\exists$ short exact seq.

$$0 \to K^p_T(k') \to K^p_T(k'(s)) \xrightarrow{d} \bigoplus_{x \in \mathbb{A}^{1,cl}_{k'}} K^{p-1}_T(k(x)) \to 0$$

$\rightsquigarrow$ $\mathbb{A}^1$-homotopy invariance. $\boxtimes$

fini.