Numerical analysis for fractional Burgers equations in supercritical cases

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1. Introduction

In the present article, we consider a fractional Burgers equation. This is a classical Burgers equation augmented with a fractional derivative of order $\alpha \in (0,1)$ which represents a kind of dissipation. Precisely, we study an initial value problem of

$$u_t + uu_x = \int_x^\infty \frac{u_y(t,y)}{(y-x)^\alpha} dy, \text{ for } t>0, \ x \in \mathbb{R}, \tag{1a}$$

$$u(0,x) = u_0(x) \text{ for } \ x \in \mathbb{R}. \tag{1b}$$

This type of equation was derived and discussed by N. Sugimoto in a series of papers [7]–[9]. These papers concerned with the investigation of viscoelastic relaxation effects of materials on wave propagation in solids, and sound propagation in train tunnels. Sugimoto derived

$$\frac{\partial f}{\partial x} - \gamma f \frac{\partial f}{\partial \tau} = \beta \frac{\partial^\alpha f}{\partial \tau^\alpha}, \tag{2}$$

where $\alpha \in (0,1)$, β and γ are positive constants, and the 'fractional derivative' term on the right hand side is defined by

$$\frac{\partial^\alpha f}{\partial \tau^\alpha}(x,\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^\tau (\tau-\sigma)^{-\alpha} \frac{\partial f}{\partial \sigma}(x,\sigma)\, d\sigma. \tag{3}$$

The variable τ is defined by $\tau = t - x/a_0$ representing retarded time in a conventional coordinate for (t,x), where a_0 is the sound speed. By suitably changing variables, (2) is reduced to (1).

Mathematical analysis of fractional Burgers equations dates back to [3], where the fractional derivative was defined via Fourier multiplier : $\dfrac{\partial^\alpha g(x)}{\partial x^\alpha} = \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}[g](\xi)](x)$ for $\alpha \in (0,2)$. They claimed nonexistence of bounded traveling wave solutions in the supercritical cases of $\alpha \in (0,1)$. For the Cauchy problem, [4] proved global existence of classical solutions for bounded initial value in subcritical cases of $\alpha \in (1,2)$. In supercritical cases, global existence of solutions for bounded initial value was shown in [1] using the notion of entropy solution. Conditions for occurrence and non-occurrence of shock formation were given in [2].

In section 2 of the present article, we review author's recent work [5], which confirms the global existence of solutions to (1) for bounded initial data. Secondly, we establish numerical methods to solve (1) by the finite difference method in section 3, and by the finite volume method combined with the Duhamel's principle in section 4, respectively. In the final section, we compare these numerical results with those obtained in [6] by splitting method using front tracking for convection phase, whose accuracy is tested against the exact solutions found in [5].

This work was supported by KAKENHI (15KT0014, 17K05313, 17K05376) and Railway Technical Research Institute.

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2. Global solution

In this section, we review the mathematical results in [5] on global existence of the solutions to the equation (1). Since the solution could develop singularities, we introduce the notion of entropy solutions following the idea of [1].

Definition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. For an initial value $u_0 \in L^\infty(\mathbb{R})$, we define an entropy solution to*

$$u_t + f(u)_x = \int_x^\infty \frac{u_y(t,y)}{(y-x)^\alpha} dy \tag{4}$$

by a function $u \in L^\infty((0,\infty) \times \mathbb{R})$ which satisfies

$$\int_0^\infty \int_{\mathbb{R}} (\eta(u)\phi_t + q(u)\phi_x)(t,x) dxdt + \int_{\mathbb{R}} \eta(u_0(x))\phi(0,x)dx \\ + \alpha \int_0^\infty \int_{\mathbb{R}} \eta'(u(t,x))\phi(t,x) \int_x^\infty \frac{u(t,y)-u(t,x)}{(y-x)^{\alpha+1}} dydxdt \geq 0, \tag{5}$$

for all nonnegative $\phi \in C_c^\infty([0,\infty) \times \mathbb{R})$, for all smooth and convex function $\eta : \mathbb{R} \to \mathbb{R}$, and for all $q : \mathbb{R} \to \mathbb{R}$ which satisfies $q' = \eta' f'$.

We just cite a basic result from [5] concerning the solution to (4) without proof.

Proposition 2.2. *If $u_0 \in L^\infty(\mathbb{R})$, then the equation* (4) *uniquely has a global entropy solution.*

3. Finite difference method

We propose a numerical scheme for (1) based on finite difference method. Hereafter we assume that the support of the initial value to (1) is bounded to the right. Reflecting on (2) from which (1) was derived, this assumption means that the support of the initial value to (2) is bounded to the left, which is naturally accepted considering that we can take into account information only up to a finite time ago. Due to the boundedness of the initial value, and the maximal principle of (1), the convective term propagates information only at a finite speed. Noting also that the fractional derivative term collects information only from the positive direction, the support of $u(t,\cdot)$ with respect to x is always bounded to the right. As far as we solve (1) only for a finite time, we may assume that the support of $u(t,\cdot)$ is always contained in $(-\infty, 0)$, and the problem is reduced to

$$u_t + uu_x = \int_x^0 \frac{u_y(t,y)}{(y-x)^\alpha} dy \text{ for } x < 0. \tag{6}$$

The fractional derivative in (6) is often referred to as the Caputo derivative of order $\alpha \in (0,1)$.

We set a domain of finite length $(-L, 0) \subset \mathbb{R}$, and divide into N grids. For simplicity, we assume that each grid has an equal width of $\Delta x = L/N$ with $k = 0, 1, \cdots, N$ corresponding to position of $x = k\Delta x - L$, but we may extend the scheme to general grid spacing.

The time derivative term is approximated by the first order forward difference, while the spatial derivative is approximated in two ways. The convective term is discretized using the upstream difference, whereas u_y in the fractional derivative is evaluated at the center of each grid, which avoids the problem of singularity. For each time step

$m = 0, 1, \cdots$ and for each spatial position $k = 1, \cdots, N$, when $u(m,k) \geq 0$, we discretize (1) as

$$\frac{u(m+1,k)-u(m,k)}{\Delta t} + u(m,k)\frac{u(m+1,k)-u(m+1,k-1)}{\Delta x} = \sum_{j=k}^{N} \frac{u(m+1,j+1)-u(m+1,j)}{\Delta x}\frac{\Delta x}{\{(j+1/2-k)\Delta x\}^{\alpha}}, \quad (7)$$

while when $u(m,k) \leq 0$, we discretize (1) as

$$\frac{u(m+1,k)-u(m,k)}{\Delta t} + u(m,k)\frac{u(m+1,k+1)-u(m+1,k)}{\Delta x} = \sum_{j=k}^{N} \frac{u(m+1,j+1)-u(m+1,j)}{\Delta x}\frac{\Delta x}{\{(j+1/2-k)\Delta x\}^{\alpha}}. \quad (8)$$

We consider the situation that $u(m,1) \geq 0$ and $u(m,0)$ is prescribed. For all m, we set $u(m,N+1) = 0$. Note that the derivative in the convective term is treated implicitly unlike the usual upwind scheme.

Putting $c_0 := \Delta x/\Delta t$ and $u_m := (u(m,1), \cdots, u(m,N))^{\top} \in \mathbb{R}^N$ for each m, the discretization (7) is expressed in a matrix form of

$$R_m u_{m+1} = u_m, \quad (9)$$

where the $N \times N$ matrix R_m is defined by

$$(R_m)_{ij} = \begin{cases} 0, & \text{if } 1 \leq j < i-1 \\ -\dfrac{u(m,i)}{c_0}, & \text{if } j = i-1 \\ 1 + (1-\delta_{i1})\dfrac{u(m,i)}{c_0} + \dfrac{\Delta t}{(\Delta x)^{\alpha}}\left(\dfrac{1}{2}\right)^{-\alpha}, & \text{if } j = i \\ -\dfrac{\Delta t}{(\Delta x)^{\alpha}}\left\{\left(j-i-\dfrac{1}{2}\right)^{-\alpha} - \left(j-i+\dfrac{1}{2}\right)^{-\alpha}\right\} & \text{if } i < j \leq N \end{cases} \quad (10)$$

when $u(m,i) \geq 0$, and

$$(R_m)_{ij} = \begin{cases} 0, & \text{if } 1 \leq j \leq i-1 \\ 1 + \dfrac{|u(m,i)|}{c_0} + \dfrac{\Delta t}{(\Delta x)^{\alpha}}\left(\dfrac{1}{2}\right)^{-\alpha}, & \text{if } j = i \\ -\dfrac{|u(m,i)|}{c_0} - \dfrac{\Delta t}{(\Delta x)^{\alpha}}\left\{\left(\dfrac{1}{2}\right)^{-\alpha} - \left(\dfrac{3}{2}\right)^{-\alpha}\right\}. & \text{if } j = i+1 \\ -\dfrac{\Delta t}{(\Delta x)^{\alpha}}\left\{\left(j-i-\dfrac{1}{2}\right)^{-\alpha} - \left(j-i+\dfrac{1}{2}\right)^{-\alpha}\right\} & \text{if } i+1 < j \leq N \end{cases} \quad (11)$$

when $u(m,i) \leq 0$.

Obviously, the matrix R_m satisfies following properties.

Proposition 3.1.

1. *R_m is a L-matrix, i.e. all diagonal entries are positive while all off-diagonal entries are nonpositive.*

2. *R_m is an irreducible matrix.*

3. *R_m is a strictly diagonally dominant matrix and hence is regular.*

These properties immediately imply the following theorem.

Theorem 3.2. *The matrix R_m^{-1} is positive for all m and hence is irreducible. In the special case when u_0 is nonnegative, u_m is nonnegative for all m.*

Proof. The latter statement is obtained by induction. □

Next we are going to show that the spectral radius of R_m^{-1} is less than one. For that purpose, let us recall a celebrated theorem of Perron-Frobenius in a brief way.

Proposition 3.3. *(Perron-Frobenius theorem)*
Let P be an irreducible nonnegative square matrix with spectral radius $\rho(P) = r$. Then the following statements hold.

1. *The number r is a positive real number and it is an eigenvalue of the matrix P, called the Perron-Frobenius eigenvalue.*

2. *The Perron-Frobenius eigenvalue r is simple. Both right and left eigenspaces associated with r are one-dimensional.*

3. *P has a right eigenvector v with eigenvalue r whose components are all positive.*

4. *The only eigenvectors whose components are all positive are those associated with the eigenvalue r.*

Lemma 3.4. *If two irreducible and nonnegative matrices A and B commute, i.e. $AB = BA$ holds, then their first eigenvectors coincide up to a factor.*

Proof. Let λ be $\rho(A)$, the spectral radius of A. By the Perron-Frobenius theorem, there exists a positive eigenvector x. Since $ABx = BAx = \lambda Bx$ and λ is a simple eigenvalue, there exists a nonnegative value μ such that $Bx = \mu x$. Again by the Perron-Frobenius theorem, any positive eigenvector of B must belong to the eigenvalue of $\rho(B)$. Thus $\mu = \rho(B)$. As the first eigenvalue is simple, the proof is complete. □

Lemma 3.5. *(Theorem 3.1 in [11]) Let $A = M - N$ be a matrix decomposition such that both A and M are regular. Then $A^{-1}N$ and $M^{-1}N$ commute.*

Lemma 3.6. *Assume the same conditions as Lemma 3.5. If $A^{-1}N$ and $M^{-1}N$ are irreducible, and if A^{-1}, M^{-1}, and N are nonnegative, then $\rho(M^{-1}N) = \dfrac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)} < 1$.*

Proof. This result owes to a portion of Theorem 3.2 in [11], but for the readers' convenience, we prove this based on knowledge already given in the present article. By Lemma 3.5, nonnegative matrices $A^{-1}N$ and $M^{-1}N$ commute. By their irreducibility, Lemma 3.4 implies that their first eigenvectors coincide up to a factor. Let us call it $x(> 0)$. Putting $\mu = \rho(A^{-1}N)(> 0)$, we have $M^{-1}Nx = (I + A^{-1}N)^{-1}A^{-1}Nx$

$= \mu(I + A^{-1}N)^{-1}x$. As $I + A^{-1}N = A^{-1}M$ is regular, $(I + A^{-1}N)^{-1}x = \dfrac{1}{1+\mu}x$ and $M^{-1}Nx = \dfrac{\mu}{1+\mu}x$. Since x belongs to the first eigenvalue of $M^{-1}N$, the proof is complete. □

Theorem 3.7. $\rho(R_m^{-1}) < 1$ *for all* m.

Proof. We regard $(R_m - I) = R_m - I$ is a matrix decomposition as in Lemma 3.5. Note that the matrix $R_m - I$ satisfies the same conditions as listed in Proposition 3.1 for R_m. Thus $R_m - I$ is regular, its inverse is positive, and hence irreducible. Now we are able to apply Lemma 3.6 to obtain the conclusion. □

Finally we show that the scheme satisfies the maximal principle for all m.

Proposition 3.8. *(Theorem 1 in [10]) Suppose* A *is a strictly diagonally dominant matrix. Let* $d_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}|$ *and* $d := \min_i d_i$. *Then* $\|A^{-1}\|_\infty \leq d^{-1}$ *holds.*

Theorem 3.9. *The scheme* (9) *obeys the maximal principle, i.e. it holds* $\|u_{m+1}\|_\infty \leq \|u_m\|_\infty$ *for all* m.

Proof. By the definition of R_m in (10) or (11), we see $\min_i \left(|(R_m)_{ii}| - \sum_{j \neq i} |(R_m)_{ij}| \right) \geq 1$. Applying Lemma 3.8, we have $\|R_m^{-1}\|_\infty \leq 1$ for all m, which implies the maximal principle.

□

4. Finite volume method

In this section, we develop a scheme for (6) based on the finite volume method. We first note that the equation (6) without convection term

$$u_t(t,x) = \int_x^0 \frac{u_y(t,y)}{(y-x)^{1/2}} dy, \quad u(0,x) = u_0(x) \tag{12}$$

is solved using the Laplace transform as

$$u(t,x) = K(t,\cdot) * u_0(\cdot)(x) = \int_x^0 K(t, x-y)\, u_0(y) dy, \tag{13}$$

where the kernel function $K(t,x)$ is given by

$$K(t,x) = \begin{cases} \dfrac{t}{2|x|^{3/2}} \exp\left(-\dfrac{\pi t^2}{4|x|} \right) & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases} \tag{14}$$

Here we list important properties of $K(t,x)$.

Proposition 4.1. (i) $K(t,x) \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$. *In particular, for any nonnegative integer* n, $\partial_x^n K(t,x)|_{x=0} = 0$.

(ii) $K(t,x) = \partial_x \left(\operatorname{erf}\left(\dfrac{\sqrt{\pi} t}{2\sqrt{|x|}} \right) \right)$ *for* $x < 0$ *and* $\int_{\mathbb{R}} K(t,x) dx = 1$, *where the error function is defined by* $\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$.

(iii) $K(t,x) \to \delta(x)$ *as* $t \to +0$ *in the sense of distribution.*

Remark 4.2. *Note that most computer languages available for scientific use are equipped with highly accurate libraries for the error function.*

By Duhamel's principle, the solution to (6) satisfies

$$\begin{aligned} u(t+\tau,x) &= K(\tau,\cdot)*u(t,\cdot)(x) - \int_0^\tau K(\tau-s,\cdot)*\partial_x f(u(t+s,\cdot))(x)ds \\ &= K(\tau,\cdot)*u(t,\cdot)(x) - \int_0^\tau \int_x^0 K(\tau-s,x-y)\,\partial_y f(u(t+s,y))\,dyds, \quad (15) \end{aligned}$$

where $f(z) := \dfrac{1}{2}z^2$ for (1). In the t-direction, we iteratively use (15) for $t = \mathit{\Delta}t, 2\mathit{\Delta}t, \cdots$ with $\tau = \mathit{\Delta}t$. In the x-direction, we assume that u is piecewise constant on $x \in \big((n-\frac{1}{2})\Delta x, (n+\frac{1}{2})\Delta x\big)$ for each integer $n = -N+1, \cdots, -1$. Accordingly, we average (15) over each section. We denote by $\overline{f}(n\mathit{\Delta}x)$ the average of a function f over $\big((n-\frac{1}{2})\Delta x, (n+\frac{1}{2})\Delta x\big)$. The first term on the right hand side of (15) yields

$$\begin{aligned} &\overline{K(\Delta t,\cdot)*u(t,\cdot)}(n\Delta x) = \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x} K(\Delta t,\cdot)*u(t,\cdot)(x)dx \\ &= \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x}\left(\int_{(n+1/2)\Delta x}^{0} + \int_y^{(n+1/2)\Delta x}\right) K(\Delta t,x-y)u(t,y)dydx \\ &= \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x}\sum_{m=n+1}^{-1} u(t,m\mathit{\Delta}x)\int_{(m-1/2)\Delta x}^{(m+1/2)\Delta x} K(\Delta t,x-y)dydx \\ &\quad + \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x} u(t,n\mathit{\Delta}x)\int_y^{(n+1/2)\Delta x} K(\Delta t,x-y)dydx \\ &= \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x}\sum_{m=n+1}^{-1} u(t,m\mathit{\Delta}x)\left[\mathrm{erf}\left(\frac{\sqrt{\pi}\mathit{\Delta}t}{2\sqrt{y-x}}\right)\right]_{y=(m+1/2)\Delta x}^{y=(m-1/2)\Delta x} dx \\ &\quad + \frac{1}{\Delta x}\int_{(n-1/2)\Delta x}^{(n+1/2)\Delta x} u(t,n\mathit{\Delta}x)\left(1-\mathrm{erf}\left(\frac{\sqrt{\pi}\mathit{\Delta}t}{2\sqrt{(n+1/2)\mathit{\Delta}x-x}}\right)\right)dx \\ &\approx \sum_{m=n+1}^{-1} u(t,m\mathit{\Delta}x)\left(\mathrm{erf}\left(\frac{\sqrt{\pi}\mathit{\Delta}t}{2\sqrt{(m-n-1/2)\mathit{\Delta}x}}\right) - \mathrm{erf}\left(\frac{\sqrt{\pi}\mathit{\Delta}t}{2\sqrt{(m-n+1/2)\mathit{\Delta}x}}\right)\right) \\ &\qquad + u(t,n\mathit{\Delta}x)\left(1-\mathrm{erf}\left(\frac{\sqrt{\pi}\mathit{\Delta}t}{\sqrt{2\mathit{\Delta}x}}\right)\right). \quad (16) \end{aligned}$$

In view of the derivation of the balance law (1) and Proposition 4.1 (i), we can justify

$$\begin{aligned} &\int_0^\tau\int_x^0 K(\tau-s,x-y)\partial_y f(u(t+s,y))dyds = \int_0^\tau\int_x^0 \partial_x K(\tau-s,x-y)f(u(t+s,y))dyds \\ &= \frac{d}{dx}\int_0^\tau\int_x^0 K(\tau-s,x-y)f(u(t+s,y))dyds = \frac{d}{dx}\int_0^\tau K(\tau-s,\cdot)*f(u(t+s,\cdot))(x)ds. \\ &\hfill (17) \end{aligned}$$

With $\tau = \mathit{\Delta}t$, we approximate the time-integral in the last term of (17) by

$$\int_0^{\mathit{\Delta}t} g(s,x)ds \approx \frac{\mathit{\Delta}t}{2}\big(g(0,x)+g(\mathit{\Delta}t,x)\big), \quad (18)$$

where we denote the integrand by $g(s,x)$. We also make use of Proposition 4.1 (iii) to obtain

$$g(t+\varDelta t,x)=K(+0,\cdot)*f(u(t+\varDelta t,\cdot))(x)=f(u(t+\varDelta t,x)). \tag{19}$$

With (17), (18), and (19), averaging the second term in the right hand side of (15) over $x\in\big((n-\frac{1}{2})\Delta x,(n+\frac{1}{2})\Delta x\big)$ is approximated as

$$\begin{aligned}
&\overline{-\int_0^{\varDelta t}\int_x^0 K(\varDelta t-s,x-y)\,\partial_y f(u(t+s,y))\,dyds}\\
&\quad\approx\frac{-\varDelta t}{2\varDelta x}\Bigg[f(u(t+\varDelta t,n\varDelta x))-f(u(t+\varDelta t,(n-1)\varDelta x))+\sum_{m=n+1}^{-1}f(u(t,m\varDelta x))\\
&\times\left\{\operatorname{erf}\left(\frac{\sqrt{\pi}\varDelta t}{2\sqrt{(m-n-1)\varDelta x}}\right)-2\operatorname{erf}\left(\frac{\sqrt{\pi}\varDelta t}{2\sqrt{(m-n)\varDelta x}}\right)+\operatorname{erf}\left(\frac{\sqrt{\pi}\varDelta t}{2\sqrt{(m-n+1)\varDelta x}}\right)\right\}\\
&\qquad\qquad+f(u(t,n\varDelta x))\left\{\operatorname{erf}\left(\frac{\sqrt{\pi}\varDelta t}{2\sqrt{\varDelta x}}\right)-1\right\}\Bigg]
\end{aligned}\tag{20}$$

in the similar way as in the derivation of (16). Eventually, our task is reduced to solving a system of algebraic equations of

$$v(n)+\frac{\Delta t}{2\Delta x}\left(f(v(n))-f(v(n-1))\right)=w(n),\quad n=-N+1,\cdots,0, \tag{21}$$

where we put $v(n)=u(t+\Delta t,n\Delta x)$, and $w(n)$ is a certain known quantity determined by $u(t,i\Delta x)$'s. Taking into account boundary conditions, (21) is solved separately for n in the ascending order. Note that the branch should be so chosen that $v(n)\to u(t,n\varDelta x)$ as $\varDelta t\to+0$.

5. Numerical results

We integrate (1) numerically by the finite difference method in section 3, the finite volume method combined with the Duhamel principle in section 4, and compare the results with those computed by a splitting method in [6]. In the splitting method, we solve the (inviscid) Burgers equation [1] for the time interval of $\varDelta t$, and then the purely diffusive equation (12) for the subsequent $\varDelta t$ alternatively. Smaller value of $\varDelta t$ leads to better reliability of the solution. In [6], we use a kind of front tracking method to solve the inviscid Burgers equation. By doing so, we know whether the solution is continuous or discontinuous even in a numerical computation.

The initial value is illustrated in Figure 1. It is nonnegative everywhere, the support is contained in $(-\infty,0)$, and the data is uniform in the vicinity of the left boundary. The steep negative slope in the middle could give rise to a blowup in the sense that $\lim_{t\to T_b-0}\inf_x u_x(t,x)=-\infty$ for a certain finite value T_b.

[1] The Burgers equation originally refers to the inviscid one, i.e. $u_t+uu_x=0$, but in this article, we call it the inviscid Burgers equation in order to differentiate from viscous Burgers equations or from fractional Burgers equations.

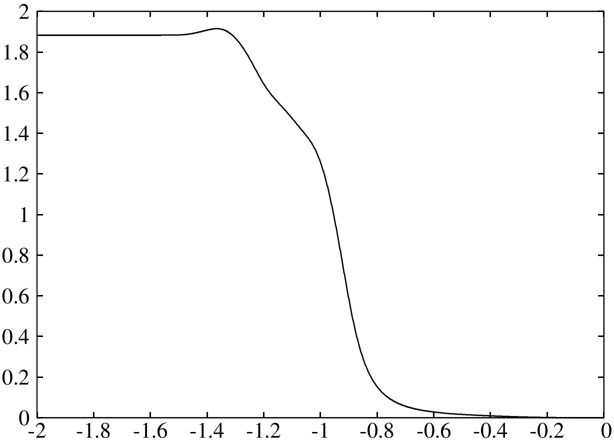
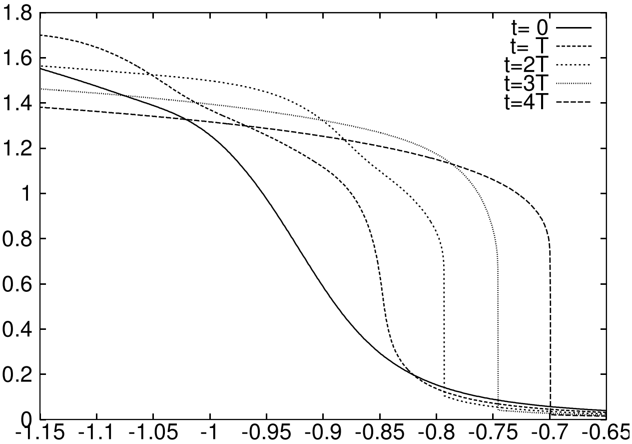
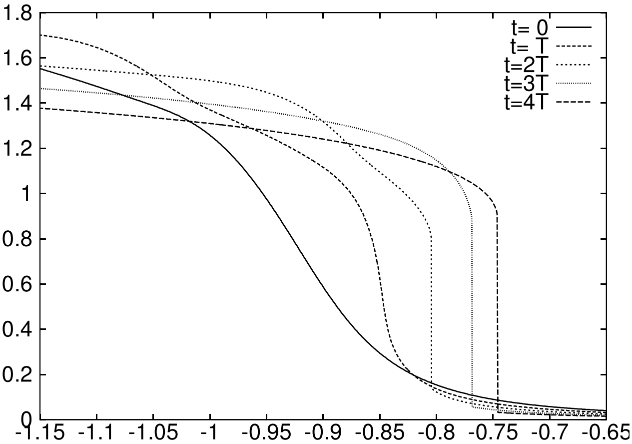


Figure 1: Initial value.



(a) large Δt

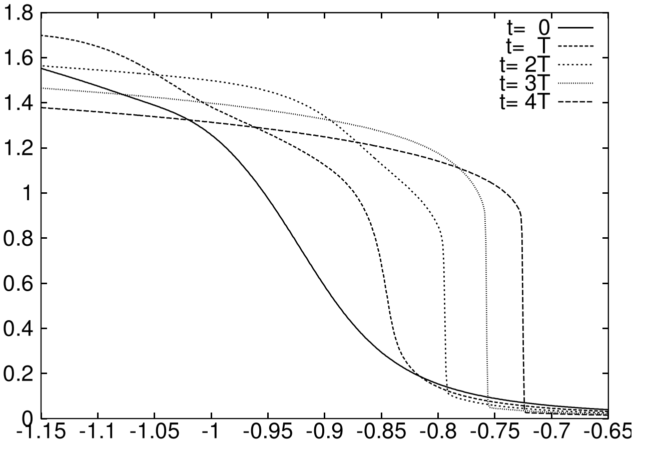


(b) small Δt

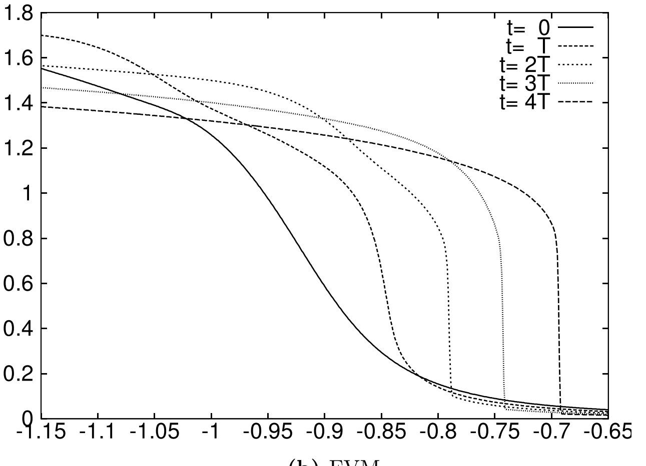
Figure 2: Evolution of profiles computed by splitting method with front tracking [6].

The two panels in Figure 2 show the spatial profile $u(t,\cdot)$ at times $t = 0, T, 2T, 3T, 4T$ computed by the splitting method for a certain time T. The upper(lower) panel in Figure 2 is computed with a large(small) alternation interval. Note the scales of Figures 1 and 2 are not the same. The graphs $u(T,\cdot)$ do not differ so much for two panels in Figure 2. However, the profile advance to the right more rapidly for large Δt after the solution contains discontinuity. This is explained in the following way. Although the solution is regularized in solving (12), smoothing effect is weak compared to the viscous Burgers equation. When we solve the inviscid Burgers equation, the solution could become discontinuous if Δt is sufficiently large. If $u(t,\cdot)$ is discontinuous at $x = d(t)$, then the location $d(t)$ moves according to the Rankine-Hugoniot condition: $\dot{d}(t) = (u(t, d(t) - 0) + u(t, d(t) + 0))/2$. In that case, the area where u_x is negative and $|u_x|$ is large moves faster to the right as compared to the case of small Δt.

Figures 3a and 3b illustrate the evolution of profiles computed by finite difference method(FDM) and finite volume method(FVM) respectively at the same times and in the same spatial domain as in Figures 2.



(a) FDM



(b) FVM

Figure 3: Evolution of profiles computed by FDM(upper) and by FVM(lower).

Figure 4 is an enlarged view of $u(2T, \cdot)$ around the location of discontinuity. The notation SFT1 refers to splitting method with large Δt in Figure 2a, while SFT2 refers to splitting method with small Δt in Figure 2b.

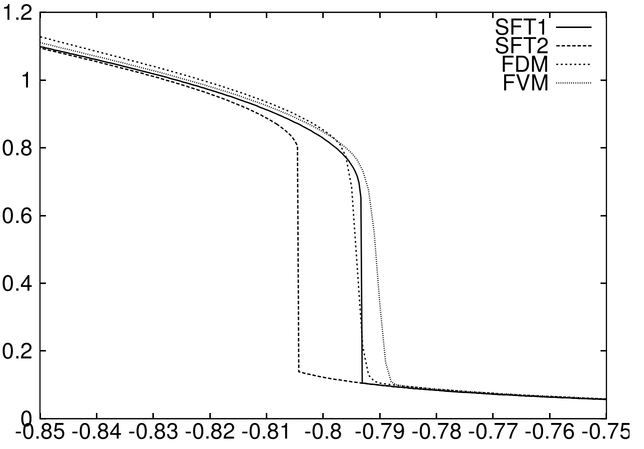


Figure 4: Enlarged view of $u(2T, \cdot)$ computed by four methods. The solid line is for SFT1, the leftmost line is for SFT2, while the rightmost line is for FVM.

Overall, it is observed that FDM yields similar profiles to SFT1 although the first order upwind scheme is often considered to be too diffusive. FDM is easy to code as compared to the front tracking method. Furthermore, it is easy to extend to the multidimensional problem via dimensional splitting. With good properties proved in section 3, FDM is found to be a good numerical method for the fractional Burgers equation. FVM also yields a similar result to other methods, but due to the averaging process, the result is smeared out a little especially in the vicinity of the discontinuity. The benefit of FVM is that the computation is faster than other methods. Though the order of the computational cost is $\mathcal{O}(N^2)$, as large as FDM, the computation time of FVM is less than one-tenth of that of FDM. This is due to the difference in the treatment of fractional derivative term: FDM solves a system of linear equations of (9) with dense matrices (10) or (11), whereas FVM solves a set of algebraic equations (21) by 'diagonalization' thanks to Proposition 4.1 (iii).

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