A SUMMARY ON THE EXISTENCE THEOREM OF DETACHED SHOCK SOLUTIONS OF POTENTIAL FLOW AND A DISCUSSION ABOUT DETACHED SHOCK SOLUTIONS OF FULL EULER SYSTEM

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Abstract. In this paper, we review a result from [1] on the existence of detached shock solutions of steady potential flow past a convex blunt body in $\mathbb{R}^2$, and summarize its proof. Furthermore, we discuss an open problem about detached shock solutions of full Euler system, and explain its difficulties.

1. Preliminaries

For a fixed constant $\gamma > 1$, called an *adiabatic exponent*, the steady compressible Euler system

$$\begin{aligned} &\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0\\ &\partial_{x_1}(\rho u_1 u_j) + \partial_{x_2}(\rho u_2 u_j) + \partial_{x_j} p = 0 \quad \text{for } j = 1, 2\\ &\partial_{x_1}(\rho u_1 B) + \partial_{x_2}(\rho u_2 B) = 0 \quad \text{for } B = \frac{1}{2}(u_1^2 + u_2^2) + \frac{\gamma p}{(\gamma - 1)\rho} \end{aligned} \tag{1.1}$$

governs two dimensional steady flow of inviscid compressible ideal polytropic gas. And, the functions (ρ, u_1, u_2, p) represent density, horizontal and vertical components of velocity, and pressure, respectively. The velocity $\mathbf{u}$ is expressed as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, for $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The function B is called the *Bernoulli invariant*, and it is a constant along each integral curve of the velocity vector field $\mathbf{u}$, provided that $\rho > 0$ holds. To simplify argument, we assume that

$$B = B_0 \tag{1.2}$$

for some constant $B_0 > 0$.

Suppose that $(\rho, \mathbf{u}, p)$ is a C^1 solution to (1.1) with satisfying $\rho > 0$, $u_1 > 0$ and (1.2). Then it satisfies

$$\begin{aligned} &\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0\\ &\partial_{x_1} u_2 - \partial_{x_2} u_1 = \frac{S\rho^{\gamma-1} S_{x_2}}{(\gamma - 1)u_1}\\ &\rho\mathbf{u} \cdot \nabla S = 0\\ &B = B_0 \end{aligned} \tag{1.3}$$

for S given by

$$S := \frac{p}{\rho^\gamma}.$$

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Here, the function S is called the *entropy*. The *vorticity* $\omega := \partial_{x_1} u_2 - \partial_{x_2} u_1$ quantifies the local rotation of the flow. If $S \equiv S_0$ for some constant $S_0 > 0$, then the system (1.3) is further simplified as

$$\begin{aligned} &\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0 \\ &\partial_{x_1} u_2 - \partial_{x_2} u_1 = 0 \\ &S = S_0 \\ &B = B_0. \end{aligned} \tag{1.4}$$

The system (1.4) is called *the steady Euler system of irrotational flow*, or *the steady Euler system of potential flow* in the sense that $\mathbf{u}$ can be represented as $\mathbf{u} = \nabla\varphi$ for a scalar function φ, called a *velocity potential function*. The local sound speed $c = c(\rho)$ and the Mach number $M = M(\rho, \mathbf{u})$ of the system (1.4) are given by

$$c(\rho) = \sqrt{\gamma S_0 \rho^{\gamma-1}}, \quad M(\rho, \mathbf{u}) = \frac{|\mathbf{u}|}{c}, \tag{1.5}$$

respectively. The flow governed by (1.4) is *subsonic* if $M(\rho, \mathbf{u}) < 1$, *sonic* if $M(\rho, \mathbf{u}) = 1$, and *supersonic* if $M(\rho, \mathbf{u}) > 1$. More interestingly, the system (1.4) is elliptic-hyperbolic mixed type if $M(\rho, \mathbf{u}) < 1$, and it is hyperbolic if $M(\rho, \mathbf{u}) > 1$.

In this paper, we review a recent result on the existence of *detached shock solutions of* (1.4) *past a blunt body* when an incoming supersonic flow is prescribed with uniform data with a horizontal velocity. And, we discuss about an open problem on the existence of detached shock solutions of (1.1) past a blunt body.

For a fixed angle $\theta_w \in (0, \frac{\pi}{2})$, let a symmetric wedge W_0 in $\mathbb{R}^2$ with the half-angle θ_w be given by

$$W_0 := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq |x_2| \cot\theta_w\}. \tag{1.6}$$

The blunt body W_b considered in this paper is given as a perturbation of W_0 as follows:

Definition 1.1. *For a fixed constant $h_0 > 0$, let a function $b : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:*

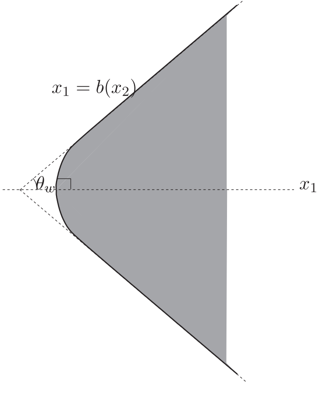

FIGURE 1.1. Blunt body W_b induced from a symmetric wedge W_0

(b_1) *$b(x_2) = b(-x_2)$ for all $x_2 \in \mathbb{R}$;*

(b_2) $b \in C^3(\mathbb{R})$;
(b_3) $b'(x_2) > 0$ *for all* $x_2 > 0$;
(b_4) $b''(x_2) \geq 0$ *for all* $x_2 \geq 0$;
(b_5) $b(x_2) = x_2 \cot \theta_w$ *for* $x_2 \geq h_0$.

For such a function b*, we define a blunt body* W_b *by*

$$W_b := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq b(x_2)\}. \tag{1.7}$$

For simplicity of notations, let us set

$$b_0 := b(0). \tag{1.8}$$

We define a domain $\mathcal{D}$ by

$$\mathcal{D} := \mathbb{R}^2 \setminus W_b.$$

We call $(\rho, u_1, u_2) \in [L^\infty(\mathcal{D})]^3$ a *weak solution of* (1.4) if the following properties are satisfied:

(s_1) $\rho > 0$ a.e. in $\mathcal{D}$;
(s_2) $B = B_0$ and $S = S_0$ pointwisely in $\mathcal{D}$;
(s_3) For any test function $\phi \in C_c^\infty(\mathbb{R}^2)$, it holds that

$$\int_{\mathcal{D}} \rho u_1 \phi_{x_1} + \rho u_2 \phi_{x_2} \, d\mathbf{x} = \int_{\mathcal{D}} u_2 \phi_{x_1} - u_1 \phi_{x_2} \, d\mathbf{x} = 0.$$

Suppose that a non self-intersecting C^1 curve Υ divides $\mathcal{D}$ into two open subdomains $\mathcal{D}^-$ and $\mathcal{D}^+$ so that $\mathcal{D}^- \cap \mathcal{D}^+ = \emptyset$ and $\mathcal{D}^- \cup \Upsilon \cup \mathcal{D}^+ = \mathcal{D}$. A weak solution of (1.4) with a shock Υ is given as a result from an integration by parts in (s_3).

Definition 1.2 (Weak solution of (1.4) with a shock Υ)**.** *We define* $(\rho, u_1, u_2) \in [L^\infty(\mathcal{D}) \cap C^0(\overline{\mathcal{D}^\pm}) \cap C^1_{\text{loc}}(\mathcal{D}^\pm)]^3$ *to be a* weak solution to (1.4) *with a* shock Υ *if the following properties are satisfied:*

(S_1) (ρ, u_1, u_2) *satisfy* (s_1)-(s_2)*, and* Υ *is* C^1;
(S_2) *In* $\mathcal{D}^\pm$, (ρ, u_1, u_2) *satisfy the equations*

$$\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0, \quad \textit{and} \quad \partial_{x_1} u_2 - \partial_{x_2} u_1 = 0 \quad \textit{pointwisely;}$$

(S_3) *For each point* $\mathbf{x}_* \in \Upsilon$*, define*

$$(\rho^+, u_1^+, u_2^+)(\mathbf{x}_*) := \lim_{\substack{\mathbf{x} \to \mathbf{x}_* \\ \mathbf{x} \in \mathcal{D}^+}} (\rho, u_1, u_2)(\mathbf{x}), \quad (\rho^-, u_1^-, u_2^-)(\mathbf{x}_*) := \lim_{\substack{\mathbf{x} \to \mathbf{x}_* \\ \mathbf{x} \in \mathcal{D}^-}} (\rho, u_1, u_2)(\mathbf{x}).$$

Then, (ρ, u_1, u_2) *satisfy the Rankine-Hugoniot conditions*

$$\rho^+(u_1^+, u_2^+) \cdot \boldsymbol{\nu} = \rho^-(u_1^-, u_2^-) \cdot \boldsymbol{\nu}, \quad \textit{and} \quad (u_1^+, u_2^+) \cdot \boldsymbol{\tau} = (u_1^-, u_2^-) \cdot \boldsymbol{\tau} \quad \textit{on } \Upsilon, \tag{1.9}$$

where $\boldsymbol{\nu}$ *is a unit normal, and* $\boldsymbol{\tau}$ *is a unit tangential on* Υ.

(S_4) *On* Υ*, we have*

$$(u_1^+, u_2^+) \cdot \boldsymbol{\nu} \neq 0 \quad (\textit{or equivalently } (u_1^-, u_2^-) \cdot \boldsymbol{\nu} \neq 0),$$

and

$$(u_1^+, u_2^+) \cdot \boldsymbol{\nu} \neq (u_1^-, u_2^-) \cdot \boldsymbol{\nu}.$$

(S_5) *On* $\partial\mathcal{D}$*, the slip boundary condition*

$$(u_1, u_2) \cdot \mathbf{n} = 0$$

holds for the inward unit normal vector field $\mathbf{n}$ *on* $\partial\mathcal{D}$.

Definition 1.3 (Entropy solution)**.** *Let* (ρ, u_1, u_2) *be a weak solution in* $\mathcal{D}$ *with a shock* Υ *in the sense of Definition 1.2. We call the solution* an entropy solution *if*

$$0 < \rho^- < \rho^+ < \infty, \quad \text{and} \quad 0 < (u_1^+, u_2^+) \cdot \boldsymbol{\nu} < (u_1^-, u_2^-) \cdot \boldsymbol{\nu} < \infty \tag{1.10}$$

hold on Υ*, where the unit normal* $\boldsymbol{\nu} = \frac{(u_1^-, u_2^-) - (u_1^+, u_2^+)}{|(u_1^-, u_2^-) - (u_1^+, u_2^+)|}$ *on* Υ *points interior to* $\mathcal{D}^+$*.*

Here, (ρ^-, u_1^-, u_2^-) *is called an incoming state.*

Given constants (γ, S_0, B_0) with $\gamma > 1$, $S_0 > 0$ and $B_0 > 0$, define a set D_∞ of incoming supersonic states by

$$D_\infty(\gamma, S_0, B_0) := \left\{ (\rho_\infty, u_\infty) \in \mathbb{R}^2 : \frac{1}{2}u_\infty^2 + \frac{\gamma S_0 \rho_\infty^{\gamma-1}}{\gamma - 1} = B_0, \; \rho_\infty > 0, \; u_\infty > \sqrt{\gamma S_0 \rho_\infty^{\gamma-1}} \right\}. \tag{1.11}$$

The set $D_\infty(\gamma, S_0, B_0)$ contains all the horizontal uniform supersonic flows with the Bernoulli constant B_0. For $(\rho_\infty, u_\infty) \in D_\infty(\gamma, S_0, B_0)$, set M_∞ as

$$M_\infty := \frac{u_\infty}{\sqrt{\gamma S_0 \rho_\infty^{\gamma-1}}}.$$

Without loss of generality, let assume that $S_0 = 1$ for the rest of the paper unless otherwise specified.

2. The existence of detached shock solutions to the system (1.4)

2.1. **The existence of detached shock solutions.**

In [1], the existence of detached shock solutions to the system (1.4) past the blunt body W_b is proved. In order to state the result more precisely, we first define Hölder norms with weight at infinity.

Definition 2.1. *Fix constants* $m \in \mathbb{Z}^+$, $\mu \in \mathbb{R}$, *and* $\alpha \in (0, 1)$.

(i) For a function $f : \mathbb{R}^+ \to \mathbb{R}$*, define*

$$\|f\|_{m,\mathbb{R}^+}^{(\mu)} := \sum_{j=0}^{m} \sup_{x_2 \in \mathbb{R}^+} (1 + |x_2|)^{j+\mu} \left| \frac{d^j}{dx_2^j} f(x_2) \right|$$

$$[f]_{m,\alpha,\mathbb{R}^+}^{(\mu)} := \sup_{x_2 \neq x_2' \in \mathbb{R}^+} (1 + \min\{|x_2|, |x_2'|\})^{m+\alpha+\mu} \frac{|\frac{d^m}{dx_2^m} f(x_2) - \frac{d^m}{dx_2^m} f(x_2')|}{|x_2 - x_2'|^\alpha}$$

$$\|f\|_{m,\alpha,\mathbb{R}^+}^{(\mu)} := \|f\|_{m,\mathbb{R}^+}^{(\mu)} + [f]_{m,\alpha,\mathbb{R}^+}^{(\mu)}.$$

(ii) Let $D \subset \mathbb{R}^2_+$ *be an open and connected domain. For points* $\mathbf{x}, \mathbf{x}' \in D$*, let* x_2*,* x_2' *denote the* x_2*-coordinates of* $\mathbf{x}, \mathbf{x}'$*, respectively. For a function* $\phi : \overline{D} \to \mathbb{R}$*, define*

$$\|\phi\|_{m,D}^{(\mu)} := \sum_{j=0}^{m} \sup_{\mathbf{x} \in D} (1 + x_2)^{j+\mu} \sum_{0 \le l \le j} |\partial_{x_1}^l \partial_{x_2}^{j-l} \phi(\mathbf{x})|$$

$$[\phi]_{m,\alpha,D}^{(\mu)} := \sup_{\mathbf{x} \neq \mathbf{x}' \in D} (1 + \min\{x_2, x_2'\})^{m+\alpha+\mu} \sum_{0 \le l \le m} \frac{|\partial_{x_1}^l \partial_{x_2}^{m-l} \phi(\mathbf{x}) - \partial_{x_1}^l \partial_{x_2}^{m-l} \phi(\mathbf{x}')|}{|\mathbf{x} - \mathbf{x}'|^\alpha}$$

$$\|\phi\|_{m,\alpha,D}^{(\mu)} := \|\phi\|_{m,D}^{(\mu)} + [\phi]_{m,\alpha,D}^{(\mu)}.$$

Since the domain $\mathcal{D}$ is symmetric about x_1-axis, the main result of [1] on the existence of detached shock solutions past W_b is stated on the upper half plane $\mathbb{R}^2_+ := \mathbb{R}^2 \cap \{x_2 \ge 0\}$.

Theorem 2.2. [1, Theorem 2.13] *Fix $\gamma > 1$ and $B_0 > 0$. And, fix $\beta \in (0,1)$.*

(a) (The existence of detached shock solutions) *For a fixed constant $d_0 > 0$, there exists a small constant $\bar{\varepsilon} > 0$ depending on (γ, B_0, d_0) so that if the incoming supersonic state $(\rho_\infty, u_\infty) \in D_\infty(\gamma, 1, B_0)$ satisfies $M_\infty = \frac{1}{\varepsilon}$ for $\varepsilon \in (0, \bar{\varepsilon}]$, then the system (1.4) has an entropy solution $(\rho, \mathbf{u})$ in $\mathbb{R}^2_+ \setminus W_b$ for $\mathbf{u} = (u_1, u_2)$ with a shock $\Upsilon_{\rm sh} = \{(f_{\rm sh}(x_2), x_2) : x_2 \geq 0\}$ in the sense of Definition 1.3 for the incoming state $(\rho_\infty, u_\infty, 0)$. And, the solution satisfies the following properties:*

(i) $f_{\rm sh}(0) = b_0 - d_0$;

(ii) *There exists a constant $\delta > 0$ depending only on (γ, B_0, d_0) such that*

$$b(x_2) - f_{\rm sh}(x_2) \geq \delta \quad \text{for all } x_2 \geq 0;$$

(iii) *Setting as $\Omega_{f_{\rm sh}} := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+ \setminus W_b : x_1 > f_{\rm sh}(x_2),\ x_2 > 0\}$, we have*

$$\lim_{\substack{|\mathbf{x}| \to \infty \\ \mathbf{x} \in \Omega_{f_{\rm sh}}}} |(\rho, \mathbf{u})(\mathbf{x}) - (\rho^\varepsilon_{\rm st}, \mathbf{u}^\varepsilon_{\rm st})| = 0, \quad \text{and} \quad \lim_{x_2 \to \infty} |f'_{\rm sh}(x_2) - s^\varepsilon_{\rm st}| = 0$$

for the uniform state $(\rho^\varepsilon_{\rm st}, \mathbf{u}^\varepsilon_{\rm st}, s^\varepsilon_{\rm st})$ uniquely determined as a strong shock state corresponding to the half-wedge angle θ_w on the shock polar curve of the incoming state (ρ_∞, u_∞). Here, $\mathbf{u}^\varepsilon_{\rm st} = (u^\varepsilon_1, u^\varepsilon_2)$ is a constant vector in $\mathbb{R}^2$.

(iv) *There exists a constant $\hat{\alpha} \in (0,1)$ depending only on θ_w, and a constant $C > 0$ depending only on (γ, B_0, d_0) such that*

$$\|f_{\rm sh} - f_0\|^{(-\beta)}_{2,\hat{\alpha},\mathbb{R}^+} + \|\mathbf{u} - \mathbf{u}^\varepsilon_{\rm st}\|^{(1-\beta)}_{1,\hat{\alpha},\Omega_{f_{\rm sh}}} \leq C\varepsilon^{\frac{2}{\gamma-1}} \tag{2.1}$$

for the functions f_0 defined by

$$f_0(x_2) := s^\varepsilon_{\rm st} x_2 + b_0 - d_0. \tag{2.2}$$

(v) *There exists a constant $\sigma \in (0,1)$ depending only on (γ, B_0, d_0) so that the Mach number $M(\rho, \mathbf{u})$ defined by (1.5) satisfies the inequality*

$$M(\rho, \mathbf{u}) \leq 1 - \sigma \quad \text{in } \overline{\Omega_{f_{\rm sh}}}.$$

In other words, the flow in $\Omega_{f_{\rm sh}}$ is subsonic, thus $\Upsilon_{\rm sh}$ is a transonic shock *in the sense that the flow changes from supersonic to subsonic across the shock $\Upsilon_{\rm sh}$.*

(b) (Convexity of detached shocks) *For a fixed constant $d_0 > 0$, let $\bar{\varepsilon}$ be from Theorem 2.2(a). Then, there exists a constant $\hat{\varepsilon} \in (0, \bar{\varepsilon}]$ depending on (γ, B_0, d_0) so that if the incoming supersonic state $(\rho_\infty, u_\infty) \in D_\infty(\gamma, B_0)$ satisfies $M_\infty = \frac{1}{\varepsilon}$ for $\varepsilon \in (0, \hat{\varepsilon}]$, then the system (1.4) has an entropy solution $(\rho, \mathbf{u})$ in $\mathbb{R}^2_+ \setminus W_b$ with a shock $\Upsilon_{\rm sh} = \{(f_{\rm sh}(x_2), x_2) : x_2 \geq 0\}$ that satisfies*

$$f''_{\rm sh}(x_2) \geq 0 \quad \text{for } x_2 > 0$$

as well as all the properties (i)–(v) stated in Theorem 2.2(a).

2.2. Discussion about Theorem 2.2 (a).

Far-field asymptotic limit: We first explain how $(\rho^\varepsilon_{\rm st}, u^\varepsilon_1, u^\varepsilon_2, s^\varepsilon_{\rm st})$ is given.

It follows from Definition 1.2 and the statement (iii) of Theorem 2.2(a) that $(\rho^\varepsilon_{\rm st}, u^\varepsilon_1, u^\varepsilon_2, s^\varepsilon_{\rm st})$ satisfies the following equations for (ρ, u_1, u_2, s):

$$\begin{aligned} &\rho(u_1 - su_2) = \rho_\infty \\ &u_1 s - u_2 = u_\infty s \\ &\frac{1}{2}((u_1)^2 + (u_2)^2) + \frac{\gamma\rho^{\gamma-1}}{\gamma - 1} = B_0. \end{aligned} \tag{2.3}$$

According to [1, Lemma 2.5], if $M_\infty(=\frac{1}{\varepsilon})$ is sufficiently large, or equivalently if ε is sufficiently small, then the set of solutions (ρ, u_1, u_2, s) to (2.3) with satisfying the entropy condition in the sense of Definition 1.3 is nonempty. Furthermore, there exist exactly two solutions $(\rho^{(1)}, u_1^{(1)}, u_2^{(1)}, s^{(1)})$ and $(\rho^{(2)}, u_1^{(2)}, u_2^{(2)}, s^{(2)})$ that satisfy the slip boundary condition

$$(u_1, u_2)\cdot \mathbf{n}_b = 0 \quad \text{on } \partial W_b \cap \{x_2 > h_0\}$$

for a unit normal $\mathbf{n}_b$ on ∂W_b. And, we have $|(u_1^{(1)}, u_2^{(1)}) - (u_\infty, 0)| \neq |(u_1^{(2)}, u_2^{(2)}) - (u_\infty, 0)|$. Without loss of generality, we assume that $|(u_1^{(1)}, u_2^{(1)}) - (u_\infty, 0)| > |(u_1^{(2)}, u_2^{(2)}) - (u_\infty, 0)|$. The state $(\rho^{(1)}, u_1^{(1)}, u_2^{(1)}, s^{(1)})$ yields a *strong shock solution* of (1.4) past the symmetric wedge W_0 of half-wedge angle θ_w, and the state $(\rho^{(2)}, u_1^{(2)}, u_2^{(2)}, s^{(2)})$ yields a *weak shock solution*. The far-field asymptotic limit $(\rho_{\rm st}^\varepsilon, \mathbf{u}_{\rm st}^\varepsilon, s_{\rm st}^\varepsilon)$ from Theorem 2.2(a) is equal to $(\rho^{(1)}, u_1^{(1)}, u_2^{(1)}, s^{(1)})$. The words *'strong'* and *'weak'* are given because we have $0 < s^{(1)} < s^{(2)} < \cot\theta_w$. Since $(\rho_{\rm st}^\varepsilon, \mathbf{u}_{\rm st}^\varepsilon)$ is the state behind a strong shock, it follows from a shock polar analysis([1, 3]) that the Mach number $M_\varepsilon = \frac{|\mathbf{u}_{\rm st}^\varepsilon|}{\sqrt{\gamma S_0 (\rho_{\rm st}^\varepsilon)^{\gamma-1}}}$ of the state $(\rho_{\rm st}^\varepsilon, \mathbf{u}_{\rm st}^\varepsilon, s_{\rm st}^\varepsilon)$ is strictly less than 1.

Outline of the proof of Theorem 2.2(a): In [1], Theorem 2.2(a) is proved by a stream function formulation. If (ρ, u_1, u_2) is a C^1 solution to (1.4) in a domain, then the equation $\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0$ in (1.4) implies that there exists a C^2-function ψ to satisfy

$$\nabla^\perp \psi = (\rho u_1, \rho u_2) \quad \text{for } \nabla^\perp\psi := (\psi_{x_2}, -\psi_{x_1}). \tag{2.4}$$

Such a function ψ is called a *stream function* in the sense that ψ is a constant along each integral curve of the momentum density vector field $\rho\mathbf{u} = \rho(u_1, u_2)$. Then, by applying the implicit function theorem, one can show that there exists a unique smooth function $\hat\rho = \hat\rho(|\mathbf{q}|^2)$ so that if the Mach number $M(= \frac{|\mathbf{u}|}{\sqrt{\gamma S_0 \rho^{\gamma-1}}})$ is less than 1, then the system (1.4) can be simplified as

$$\operatorname{div}\left(\frac{\nabla\psi}{\hat\rho(|\nabla\psi|^2)}\right) = 0. \tag{2.5}$$

And, Theorem 2.2 (a) can be proved by solving the following free boundary problem for $(\psi, f_{\rm sh})$ in $\mathbb{R}^2_+ \setminus W_b$:

Problem 2.3 (Stream function formulation of free boundary problem)**.** *Fix a constant $d_0 > 0$. Find a function $f_{\rm sh} \in C^1_{\rm loc}(\mathbb{R}^+)$ with satisfying $f_{\rm sh}(x_2) < b(x_2)$ for $x_2 \geq 0$ and a function $\psi \in C^1_{\rm loc}(\overline{\Omega_{f_{\rm sh}}}) \cap C^2_{\rm loc}(\Omega_{f_{\rm sh}})$ so that the following properties hold:*

(i)

$$|\nabla\psi| < \left(\frac{2(\gamma-1)}{\gamma+1}B_0\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{in} \quad \overline{\Omega_{f_{\rm sh}}} \quad (\Longleftrightarrow M \neq 1 \quad \text{in} \quad \overline{\Omega_{f_{\rm sh}}})$$

for $\Omega_{f_{\rm sh}} := \{(x_1, x_2) \in \mathbb{R}^2_+ : f_{\rm sh}(x_2) < x_1 < b(x_2)\}$

(ii) *(Equation for ψ)*

$$\operatorname{div}\left(\frac{\nabla\psi}{\hat\rho(|\nabla\psi|^2)}\right) = 0 \quad \text{in} \quad \Omega_{f_{\rm sh}}$$

(iii) *(Boundary conditions for ψ) Define*

$$\Upsilon_{\rm sh} := \{(f_{\rm sh}(x_2), x_2) : x_2 \geq 0\}, \quad \Gamma_{\rm sym} := \{(x_1, 0) : f_{\rm sh}(0) < x_1 < b(0)\},$$
$$\Gamma_b := \{(b(x_2), x_2) : x_2 \geq 0\}.$$

Then ψ *satisfies the following boundary conditions:*

$$\begin{aligned} \psi &= \rho_\infty u_\infty x_2 \quad \text{on } \Upsilon_{\mathrm{sh}}, \\ \psi &= 0 \quad \text{on } \Gamma_{\mathrm{sym}} \cup \Gamma_b. \end{aligned} \tag{2.6}$$

(Asymptotic boundary condition) In addition, ψ *satisfies*

$$\lim_{\substack{|\mathbf{x}|\to\infty \\ \mathbf{x}\in\Omega_{f_{\mathrm{sh}}}}} |\nabla^\perp\psi(\mathbf{x}) - \rho^\varepsilon_{\mathrm{st}}\mathbf{u}^\varepsilon_{\mathrm{st}}| = 0. \tag{2.7}$$

(iv) *(Free boundary condition)*

$$\begin{aligned} f'_{\mathrm{sh}}(x_2) &= \frac{\left(\psi_{x_1}/\hat\rho(|\nabla\psi|^2)\right)(f_{\mathrm{sh}}(x_2),x_2)}{\left(\psi_{x_2}/\hat\rho(|\nabla\psi|^2)\right)(f_{\mathrm{sh}}(x_2),x_2) - u_\infty} \quad \text{for all } x_2 > 0, \\ f_{\mathrm{sh}}(0) &= b_0 - d_0. \end{aligned} \tag{2.8}$$

In [1], Problem 2.3 is solved in two steps:

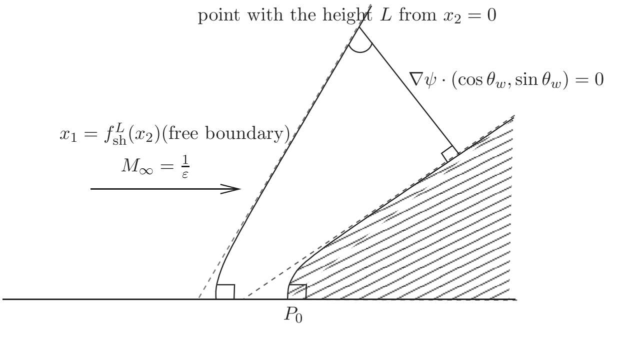

FIGURE 2.1. A cut-off domain

Step 1. Given a sufficiently large constant L, a free boundary problem for $(\psi^L, f^L_{\mathrm{sh}})$ is formulated with the additional boundary condition $\nabla\psi^L\cdot(\cos\theta_w, \sin\theta_w) = 0$ given on a cut-off boundary, where the cut-off boundary has an end point away from the boundary of the blunt body W_b with its height L from the line $x_2 = 0$ (Fig. 2.1). Here, this end point is to be determined in solving the free boundary problem. For convenience, we call L *the height of the cut-off boundary*.

And, the free boundary problem in the cut-off domain is solved by applying Schauder fixed point theorem under the assumption of largeness of M_∞ depending only on (γ, B_0, S_0, d_0).

In this step, a local uniqueness of a solution to the free boundary problem can be additionally achieved by applying the contraction mapping principle if M_∞ is sufficiently large. But, the largeness of M_∞ may depend on L to guarantee the local uniqueness.

Step 2. Fix a sequence $\{L_n\}_{n=1}^\infty$ so that each L_n is sufficiently large with $L_n < L_{n+1}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} L_n = \infty$. For each $n \in \mathbb{N}$, one can solve the free boundary problem formulated in step 1 in a cut-off domain with a cut-off boundary of the height L_n. Let $(\psi_n, f_{\mathrm{sh},n})$ be a solution to the free boundary problem. Then, one can extract a subsequence from $\{f_{\mathrm{sh},n}\}_{n=1}^\infty$ so that it converges to a function $f_{\mathrm{sh},\infty} : \mathbb{R}^+ \to \mathbb{R}$ in C^2 on any compact subset of $\mathbb{R}^+$. This procedure is done by applying Arzelá-Ascoli theorem and a diagonal argument. Then, a solution $(\psi, f_{\rm sh})$ to Problem 2.3 can be constructed by using the limit function $f_{{\rm sh},\infty}$, the sequence $\{\psi_n\}$ and a limiting argument.

Further discussion on Theorem 2.2(a): Note that Theorem 2.2(a) cannot guarantee the uniqueness of a detached shock solution. Suppose that $(\rho, \mathbf{u}, f_{\rm sh})$ and $(\tilde\rho, \tilde{\mathbf{u}}, \tilde f_{\rm sh})$ are two detached shock solutions that satisfy all the properties (i)–(v) stated in Theorem 2.2 (a). Then, one can compute functions ψ and $\tilde\psi$ from $(\rho, \mathbf{u}, f_{\rm sh})$ and $(\tilde\rho, \tilde{\mathbf{u}}, \tilde f_{\rm sh})$, respectively, so that $(\psi, f_{\rm sh})$ and $(\tilde\psi, \tilde f_{\rm sh})$ solve Problem 2.3. If we had

$$\lim_{x_2\to\infty} |f_{\rm sh}(x_2) - \tilde f_{\rm sh}(x_2)| = 0,$$

then it would follow from a contracting argument that $f_{\rm sh} = \tilde f_{\rm sh}$ on $\mathbb{R}^+$, thus $\psi = \tilde\psi$ in $\Omega_{f_{\rm sh}} (= \Omega_{\tilde f_{\rm sh}})$ for sufficiently large M_∞. But, the best estimate of $|f_{\rm sh}(x_2) - \tilde f_{\rm sh}(x_2)|$ obtained from statement (iv) of Theorem 2.2(a) is

$$|f_{\rm sh}(x_2) - \tilde f_{\rm sh}(x_2)| \le C x_2^{\beta}$$

for some constant $C > 0$.

And, for each $(\rho_\infty, u_\infty) \in D_\infty(\gamma, 1, B_0)$, one can construct a family of detached shock solutions of (1.4) with different values of d_0. A qualitative analysis shows that if $d_0 \ge \underline{d}$ for some $\underline{d} > 0$, then the estimate constants $(\bar\varepsilon, \delta, C, \sigma)$ in Theorem 2.2(a) can be chosen depending only on $(\gamma, B_0, \underline{d})$. Therefore, Theorem 2.2(a) implies that there exists a small constant $\varepsilon_* > 0$ depending on $(\gamma, B_0, \underline{d})$ so that if $(\rho_\infty, u_\infty) \in D_\infty(\gamma, 1, B_0)$ satisfies $M_\infty \ge \frac{1}{\varepsilon_*}$, then for each $d_0 \ge \underline{d}$, there exists at least one detached shock solutions $(\rho, \mathbf{u}, f_{\rm sh})$ with

$$f_{\rm sh} = b_0 - d_0.$$

This yields infinitely many detached shock solutions for a fixed incoming supersonic data. In order to pick *a physically admissible detached shock solution*, a further analysis on structural or dynamical stability of detached shock solutions would be necessary.

2.3. **Discussion about Theorem 2.2 (b).**

Outline of the proof of Theorem 2.2(b): In [1], Theorem 2.2 (b) is proved in four steps.

Step 1. For a fixed a constant L sufficiently large, let $(\psi, f_{\rm sh})$ be a solution to the free boundary problem in a cut-off domain with a cut-off boundary of the height L from the line $x_2 = 0$. See §2.2 for the description on how a cut-off free boundary problem is formulated to solve Problem 2.3.

For $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ given by

$$u_1\mathbf{e}_1 + u_2\mathbf{e}_2 := \frac{\nabla^\perp\psi}{\hat\rho(|\nabla\psi|^2)},$$

it is shown in [1, Lemma 6.3] that $u_1 > 0$ holds away from the vertex point P_0 of the blunt body W_b. See Fig. 2.1. This implies that the speed $|\mathbf{u}|$ is strictly positive away from P_0. In fact, P_0 is a stagnation point, that is, $|\mathbf{u}(P_0)| = 0$. This can be checked by using the boundary condition (2.6), which corresponds to the slip boundary condition of $\mathbf{u}$ on $\Gamma_{\rm sym} \cup \Gamma_b$, and C^1 regularity of ψ up to the boundary, which implies the continuity of $\mathbf{u}$.

Step 2. Away from the point P_0, define

$$\Theta := \arctan\frac{u_2}{u_1}, \quad Q := \ln|\mathbf{u}|.$$

By differentiating the equation $B = B_0$ in the direction of $\mathbf{u}$, we get the expression

$$\mathbf{u} \cdot \nabla\rho = -\frac{|\mathbf{u}|}{\rho^{\gamma-2}}\mathbf{u} \cdot \nabla|\mathbf{u}|. \tag{2.9}$$

Next, we rewrite the first equation in (1.4) as $\nabla \cdot \mathbf{u} + \frac{\mathbf{u}\cdot\nabla\rho}{\rho} = 0$, then combine (2.9) with this equation to get

$$\frac{1}{|\mathbf{u}|}\nabla \cdot \mathbf{u} - \frac{1}{\rho^{\gamma-1}}\mathbf{u} \cdot \nabla|\mathbf{u}| = 0. \tag{2.10}$$

We reduce the system (1.4) into the system of (2.10) and the second equation in (1.4), and rewrite the reduced system in terms of (Θ, Q) away from P_0 to get two differential equations for (Θ, Q). From this rewritten system, one can directly derive a second order differential equation as follows:

$$\sum_{i,j=1}^{2} \partial_{x_i}(a_{ij}\partial_{x_j}Q) = 0 \quad \text{away from } P_0 \tag{2.11}$$

for

$$a_{11} = 1 - M^2\cos^2\Theta, \quad a_{12} = -a_{21} = M^2\sin\Theta\cos\Theta, \quad a_{22} = 1 - M^2\sin^2\Theta.$$

And, the equation (2.11) is uniformly elliptic in $\{x_1 > f_{\rm sh}(x_2)\}$ as we seek for a subsonic flow behind a detached shock (the statement (v) of Theorem 2.2(a)). Then, by using maximum principle and Hopf's lemma, it can be shown that if Q has a local extremum at a point P_*, then P_* cannot lie

- in the interior of subsonic region $\{x_1 > f_{\rm sh}(x_2)\}$;
- on the cut-off boundary;
- on the shock $\Gamma_{\rm sh} := \{x_1 = f_{\rm sh}(x_2) : 0 \le x_2 \le L\}$.

This implies that if $|\mathbf{u}|(= e^Q)$ has a local extremum at a point P_*, then P_* must lie on either the boundary of the blunt body W_b, or on the symmetric line $\Gamma_{\rm sym}$.

Step 3. A direct computation with using the Rankine-Hugoniot conditions (1.9) yields that

$$\operatorname{sgn} f''_{\rm sh}(x_2) = \operatorname{sgn} \frac{d|\mathbf{u}|}{dx_2}(f_{\rm sh}(x_2), x_2) \quad \text{for } 0 < x_2 < L.$$

Therefore, if it is proved that

$$\frac{d}{dx_2}|\mathbf{u}(f_{\rm sh}(x_2), x_2)| \ge 0 \quad \text{for } 0 < x_2 < L, \tag{2.12}$$

then it directly implies that

$$f''_{\rm sh}(x_2) > 0 \quad \text{for } 0 < x_2 < L. \tag{2.13}$$

The inequality (2.12) can be proved by using the result established in Step 2. Throughout Step 1 to Step 3, the main tools are the maximum principle and Hopf's lemma. In order to prove (2.12), however, it requires an additional observation. More precisely, the convexity of the blunt body W_b, given in the statement (b_4) of Definition 1.1 plays an important role in proving (2.12).

Remark 2.4. *The analysis in* [1] *shows that the convexity of the blunt body* W_b *is a sufficient condition to establish* (2.12). *But, it is unclear whether the condition* (b_4) *in Definition 1.1 can be removed in proving* (2.12).

Step 4. Finally, Theorem 2.2(b) is proved by a limiting argument similar to Step 2 in §2.2.

3. Discussion about detached shock solutions to the system (1.1)

A weak solution of (1.1) with a shock Υ is defined almost same as Definition 1.2 except that its entropy$(=\frac{p}{\rho^\gamma})$ jumps across the shock Υ in general, thus the vorticity$(=\nabla\times\mathbf{u})$ is generated across the shock Υ even though an incoming flow is irrotational.

As in §1, suppose that a non self-intersecting C^1-curve Υ divides $\mathcal{D}$ into two open subdomains $\mathcal{D}^-$ and $\mathcal{D}^+$ so that $\mathcal{D}^-\cap\mathcal{D}^+=\emptyset$ and $\mathcal{D}^-\cup\Upsilon\cup\mathcal{D}^+=\mathcal{D}$.

Definition 3.1 (Weak solution of (1.1) with a shock Υ). *We define* $(\rho,u_1,u_2,p)\in[L^\infty(\mathcal{D})\cap C^0(\overline{\mathcal{D}^\pm})\cap C^1_{\rm loc}(\mathcal{D}^\pm)]^4$ *to be a* weak solution to (1.1) *with a* shock Υ *if the following properties are satisfied:*

(S_1') (ρ,u_1,u_2,p) *is a weak solution to* (1.1) *in* $\mathcal{D}$ *in the sense of distribution, and* Υ *is* C^1*;*

(S_2') *In* $\mathcal{D}^\pm$, (ρ,u_1,u_2,p) *satisfy the equations stated in* (1.1) *pointwisely;*

(S_3') *For each point* $\mathbf{x}_*\in\Upsilon$, *define*

$$(\rho^+,u_1^+,u_2^+,p^+)(\mathbf{x}_*):=\lim_{\substack{\mathbf{x}\to\mathbf{x}_*\\ \mathbf{x}\in\mathcal{D}^+}}(\rho,u_1,u_2,p)(\mathbf{x}),\quad (\rho^-,u_1^-,u_2^-,p^-)(\mathbf{x}_*):=\lim_{\substack{\mathbf{x}\to\mathbf{x}_*\\ \mathbf{x}\in\mathcal{D}^-}}(\rho,u_1,u_2,p)(\mathbf{x}).$$

Then, (ρ,u_1,u_2,p) *satisfy the following Rankine-Hugoniot conditions on* Υ*:*

$$\begin{aligned}
&\rho^+(u_1^+,u_2^+)\cdot\boldsymbol{\nu}=\rho^-(u_1^-,u_2^-)\cdot\boldsymbol{\nu}\\
&(u_1^+,u_2^+)\cdot\boldsymbol{\tau}=(u_1^-,u_2^-)\cdot\boldsymbol{\tau},\\
&\rho^+|(u_1^+,u_2^+)\cdot\boldsymbol{\nu}|^2+p^+=\rho^-|(u_1^-,u_2^-)\cdot\boldsymbol{\nu}|^2+p^-\\
&\frac12|(u_1^+,u_2^+)\cdot\boldsymbol{\nu}|^2+\frac{\gamma p^+}{(\gamma-1)\rho^+}=\frac12|(u_1^-,u_2^-)\cdot\boldsymbol{\nu}|^2+\frac{\gamma p^-}{(\gamma-1)\rho^-}
\end{aligned}$$

where $\boldsymbol{\nu}$ *is a unit normal, and* $\boldsymbol{\tau}$ *is a unit tangential on* Υ.

(S_4') *On* Υ, *we have*

$$(u_1^+,u_2^+)\cdot\boldsymbol{\nu}\neq0\quad\textit{(or equivalently }(u_1^-,u_2^-)\cdot\boldsymbol{\nu}\neq0\textit{)},$$

and

$$(u_1^+,u_2^+)\cdot\boldsymbol{\nu}\neq(u_1^-,u_2^-)\cdot\boldsymbol{\nu}.$$

(S_5') *On* $\partial\mathcal{D}$, *the slip boundary condition*

$$(u_1,u_2)\cdot\mathbf{n}=0$$

holds for the inward unit normal vector field $\mathbf{n}$ *on* $\partial\mathcal{D}$.

If $(\rho,\mathbf{u},p)$ with $\mathbf{u}=(u_1,u_2)$ is a weak solution of (1.1) with a shock Υ, and if it is an entropy solution in the sense of Definition 1.3, then a direct computation with using (S_3') stated in Definition 3.1 yields that the entropy $S^+(:=\frac{p^+}{(\rho^+)^\gamma})$ of the state $(\rho^+,\mathbf{u}^+,p^+)$ is given by

$$S^+=\frac{\frac{2\gamma}{\gamma+1}\rho^-(\mathbf{u}^-\cdot\boldsymbol{\nu})^2(1+\frac{p^-}{\rho^-(\mathbf{u}^-\cdot\boldsymbol{\nu})^2})+p^-}{(\rho^+)^\gamma}\quad\text{for}\quad\rho^+=\frac{\rho^-(\mathbf{u}^-\cdot\boldsymbol{\nu})^2}{\frac{2(\gamma-1)}{\gamma+1}(\frac12(\mathbf{u}^-\cdot\boldsymbol{\nu})^2+\frac{\gamma p^-}{(\gamma-1)\rho^-})}\quad\text{on }\Upsilon. \tag{3.1}$$

Therefore, even if $(\rho^-,\mathbf{u}^-,p^-)$ is a uniform state, the entropy S^+ behind a shock Υ can be a non-constant function unless the shock Υ is a straight line so that $\mathbf{u}^-\cdot\boldsymbol{\nu}$ is a constant along Υ. This observation combined with the vorticity equation

$$\nabla\times\mathbf{u}=\frac{S\rho^{\gamma-1}S_{x_2}}{(\gamma-1)u_1}$$

stated in (1.3) implies that the vorticity is generated across a shock in general even if the incoming supersonic flow is irrotational. So it is natural to employ the system (1.1) in order to precisely analyze shock phenomena. Thus, we are led to the following questions:

Question 1. Does the system (1.1) have an entropy solution (ρ, u_1, u_2, p) with a *detached shock* $\Upsilon_{\rm sh}$ in $\mathbb{R}^2 \setminus W_b$?

Question 2. If it does, is the shock $\Upsilon_{\rm sh}$ convex?

It is our conjecture that the system (1.1) has an entropy solution with a detached shock in $\mathbb{R}^2 \setminus W_b$. But there is a difficulty. To construct a detached shock solution, the first step would be to find a far-field asymptotic limit of the solution. But, differently from the case of irrotational flow, we cannot choose a strong shock solution (which is a piecewise constant solution with a straight shock) given from a shock polar analysis as a far-field asymptotic limit. A detached shock $\Upsilon_{\rm sh}$ is a curve not a straight line (this can be easily checked by a local analysis with using Definition 1.1 and Rankine-Hugoniot conditions, stated in (S_3') of Definition 3.1) so the entropy S^+ given by (3.1) is a non-constant function along the shock $\Upsilon_{\rm sh}$. Therefore, the far-field asymptotic limit must be a non-constant vector field because the entropy behind a shock is given as a solution of the transport equation $\rho\mathbf{u} \cdot \nabla S = 0$.

The convexity of a detached shock $\Upsilon_{\rm sh}$ seems even more difficult to prove for the case of the system (1.1) than the case of irrotational flow. The proof of Theorem 2.2(b) in [1] significantly relies on the fact that the entropy is assumed to be globally constant. The constant entropy yields the homogeneous differential equation $\partial_{x_1} u_2 - \partial_{x_2} u_1 = 0$ which represents a zero-vorticity state. And, this equation is used to derive several homogeneous second order elliptic differential equations of physical variables such as u_1, u_2 and the speed $\sqrt{u_1^2 + u_2^2}$ in proving Theorem 2.2(b). See Eq. (2.11) for an example. By applying the maximum principle and Hopf's lemma to those equations, a non-vanishing property or a monotonicity of physical variables such as u_1, u_2 and the Mach number M are obtained. And, these properties are key ingredients in proving the convexity of a detached shock. For the system (1.1), on the other hand, the vorticity($= \partial_{x_1} u_2 - \partial_{x_2} u_1$) is generally nonzero behind a shock, and its sign is same as the sign of $\frac{S_{x_2}}{u_1}$. As we seek for a detached shock solution with $u_1 > 0$ away from the vertex point P_0 of the blunt body, the vorticity equation implies that the sign of the vorticity entirely depends on the sign of S_{x_2}. Since the value of S is determined by (3.1) and the transport equation $\rho\mathbf{u} \cdot \nabla S = 0$, one can speculate that the sign of S_{x_2} depends on how the normal direction of a shock curve $\Upsilon_{\rm sh}$ changes. Therefore, it may be difficult to prove the existence of a detached shock $\Upsilon_{\rm sh}$ past the blunt body W_b and the convexity of $\Upsilon_{\rm sh}$ separately. Instead, we should try to construct a detached shock solution of (1.1) past the blunt body W_b with a convex shock $\Upsilon_{\rm sh}$. This would require a new iteration method. In addition, we should investigate whether the convexity of detached shock past a convex blunt body is an inevitable consequence. Many examples of convex detached shocks are observed in nature. But no rigorous understanding on their mechanisms is given up to this day.

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