Cylinders in canonical del Pezzo fibrations

Masatomo Sawahara

Saitama University

October 21, 2020

Notation

- Unless otherwise stated, all varieties are defined over k, where
  k: field of characteristic zero
  (NOT necessarily an algebraic closed field).
- $\overline{k}$: algebraic closure of k.
- A Fano variety means a normal projective variety,
  whose anti-canonical divisor is ample.
- A del Pezzo surface means a Fano variety of dimension two.

Cylinder

Definition.

X: normal algebraic variety, U: (Zariski) open subset of X.
U: cylinder (on X) $\underset{\text{def.}}{\Longleftrightarrow}$ $U \simeq \mathbb{A}^1_k \times Z$ for some variety Z.

Why cylinder?

There are many applications (especially **unipotent group actions**).

Theorem. (Kishimoto-Prokhorov-Zaidenberg '11)

Assume $k = \overline{k}$.
X: normal projective variety, H: ample $\mathbb{Q}$-divisor on X.
Then the following two assertions are equivalent:

1. *X contains an H-polar cylinder U, i.e., U is a cylinder and $\exists\, D$: eff. $\mathbb{Q}$-div. on X s.t. $D \sim_{\mathbb{Q}} H$ and $U = X \backslash \mathrm{Supp}(D)$.*
2. *$\hat{X} := \mathrm{Spec}\left(\bigoplus_{i \geq 0} H^0(X, \mathscr{O}_X(iH))\right)$ admits an eff. $\mathbb{G}_a$-action.*

Example. (Cheltsov-Park-Won '16)

Assume $k = \overline{k}$.
They proved that:

$$\mathbb{P}^3_k \supseteq S := (x^3 + y^3 + z^3 + w^3 = 0) \supseteq \nexists\, (-K_S)\text{-polar cylinder.}$$

$\therefore\ \mathbb{A}^4_k \supseteq \hat{S} = (x^3 + y^3 + z^3 + w^3 = 0)\ \nexists \curvearrowleft$ eff. $\mathbb{G}_a$-action.

How to find cylinders? –Cylinders in MFS–

V: normal projective variety with at most $\mathbb{Q}$-factorial klt singularities $/\mathbb{C}$.

Theorem. (Birkar-Cascini-Hacon-McKernan '10)

K_V: NOT pseudo-effective $\Longrightarrow$
$\exists V \dashrightarrow X$; minimal model program (MMP),
which ends with a Mori fiber space (MFS) $f : X \to Y$.

Lemma. (Dubouloz-Kishimoto '19)

$V \dashrightarrow X$; MMP.
X contains a cylinder $\Longrightarrow$ V contains a cylinder.

Question.

Look for cylinders contained in a projective variety with MFS structure.

Vertical cylinder

$f : X \to Y$; dominant morphism between normal varieties.

Definition.

$X \supseteq U \simeq \mathbb{A}^1_k \times Z$: cylinder.
U : vertical cylinder w.r.t. f $\underset{\text{def.}}{\iff}$ $\exists g : Z \to Y$ s.t. $f|_U = g \circ pr_Z$.

$$\begin{array}{ccc} X & \hookleftarrow & U \\ {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle pr_Z} \\ Y & \xleftarrow{\;g\;} & Z \end{array}$$

X_η : generic fiber of f.

Lemma.

f admits a vertical cylinder $\overset{\text{iff}}{\iff}$ X_η contains a cylinder.

Generic fiber of MFS

$f : X \to Y$; MFS $/\mathbb{C}$, X_η : generic fiber of f.

- X_η: Fano variety of rank one over $\mathbb{C}(Y)$ and of dimension $\dim X - \dim Y$.
- Note that $\mathbb{C}(Y)$ is NOT algebraically closed unless $\dim Y = 0$.

Question.

V: Fano variety of rank one over k.
In which case does V contain a cylinder?

Theorem. (Dubouloz-Kishimoto '18)

*S: **smooth** del Pezzo surface of rank one over k.*
$d := (-K_S)^2 \in \{1, \dots, 6, 8, 9\}$: degree of S.
$S \supseteq \exists$ cylinder $\overset{iff}{\Longleftrightarrow}$ $d \geq 5$ and $S(k) \neq \emptyset$.
(Hence, the existence of cylinders contained in S is completely controlled by means of degree of S and k-rational points on S.)

7 / 10

Canonical del Pezzo fibration

- To state the result, we need to prepare some definitions.

Definition.

S: del Pezzo surface over k.
S: canonical del Pezzo surface $\underset{def.}{\Longleftrightarrow}$ $S_{\overline{k}}$ has at most Du Val singularities.

Definition.

$f : X \to Y$; dominant projective morphism.
f; canonical del Pezzo fibration $\underset{def.}{\Longleftrightarrow}$ the following conditions hold:

- *X has at most **canonical singularities**.*
- *The generic fiber X_η: del Pezzo surface of rank one.*

Question.

S: canonical del Pezzo surface of rank one over k.
In which case does S contain a cylinder?

8 / 10

Main Result

Theorem. (S.)

S: __canonical__ del Pezzo surface of rank one over k.
The existence of cylinders contained in S is completely controlled by means of degree of S and the action of $\mathrm{Gal}(\overline{k}/k)$ on the set of exceptional curves appearing the minimal resolution of $S_{\overline{k}}$ except for few cases.
(Precise statement being a bit long, we omit it.)

Example. (S.)

Assume $k = \mathbb{C}(t)$ and $\mathbb{P}(1,1,1,2) = \mathrm{Proj}(k[x,y,z,w])$.
$\mathbb{P}(1,1,1,2) \supseteq S_n := (t^n w^2 + x^2y^2 + yz^3 = 0)$, here $n \in \mathbb{Z}$.

- S_n: canonical del Pezzo surface of degree 2.
- $S_{n,\overline{k}}$ has exactly two singular points of A_2 and A_5-type respectively.
- Namely, S_n is of rank one.

$S_n \supseteq \exists$ cylinder $\overset{\text{iff}}{\Longleftrightarrow}$ n: even.

References

[BCHM] C. Birkar, P. Cascini, C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Am. Math. Soc., **23** (2010), 405–468.

[CPW1] I. Cheltsov, J. Park and J. Won, *Affine cones over smooth cubic surfaces*, J. Eur. Math. Soc., **18** (2016), 1537–1564.

[CPW2] I. Cheltsov, J. Park and J. Won, *Cylinders in singular del Pezzo surfaces*, Comp. Math., **152** (2016), 1198–1224.

[DK1] A. Dubouloz and T. Kishimoto, *Cylinders in del Pezzo fibrations*, Israel J. Math., **225** (2018), 797–815.

[DK2] A. Dubouloz and T. Kishimoto, *Deformations of $\mathbb{A}^1$-cylindrical varieties*, Math. Ann., **373** (2019), 1135–1149.

[KPZ] T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Group actions on affine cones*, In: *Affine Algebraic Geometry*, 123–163, Peter Russell's Festschrift, CRM Proc. Lecture Notes **54**, Amer. Math. Soc., Providence, RI, 2011.

[S1] M. Sawahara, *Cylinders in weak del Pezzo fibrations*, submitted.

[S2] M. Sawahara, *Cylinders in canonical del Pezzo fibrations*, in preparation.