Rapid energy decay of the Navier-Stokes flow by the external force

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1 Introduction

Let $n \geq 3$. We consider the incompressible Navier-Stokes equations in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = \nabla \cdot f & \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = a & \text{in } \mathbb{R}^n, \end{cases} \tag{N-S}$$

where $u = u(x,t) = \big(u_1(x,t), \dots, u_n(x,t)\big)$ and $\pi = \pi(x,t)$ denote the unknown velocity and the pressure of the fluid at $(x,t) \in \mathbb{R}^n \times (0,\infty)$, respectively, while, $f = f(x,t) = \big(f_{k\ell}(x,t)\big)_{k,\ell=1,\dots,n}$ denotes the external forcing tensor and $a = a(x) = \big(a_1(x), \dots, a_n(x)\big)$ denotes the given initial data.

After the distinguished work of Leray [8], the time decay problem is one of main interests in mathematical fluid mechanics. Masuda [9], Schonbek [14], Kajikiya and Miyakawa [6] and Wiegner [16], for instance, gave pioneering works on this direction. Nowadays, it is well known that the optimal decay rate for a weak solution is described as

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{n+2}{4}}, \qquad t > 0, \tag{1.1}$$

for initial data $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ which satisfies $\int_{\mathbb{R}^n} (1+|x|)|a(x)|\,dx < \infty$. Then Fujigaki-Miyakawa [4] clarified that the decay rate as in (1.1) actually describes decay rate of the nonlinear terms, deriving the asymptotic expansion of the linear part and of the nonlinear part as follows

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\| u_j(t) + \sum_{k=1}^{n}(\partial_k E_t)(\cdot)\int_{\mathbb{R}^n} y_k a_j(y)\,dy + \sum_{\ell,k=1}^{n} F_{\ell k,j}(\cdot,t)\int_0^\infty\int_{\mathbb{R}^n}(u_\ell u_k)(y,s)\,dyds\Big\|_q = 0$$

for all $j = 1,\dots,n$ and for all $1 \le q \le \infty$, where $E_t(x) = (4\pi t)^{-n/2}\exp(\frac{|x|^2}{4t})$ and $F_{\ell k,j}(x,t) = \partial_\ell E_t(x)\delta_{\ell k} + \int_t^\infty \partial_\ell\partial_k\partial_j E_s(x)\,ds$. Once the principal terms is revealed, Miyakawa and Schonbek [12] revealed the necessary and sufficient condition that the first order principal terms vanish as following:

$$\int_{\mathbb{R}^n} y_k a_j(y)\,dy = 0 \qquad \text{for } j,k = 1,\dots,n, \tag{1.2}$$

and there exists some constant $c \in \mathbb{R}$ such that

$$\int_0^\infty\int_{\mathbb{R}^n}(u_k u_\ell)(y,s)\,dyds = c\delta_{k\ell} \qquad \text{for } k,\ell = 1,\dots,n. \tag{1.3}$$

From the above, the condition (1.3) seems to make difficulty to obtain a rapid decay since we need information of the (unknown) flow over whole space-time region. Therefore, for this difficulty, some group action on the flow plays an important role in verification of (1.3). Indeed, the first author introduced so-called cyclic symmetry of the flow, i.e.,

(a) u_j is odd in x_j and even in each other variables,

(b) $u_1(x_1,\dots,x_n) = u_2(x_n,x_1,\dots,x_{n-1}) = \cdots = u_n(x_2,\dots,x_n,x_1)$.

With the aid of (a) and (b), the first author [1], Miyakawa [10, 11] derived the rapid decay with the rate which corresponds to the second order terms or to the third order terms in the asymptotic expansion of the flow. Later, the second author and Tsutsui [13] gave a generalization of [2], [10, 11] with weighted Hardy spaces. On the other hand, more specific group action was discussed by the first author [3]. However, the symmetry, like (a) and (b), seems to be somehow artificial. Moreover, it is natural question that any non symmetric flow has a chance to evolve the rapid-decay flow, cancelling the slow decay factors.

The aim of this article is to enlarge the possibility of rapid decay without any symmetry. As is mentioned above, the essential difficulty is still the verification of (1.3). In stead of the cyclic symmetry, for any initial state we try to control the flow by the external force. In other words, for any initial data which is small in a suitable sense, we find a associated external force and a corresponding solution of the Navier-Stokes equations. This approach seems to be natural and reasonable in not only mathematical analysis but also physics or engineering, since the flow is forced to be calm down by a artificial forcing term if initial state is given.

Our essential idea is to construct a force $\nabla \cdot f$ so that

$$\int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}f(s) \sim \sum_{k,\ell=1}^{n} F_{\ell k,j}(\cdot,t)\int_0^\infty\int_{\mathbb{R}^n}(u_\ell u_k)(y,s)\,dyds \quad \text{for large } t>0, \tag{1.4}$$

for a direct counteraction of the leading terms of nonlinear Duhamel term, where $\mathbb{P}$ is so-called the Leray-Hopf, the Weyl-Helmholtz or the Fujita-Kato projection on to solenoidal vectors. For the realization of (1.4), we introduce the following iteration as a computable procedure:

$$u^{(m)}(t) = e^{t\Delta}a + \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}f^{(m)}(s)\,ds - \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}[u^{(m)}\otimes u^{(m)}]\,ds, \tag{1.5}$$

$m = 1, 2, \ldots$. Here, the forcing tensor $f^{(m)} = (f_{k\ell}^{(m)})$ is given by $f^{(0)} \equiv 0$ and

$$f_{k\ell}^{(m)}(x,t) = \begin{cases} c_{k\ell}^{(m-1)}\phi(x,t), & k \neq \ell, \\ (c_{kk} - \bar{c}^{(m-1)})\phi(x,t), & k = \ell, \end{cases}$$

for some function $\phi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, where $c_{k\ell}^{(m)} = \int_0^\infty\int_{\mathbb{R}^n}(u_k^{(m)}u_\ell^{(m)})(y,s)\,dyds$ and $\bar{c}^{(m-1)} = c_{11}^{(m)} + \cdots + c_{nn}^{(m)}$. We note that since we are able to take ϕ compact supported in both space and time, in order to control of the flow, it is enough that the force is applied to finite time and bounded space region.

In our scheme, we have a difficulty that we need to derive a bound C, independent of m, such that

$$c_{k\ell}^{(m)} = \int_0^\infty\int_{\mathbb{R}^n}(u_k^{(m)}u_\ell^{(m)})(y,s)\,dyds \left(\leq \int_0^\infty \|u^{(m)}(s)\|_2^2\,ds\right) < C, \tag{1.6}$$

since the size of $f^{(m)}$, i.e., one of $c_{k\ell}^{(m-1)}$ determines the global existence of $u^{(m)}$ according to Fujita-Kato method. To begin with, we try to use time decay like (1.1) which is derived by the Fourier splitting technique. However, it is unable to be adopted since the constant as in (1.1) depends on the solution u itself. See, for instance, [14, 6, 16].

Due to the above difficulty, we should develop a new approach and we should establish the time decay estimate like (1.1) with the constant C which depends only on the given data a, ϕ and the dimension n. For this purpose, the weighted Hardy spaces is effective. Indeed, the second author and Tsutsui [13] introduced the weighted Hardy space to derive a higher order asymptotic expansion, since the weighted Hardy space enables us to deal with higher order weights and to obtain more rapid decay compared with the weighted Lebesgue spaces. With the aid of the weighted Hardy norm, we make a specific refinement of the Fujita-Kato iteration scheme which gives a bound as in (1.6) and yields the convergence our procedure as (1.5).

2 Results

Before stating results, we introduce the following notations and some function spaces. Let $C_0^\infty(\Omega)$ denote the set of all C^∞-functions (or vectors) with compact support in a connected set Ω. Let $C_{0,\sigma}^\infty(\mathbb{R}^n)$ denote the set of all C^∞-solenoidal vectors ϕ with compact support in $\mathbb{R}^n$, i.e., $\operatorname{div}\phi = 0$ in $\mathbb{R}^n$. $L^r_\sigma(\mathbb{R}^n)$ is the closure of $C_{0,\sigma}^\infty(\mathbb{R}^n)$ with respect to the L^r-norm $\|\cdot\|_r$, $1 < r < \infty$. $L^r(\mathbb{R}^n)$ and $W^{m,r}(\mathbb{R}^n)$ denote the usual (vector-valued) L^r-Lebesgue space and L^r-Sobolev space over $\mathbb{R}^n$, respectively. Moreover, $\mathscr{S}(\mathbb{R}^n)$ denotes the set of all of the Schwartz functions. $\mathscr{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions. When X is a Banach space, $\|\cdot\|_X$ denotes the norm on X. Moreover, $C(I;X)$, $BC(I;X)$ and $L^r(I;X)$ denote the X-valued continuous and bounded continuous functions over the interval $I \subset \mathbb{R}$, and X-valued L^r functions, respectively.

We introduce the weighted Hardy space $H^p_\alpha(\mathbb{R}^n)$ with the homogeneous power weight $w(x) = |x|^\alpha$ for $\alpha \in \mathbb{R}$ and for $0 < p \le \infty$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Then the maximal function of f is denoted by $M_\varphi[f](x) = \sup_{\lambda>0} |\varphi_\lambda * f(x)|$ where $\varphi_\lambda(x) = \lambda^{-n}\varphi(x/\lambda)$ for $\lambda > 0$. Then we define $H^p_\alpha(\mathbb{R}^n)$ as

$$H^p_\alpha(\mathbb{R}^n) := \{f \in \mathscr{S}'(\mathbb{R}^n); \|f\|_{H^p_\alpha} < \infty\},$$

where

$$\|f\|_{H^p_\alpha} = \|M_\varphi[f]\|_{L^p_\alpha} = \left(\int_{\mathbb{R}^n} |M_\varphi[f](x)|^p |x|^{\alpha p}\,dx\right)^{\frac{1}{p}}.$$

Here we note that $L^p_\alpha(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n); \int_{R^n} |f(x)|^p|x|^{\alpha p}dx < \infty\}$ and note that $L^\infty_\alpha(\mathbb{R}^n)$ denotes $L^\infty(\mathbb{R}^n)$. Furthermore, we note that in the case

$1 < p < \infty$ and $-n/p < \alpha < \infty$, $H^p_\alpha(\mathbb{R}^n) = L^p_\alpha(\mathbb{R}^n)$ if and only if $\alpha < n(1-1/p)$. See also [13].

Theorem 2.1 (Fujigaki-Miyakawa [4]). *Let* $a \in L^1(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$ *and* $f \in C_0^\infty\big(\mathbb{R}^n \times [0,\infty)\big)$. *Suppose* $u \in BC\big([0,\infty); L^n_\sigma(\mathbb{R}^n)\big)$ *is a global mild solution of (N-S). If* $\int_{\mathbb{R}^n} |x||a(x)|\,dx < \infty$, *then it holds that*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\| u_j(t) + \sum_{k=1}^n (\partial_k E_t)(\cdot)\int_{\mathbb{R}^n} y_k a_j(y)\,dy - \sum_{k,\ell=1}^n F_{\ell k,j}(\cdot,t)\int_0^\infty\int_{\mathbb{R}^n} f_{k\ell}(y,s)\,dyds + \sum_{k,\ell=1}^n F_{\ell,j,k}(\cdot,t)\int_0^\infty\int_{\mathbb{R}^n}(u_\ell u_k)(y,s)\,dyds\Big\|_q = 0$$

for $1 \le q \le \infty$.

Remark 2.1. *Though [4] dealt with only the case* $f \equiv 0$, *the proof is essentially same. The derivation of the leading order term for the Duhamel term of* f *is just analogy of that for the nonlinear term.*

As an immediate consequence from Miyakawa and Schonbek [12], the condition associated with (1.3) is modified as follows:

Corollary 2.2. *Suppose* $a \in L^n_\sigma(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n}(1+|x|)|a(x)|\,dx < \infty$, $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ *and* u *is a global mild solution of (N-S). Then it holds that for* $1 \le q \le \infty$

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|u(t)\|_q = 0$$

if and only if

$$\int_{\mathbb{R}^n} y_k a(y)\,dy = 0 \tag{2.1}$$

and

$$\int_0^\infty\int_{\mathbb{R}^n} f_{k\ell}(y,s)\,dyds - \int_0^\infty\int_{\mathbb{R}^n}(u_\ell u_k)(y,s)\,dyds = c\delta_{k\ell} \tag{2.2}$$

for some $c \in \mathbb{R}$ *for all* $k,\ell = 1,\ldots,n$.

Furthermore, if (2.1)*and* (2.2) *does not hold then*

$$\liminf_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|u(t)\|_q > 0$$

for $1 \le q \le \infty$.

Remark 2.2. *we note that if*

$$\int_0^\infty \int_{\mathbb{R}^n} f_{k\ell}(y,s)\,dyds \neq \int_0^\infty \int_{\mathbb{R}^n} f_{\ell k}(y,s)\,dyds \qquad \text{for some } k \text{ and } \ell,$$

then the condition (2.2) *does not hold, i.e.,*

$$\liminf_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|u(t)\|_q > 0, \qquad 1 \le q \le \infty.$$

Hence, if the tensor F is not symmetry in the above sense, we never expect rapid time decay even though no matter fast F decays at spatial infinity and time infinity.

Corollary 2.2 is just an analogy of [12], but enables us to control the flow by the external forces. Indeed, for a general initial data, we derive a rapid energy decay $\|u(t)\|_2 = o(t^{-\frac{n+2}{4}})$ as $t \to \infty$ by a suitable external force.

Theorem 2.3. *Let $1 < \gamma < \frac{2n}{n+2}$. Then there exists $\delta = \delta(n,\gamma) > 0$ with the following property. If $a \in H_1^\gamma(\mathbb{R}^n) \cap L_\sigma^n(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n}(1+|x|)|a(x)|\,dx < \infty$, $\int_{\mathbb{R}^n} x_k a(x)\,dx = 0$ for all $k = 1,\dots,n$ and*

$$\max\left\{\|a\|_n, \|a\|_2, \|a\|_{H_1^\gamma}\right\} \le \delta,$$

then there exists $f \in C_0^\infty\big(\mathbb{R}^n \times [0,\infty)\big)$ and a solution u of (N-S) such that

$$\|u(t)\|_2 = o(t^{-\frac{n+2}{4}}) \text{ as } t \to \infty.$$

Remark 2.3. *(i) Since f has a compact support in the time interval, after the effect of the force vanishes, the flow is governed by*

$$u(t) = e^{t\Delta}u(t_0) - \int_{t_0}^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}(u\otimes u)(s)\,ds, \qquad t > t_0.$$

So we have two possibilities.

The first case is that the first order moment of $u(t)$ is preserved, i.e., $\int_{\mathbb{R}^n}(1+|x|)|u(x,t)|\,dx < \infty$ for $t \ge t_0$. In this case, due to the [12], the condition (1.3) *implies*

$$\int_{\mathbb{R}^n}(u_\ell u_k)(x,t)\,dx = 0 \qquad \text{for } k \neq \ell, \quad t \ge t_0,$$

which is discussed in [2] and is realized under the cyclic symmetry, see [11].

Another case is that the moment of $u(t)$ is not preserved. In this case, the linear part $e^{t\Delta}u(t_0)$ and the nonlinear part $\int_{t_0}^t \nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes u)\,ds$ make a nontrivial interaction and slow decay factors are cancelled.

(ii) The range of the support of f is determined only by δ and the initial data a.

Since $\|\cdot\|_2$ and $\|\cdot\|_{H_1^\gamma}$ are not scale invariant norms for (N-S), the scale argument implies the following corollary.

Corollary 2.4. *Let $1 < \gamma < \frac{2n}{n+2}$ and $\delta = \delta(n,\gamma) > 0$ be the same as in Theorem 2.3. If $a \in H_1^\gamma(\mathbb{R}^n) \cap L_\sigma^n(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n}(1+|x|)|a(x)|\,dx < \infty$, $\int_{\mathbb{R}^n} x_k a(x)\,dx = 0$ for all $k = 1,\dots,n$ and*

$$\|a\|_n < \delta,$$

then the conclusion of Theorem 2.3 holds.

3 Preliminaries

The following is well-known L^p-L^p estimate, whose constants play an important role in our approach.

Proposition 3.1 ([15, Corollary 1.1][13, Lemma 3.2])**.** *Let $1 \le p \le q < \infty$.*

(i) Then there exists a constant $C_{qp} > 0$ such that

$$\|e^{t\Delta}a\|_q \le C_{qp} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p, \qquad t > 0, \tag{3.1}$$

$$\|\nabla e^{t\Delta}a\|_q \le C_{qp} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|a\|_p, \qquad t > 0. \tag{3.2}$$

for a function, velocity vector or tensor $a \in L^p(\mathbb{R}^n)$.

(ii) For $-\frac{n}{q} < \beta \le \alpha < \infty$ there exists a constant $C_{q,p}^{\alpha,\beta} > 0$ such that

$$\|e^{t\Delta}a\|_{H_\beta^q} \le C_{q,p}^{\alpha,\beta} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha-\beta}{2}}\|a\|_{H_\alpha^p}, \quad t > 0.$$

In this article, for $1 < r < \infty$, the projection $\mathbb{P} : L^r(\mathbb{R}^n) \to L_\sigma^r(\mathbb{R}^n)$ satisfies $\|\mathbb{P}u\|_r = A_r\|u\|_r$ for all $u \in L^r(\mathbb{R}^n)$ with some constant $A_r > 0$.

4 Sketch of proof

Sequentially regenerating forces $f^{(m)}$, $m = 1, 2, \dots$, we construct an associated solution $u^{(m)}$ of the Navier-Stokes equations:

$$u^{(m)} = e^{t\Delta}a + \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}f^{(m)}(s)\,ds - \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}[u^{(m)}\otimes u^{(m)}](s)\,ds, \tag{N-S$_m$}$$

for $m = 1, 2, \ldots$. Here, denoting $u^{(0)}$ is a Navier-Stokes flow with $f \equiv 0$, we put for $m = 1, 2, \ldots,$

$$f_{k\ell}^{(m)}(x,t) = \begin{cases} c_{k\ell}^{(m-1)}\phi(x,t) & k \neq \ell, \\ (c_{kk}^{(m-1)} - \bar{c}^{(m-1)})\phi(x,t) & k = \ell, \end{cases}$$
$$c_{k\ell}^{(m)} = \int_0^\infty \int_{\mathbb{R}^n} u_k^{(m)} u_\ell^{(m)}\, dyds, \qquad k, \ell = 1, \ldots, n,$$
$$\bar{c}^{(m)} = c_{11}^{(m)} + \cdots + c_{nn}^{(m)}.$$

So we construct each solution $u^{(m)}(t)$ of (N-S$_m$) by Fujita-Kato method.

We put for $m = 1, 2, \ldots,$

$$\begin{aligned} K_0^{(m)} &= \sup_{0<t<\infty} t^{\frac{1}{2}-\frac{n}{2r}} \|e^{t\Delta}a\|_r + \sup_{0<t<\infty} t^{\frac{1}{2}-\frac{n}{2r}} \left\| \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}f^{(m)}(s)\, ds \right\|_r \\ &= K_0^{(0)} + \sup_{0<t<\infty} t^{\frac{1}{2}-\frac{n}{2r}} \left\| \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}f^{(m)}(s)\, ds \right\|_r. \end{aligned}$$

Then if $K_0^{(m)}$ is suitably small, we obtain a solution $u^{(m)}$ of (N-S$_m$) with uniform bounds.

On the other hand, the essential idea is that $|c_{k\ell}^{(0)}| < 1$ and $|c_{kk}^{(0)} - \bar{c}^{(0)}| < 1$ for $k, \ell = 1, \ldots, n$. Moreover, assuming $|c_{k\ell}^{(m-1)}| < 1$ and $|c_{kk}^{(m-1)} - \bar{c}^{(m-1)}| < 1$ for $k, \ell = 1, \ldots, n$, we investigate a suitable smallness for a and ϕ yields $|c_{k\ell}^{(m)}| < 1$ and $|c_{kk}^{(m)} - \bar{c}^{(m)}| < 1$ for $k, \ell = 1, \ldots, n$, inductively for $m = 1, 2, \ldots$.

In order to observe this, we note that

$$\begin{aligned} \left\| \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}f^{(m)}(s)\, ds \right\|_r &\leq C_{rn}A_n \int_0^t (t-s)^{-1+\frac{n}{2r}} s^{-\frac{1}{2}}\, ds \sup_{0<s<\infty} s^{\frac{1}{2}} \|f^{(m)}(s)\|_n \\ &= t^{-\frac{1}{2}+\frac{n}{2r}} C_{rn}A_n B\Big(\frac{n}{2r}, \frac{1}{2}\Big) \sup_{0<s<\infty} s^{\frac{1}{2}} \|f^{(m)}(s)\|_n, \end{aligned}$$

and by the assumption $|c_{k\ell}^{(m-1)}| < 1$ and $|c_{kk}^{(m-1)} - \bar{c}^{(m-1)}| < 1$ for $k, \ell = 1, \ldots, n$, we see that

$$\|f^{(m)}(s)\|_p \leq \sum_{k\neq\ell} |c_{k\ell}^{(m-1)}| \|\phi(s)\|_p + \sum_{k=1}^n |c_{kk}^{(m-1)} - \bar{c}^{(m-1)}| \|\phi(s)\|_p \leq n^2 \|\phi(s)\|_p,$$

for $1 \leq p < \infty$. So we obtain that

$$K_0^{(m)} \leq K_0^{(0)} + n^2 C_1 \sup_{0<s<\infty} s^{\frac{1}{2}} \|\phi(s)\|_n.$$

Hence, if

$$K_0^{(m)} \le K_0^{(0)} + n^2 C_1 \sup_{0<s<\infty} s^{\frac{1}{2}} \|\phi(s)\|_n \ll 1, \tag{4.1}$$

then we obtain a solution $u^{(m)}$ of (N-S$_m$). Since $f \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, with the aid of local solvability and uniqueness for (N-S$_m$), $u^{(m)}$ is actually a strong solution of the Navier-Stokes equations for $f^{(m)}$. Moreover, by the itertion in $BC([0,\infty); L^2(\mathbb{R}^n))$ we derive $|c_{k\ell}^{(m)}| < 1$ under suitable smallness on initial data a.

Finally, we consider the convergence of $u^{(m)}$ in $BC\big([0,\infty); L^2(\mathbb{R}^n)\big)$ as $m \to \infty$. We note that

$$\begin{aligned} u^{(m+1)}(t) - u^{(m)}(t) &= \int_0^t \nabla \cdot e^{(t-s)\Delta} \mathbb{P}[f^{(m+1)} - f^{(m)}](s)\, ds \\ &\quad + \int_0^t \nabla \cdot e^{(t-s)\Delta} \mathbb{P}\big[(u^{(m+1)} - u^{(m)}) \otimes u^{(m+1)}\big](s)\, ds \\ &\quad + \int_0^t \nabla \cdot e^{(t-s)\Delta} \mathbb{P}\big[u^{(m)} \otimes (u^{(m+1)} - u^{(m)})\big](s)\, ds \\ &=: \mathcal{I}_1^{(m)}(t) + \mathcal{I}_2^{(m)}(t) + \mathcal{I}_3^{(m)}(t). \end{aligned}$$

Then we have

$$\|\mathcal{I}_2(t)\|_2 \le C_2 K_0^{(m+1)} \sup_{0<s<\infty} \|u^{(m+1)}(s) - u^{(m)}(s)\|_2,$$

By the same manner, we have that

$$\|\mathcal{I}_3^{(m)}(t)\|_2 \le C_2 K_0^{(m)} \sup_{0<s<\infty} \|u^{(m+1)}(s) - u^{(m)}(s)\|_2.$$

Next, we estimate $\mathcal{I}_1^{(m)}$ as

$$\|\mathcal{I}_1^{(m)}(t)\|_2 \le C \sup_{0<s<\infty} s^{\frac{1}{2} - \frac{n}{2r}} \|f^{(m+1)}(s) - f^{(m)}(s)\|_\rho.$$

Here, we see that

$$\begin{aligned} \|f^{(m+1)}(s) - f^{(m)}(s)\|_\rho &\le \sum_{k \ne \ell} C_3 \big|c_{k\ell}^{(m)} - c_{k\ell}^{(m-1)}\big| \|\phi(s)\|_n \\ &\quad + \sum_{k=1}^n C_3 \big|(c_{kk}^{(m)} - \overline{c}_{kk}^{(m)}) - (c_{kk}^{(m-1)} - \overline{c}_{kk}^{(m-1)})\big| \|\phi(s)\|_n \end{aligned}$$

Moreover, we obtain that

$$\sum_{k\neq\ell}\big|c_{k\ell}^{(m)} - c_{k\ell}^{(m-1)}\big| \le C_4 \sup_{0<s<\infty} \|u^{(m)}(s) - u^{(m-1)}(s)\|_2,$$

and that

$$\sum_{k=1}^{n}\big|(c_{kk}^{(m)} - \bar{c}_{kk}^{(m)}) - (c_{kk}^{(m-1)} - \bar{c}_{kk}^{(m-1)})\big| \le C_4 \sup_{0<s<\infty} \|u^{(m)}(s) - u^{(m-1)}(s)\|_2$$

Therefore we obtain that $\|\mathcal{I}_1^{(m)}(t)\|_2 \le 2C_4 \sup_{0<s<\infty} s^{\frac{1}{2}-\frac{n}{2r}}\|\phi(s)\|_n \sup_{0<s<\infty} \|u^{(m)}(s) - u^{(m-1)}(s)\|_2$. Hence if $2C_2K_0^{(m)} < 1/4$, $2C_2K_0^{(m+1)} < 1/4$ and $C_4 \sup_{0<s<\infty} s^{\frac{1}{2}-\frac{n}{2r}}\|\phi(s)\|_n < 1/4$ then

$$\sup_{0<t<\infty} \|u^{(m+1)}(t) - u^{(m)}(t)\|_2 \le \frac{1}{4} \sup_{0<s<\infty} \|u^{(m)}(s) - u^{(m-1)}(s)\|_2 + \frac{1}{2} \sup_{0<s<\infty} \|u^{(m+1)}(s) - u^{(m)}(s)\|_2.$$

Hence we see that

$$\sup_{0<t<\infty} \|u^{(m+1)}(t) - u^{(m)}(t)\|_2 \le \frac{1}{2} \sup_{0<s<\infty} \|u^{(m)}(s) - u^{(m-1)}(s)\|_2,$$

for $m = 1, 2, \dots$. This implies $u^{(m)}$ converges to some v in $BC\big([0,\infty); L^2(\mathbb{R}^n)\big)$ as $m \to \infty$. By the same manner, we are able to show $u^{(m)}$ converges to v in $BC\big([0,\infty); L^r(\mathbb{R}^n)\big)$ as $m \to \infty$.

As a conclusion, the limit functions are desired the solution and the external force.

References

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