Cohomology of conical symplectic resolutions

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Definition

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A smooth symplectic variety (X, ω) with an action of $\mathbb{S} = \mathbb{C}^*$ is a conical if X is the following coniditions,

1. $\phi : X \to X_0 = \mathrm{Spec}\mathbb{C}[X]$ is projective birational.
2. $s^*\omega = s^n\omega$ for $s \in \mathbb{C}^*$.
3. $\mathbb{C}^*$ acts on $\mathbb{C}[X]$ with only non-negative weights and $\mathbb{C}[X]^{\mathbb{C}^*} = \mathbb{C}$

Example

Let X be a crepant resolution of $\mathbb{C}^2/G$, where a group G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then the Hilbert scheme on n-points of X is a conical symplectic resolution.

Symplectic dual

Consider a conical symplectic resolution X with a Hamiltonian action of $\mathbb{T} = \mathbb{C}^*$, commuting $\mathbb{S}$, such that $X^{\mathbb{T}}$ is finite.
Let $\mathbb{W}$ be a Weyl group of the group of Hamiltonian symplectic morphisms of X and W be a Namikawa Weyl group of X.

Definition

A symplectic duality from X to $X^!$ consists of

1. There exists a order reversing bijection from the index set of $X^{\mathbb{T}}$ to the index set of $(X^!)^{\mathbb{T}}$.
2. $W \simeq \mathbb{W}^!$ and $W^! \simeq \mathbb{W}$
3. etc

Examples

- The Hilbert scheme on n-points of $\mathbb{C}^2$ is self dual.
- An affine A type quiver variety and other affine A type quiver variety are symplectic dual.
- A hypertoric variety and other hypertoric variety are symplectic dual.

Conjecture

Conjecture (Hikita,2015)

If $X \to X_0$ and $X' \to X_0'$ are symplectic dual, then we have the following graded algebra isomorphim:

$$H^*(X) \simeq C[X`{}_0^{\mathbb{T}}]$$

$$H^*(X') \simeq C[X_0^{\mathbb{T}}]$$

Theorem (Hikita,2015)

In the case of the Hilbert scheme on n-points of $\mathbb{C}^2$ and a hypertoric variety, type A S3-variety, this conjecture is true.

Case of the framed moduli space

Let $M(r, n)$ be the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank r and $c_2 = n$. Then $M(r, n)$ is a conical symplectic resolution .

Theorem (Nakajima-Yoshioka,2003)

we have the following equations:

$$\sum_d \mathrm{dim} H^d(M(r,n))t^d = \sum_{(Y_1,\cdots,Y_r)} t^{2(rn-(\sum_j jl(Y_j)))}$$

where $(Y_1 \cdots , Y_r)$ is $r-$tuples Young diagram running over $\sum |Y_j| = n$, $l(Y_j)$ is a length of Young diagram Y_j

Main result

Let X_r be a crepant resolution of $\mathbb{C}^2/(\mathbb{Z}/r\mathbb{Z})$.

Theorem (Ben-Licata-Proudfoot-Webster,2014)

$\mathrm{Hilb}^n(X_r)$ and $M(r,n)$ are symplectic dual.

Theorem (Hatano)

There exists a graded vector space isomorphism

$$H^*(M(r,n)) \simeq \mathbb{C}[((\mathrm{Hilb}X_r)_0)^{\mathbb{C}^*}].$$