On liftability of Du Val del Pezzo surfaces in positive characteristic

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Kinosaki Algebraic Geometry Symposium 2020, October

Pathologies on Du Val del Pezzo surfaces

All varieties are defined over $k = \bar{k}$ in characteristic $p \geq 0$.

Definition

X is a Du Val del Pezzo surface $\overset{\text{def}}{\Longleftrightarrow}$ X is a normal projective surface with only Du Val singularities s.t. $-K_X$ is ample.

Notation

If X has three A_1-singularities and one D_4-singularity, then we write $\mathrm{Dyn}(X) = 3A_1 + D_4$ or $X = X(3A_1 + D_4)$.

Pathological Phenomena in $p > 0$

Keel–M^{c}Kernan constructed a Du Val del Pezzo surface $X := X(7A_1)$ in $p = 2$.

- $7A_1 \notin$ Dynkin type in $p = 0$.
- Violating Bertini's Theorem: all members of $|-K_X|$ are singular.
- Violating Kodaira type Theorem: $\exists A$: ample $\mathbb{Z}$-divisor s.t. $H^1(\mathcal{O}_X(K_X + A)) \neq 0$ (Cascini-Tanaka).

Log liftability

It is believed that **"Pathologies in $p > 0 \leftrightarrow$ Non liftability to the Witt ring"**

Unfortunately, all Du Val del Pezzo surfaces lift to the Witt ring. Therefore, we consider stronger liftability.

Definition (log lift)

X : Du Val del Pezzo surface in $p > 0$.
X is log liftable $\overset{\text{def}}{\Longleftrightarrow}$ (Y, E) lifts to the Witt ring $W(k)$, where $\pi\colon Y \to X$ is the minimal resolution with reduced exceptional divisor E.

As we will see later, $X(7A_1)$ is not log liftable.

Pathologies and log liftability

We introduce the following properties.
For a Du Val del Pezzo surface X in $p > 0$, we say that X satisfies:

- (ND) if $\nexists X_{\mathbb{C}}$: Du Val del Pezzo surface over $\mathbb{C}$ s.t. $\mathrm{Dyn}(X) = \mathrm{Dyn}(X_{\mathbb{C}})$ and $\rho(X) = \rho(X_{\mathbb{C}})$.
- (NB) if all members of $|-K_X|$ are singular.
- (NK) if $\exists A$: ample $\mathbb{Z}$-divisor s.t. $H^1(\mathcal{O}_X(K_X + A)) \neq 0$.
- (NL) if X is not log liftable.

Lemma

We have the following implications.

$$\begin{array}{ccccc} ND & \Longrightarrow & NL & \Longrightarrow & NB \\ & & \Uparrow & & \\ & & NK & & \end{array}$$

By the above lemma, we should focus on Du Val del Pezzo surfaces satisfying NB.

Main Result

Theorem

Du Val del Pezzo surfaces satisfying NB are as follows.

Table 1:

K_X^2	$K_X^2 = 1$									$K_X^2 = 2$			
$\mathrm{Dyn}(X)$	E_8	D_8	$A_1 + E_7$	$2D_4$	$2A_1 + D_6$	$4A_1 + D_4$	$8A_1$	$A_2 + E_6$	$4A_2$	E_7	$A_1 + D_6$	$3A_1 + D_4$	$7A_1$
Characteristic	$p = 2, 3$	$p = 2$				$p = 2$		$p = 3$		$p = 2$			
No. of isomorphism classes	1	1	1	∞	1	∞	∞	1	1	1	1	1	1
NL?	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\times$	$\bigcirc$	$\times$	$\times$	$\times$	$\bigcirc$
ND?	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$
NK?	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$

Remark

(NL) $\iff$ $\mathrm{Dyn}(X) = 4A_1 + D_4, 8A_1, 7A_1,$ or $(p, \mathrm{Dyn}(X)) = (3, 4A_2)$.

Non log liftability of $X(4A_2)$

Sketch of proof

Suppose that $X := X(4A_2)$ is log liftable. Let $\pi\colon Y \to X$ be the minimal resolution with reduced exceptional divisor E and (Y_K, E_K) be the generic fiber of $W(k)$-lifting of (Y, E). Then the blow-up $Z_K \to Y_K$ at the base point of $|-K_{Y_K}|$ gives the anti-canonical morphism $f_K\colon Z_K \to \mathbb{P}^1_K$. Let $G_{K,i} \subset Z_K$ be the strict transform of $E_{K,i}$, where $E_{K,i}$ $(i = 1, 2, 3, 4)$ are connected component of E_K. Then $f_K(G_{K,i})$ is a K-rational point for each i. By using Beauville's result, we have $f_K(G_i) = \omega$ and hence $\omega \in K$, where ω is a primitive cube root of unity. This is a contradiction.

$$\begin{array}{ccc}
(Z_K, G_K = \sum_{i=1}^4 G_{K,i}) & \longrightarrow & (Y_K, E_K := \sum_{i=1}^4 E_{K,i}) \\
 & \searrow^{f_K} & \vdots\ \phi_{|-K_{Y_K}|} \\
 & & \mathbb{P}^1_K
\end{array}$$