

Classifying the irreducible components of moduli stacks of torsion-free sheaves on K3 surfaces and an application to Brill-Noether theory

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Introduction

Purpose

Irr decomp of **moduli stacks of torsion-free sheaves**
of rk 2 on K3 surfaces of $\rho = 1$
&
irr decomp of **Brill-Noether(BN) locus** on Hilbert schs of pts.

※ Moduli stacks can parametrize **unstable sheaves**.

Previous research

- The case of ruled surfaces
→ C.Walter (1995)
- Stratification of moduli stacks
→ V.Hoskins (2018) or T.L.Gomez, I.Sols and A.Zamora (2015)

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Mukai vector

X : Proj K3 surf/ \mathbb{C} of $\rho = 1$, $E \in \text{Coh}(X)$

1. $\nu(E) := (\text{rk}(E), c_1(E), \text{ch}_2(E) + \text{rk}(E)) \in \mathbb{Z} \oplus \text{Pic}(X) \oplus \mathbb{Z}$
2. $\langle \nu, w \rangle := -[\nu]_0[w]_2 + [\nu]_1[w]_1 - [\nu]_2[w]_0 \in \mathbb{Z}$
, where $\nu := ([\nu]_0, [\nu]_1, [\nu]_2) \in \mathbb{Z} \oplus \text{Pic}(X) \oplus \mathbb{Z}$

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Moduli stacks

3. $\mathcal{M}(\nu)$:

Ob. flat family \mathcal{E}/U paramet torsion-free sheaves w/
Mukai vector ν

Mor. $(\varphi, \alpha) : \mathcal{E}/U \rightarrow \mathcal{E}'/U'$
 $(\varphi : U \rightarrow U' : \text{mor of schs}, \alpha : \mathcal{E} \rightarrow (\text{id}_X \times \varphi)^* \mathcal{E}' : \text{iso})$

4. $\mathcal{M}_{(\nu_1, \nu_2)}^{\text{HN}}(\nu) := \left\{ E \in \mathcal{M}(\nu) \mid \begin{array}{l} \exists (0 \subset E_1 \subset E) : \text{HN-filtration} \\ \text{s.t. } \nu(E_1) = \nu_1, \nu(E/E_1) = \nu_2 \end{array} \right\}$

5. $\mathcal{M}^{\text{ss}}(\nu) := \{ E \in \mathcal{M}(\nu) \mid E : \text{semistable} \}$

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Irreducible decomposition of $\mathcal{M}(v)$

Main Theorem 1

v_0 : primitive Mukai vector

$v := mv_0$ ($m \in \mathbb{Z}$)

Assume $[v]_0 = 2$ & v satisfies one of (a) \sim (c)

- (a) : $\langle v, v \rangle > 0$, (b) : $\langle v, v \rangle = 0, -2$ and v is primitive ,
 (c) : $\langle v, v \rangle < -2$ and $\langle v_0, v_0 \rangle \neq -2$.

Then,

$$\mathcal{M}(v) = \begin{cases} \overline{\mathcal{M}^{\text{ss}}(v)} \cup \bigcup_{\langle v_1, v_2 \rangle \leq 1} \overline{\mathcal{M}_{(v_1, v_2)}^{\text{HN}}(v)} & \text{(a), (b)} \\ \bigcup \overline{\mathcal{M}_{(v_1, v_2)}^{\text{HN}}(v)} & \text{(c)} \end{cases}$$

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Irreducible decomposition of $\mathcal{M}(v)$

Remark

For proof of Thm 1,

- theory of stratification via HN-filt
- theory of moduli sps of sheaves on K3 surfs by K.Yoshioka

are important.

\rightsquigarrow We get the relation between the stratas by calculating dims etc..

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Application to BN theory

Definition (BN locus of Hilbert schs of pts)

D : eff div on X

$N \in \mathbb{N}$ s.t. $N \leq h^0(\mathcal{O}(D))$

$$W_N^i(D) := \{Z \in \text{Hilb}^N(X) \mid h^1(\mathcal{I}_Z(D)) \geq i + 1\}$$

Remark

$H^0(\mathcal{I}_Z(D)) - \{0\}/\mathbb{C}^* = \text{eff divs lin equiv to } D \text{ passing through } Z.$

For general $Z \in \text{Hilb}^N(X),$

$$h^0(\mathcal{I}_Z(D)) = h^0(\mathcal{O}_X(D)) - \ell(\mathcal{O}_Z) = \text{expected dimension.}$$

But, for $Z \in W_N^i(D),$

$$h^0(\mathcal{I}_Z(D)) > h^0(\mathcal{O}_X(D)) - \ell(\mathcal{O}_Z).$$

$$\ast \quad h^0(\mathcal{I}_Z(D)) = h^0(\mathcal{O}_X(D)) - \ell(\mathcal{O}_Z) + h^1(\mathcal{I}_Z(D)).$$

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Application to BN theory

Main Theorem 2

$D := nH$, $\nu := (2, nH, \frac{n^2 H^2}{2} - N + 2)$, where H : amp gen of $\text{Pic}(X)$.

If $\langle \nu, \nu \rangle > 0$, there exists the following 1 to 1 corresp

$$\left\{ \text{the irr comps of } W_N^0(D) \right\}$$

1 to 1 \updownarrow

$$\left\{ \overline{\mathcal{M}_{(\nu_1, \nu_2)}^{\text{HN}}}(\nu) \mid (\nu_1, \nu_2) \text{ satisfying } (*) \right\} \cup \left\{ \overline{\mathcal{M}^{\text{SS}}}(\nu) \right\}$$

$$\subsetneq \left\{ \text{the irr comps of } \mathcal{M}(\nu) \right\}.$$

$$[\nu_1]_1, [\nu_2]_1 \neq 0 : \text{effective, } \langle \nu_1, \nu_2 \rangle \leq 1, [\nu_2]_2 \geq -1 \quad (*)$$

Remark

- If Z : general in $W_N^0(D)$, the corresp is given by ext'n

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{I}_Z(D) \rightarrow 0.$$

- Thm 2 \rightsquigarrow (non-emptiness,) the dims and the num of irr comps of $W_N^0(D)$.

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Thank you for listening !