Large Time Behavior of Solutions to the Nonlinear Hyperbolic Relaxation System with Slowly Decaying Data

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1 Introduction

This paper is concerned with the large time asymptotic behavior of the global solutions to the initial value problem for the following system:

$$\begin{aligned} &u_t + v_x = 0, \quad v_t + u_x = f(u) - v, \ x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function. This system is a typical example of hyperbolic system of conservation laws with relaxation called Jin-Xin model, which describes many physical phenomena such as non-equilibrium gas dynamics, magnetohydrodynamics, viscoelasticity and flood flow with friction (see e.g. [11, 22]).

If we eliminate v from (1.1), we obtain the following damped wave equation with a nonlinear convection term:

$$\begin{aligned} &u_{tt} - u_{xx} + u_t + (f(u))_x = 0, \ x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where the initial data $u_1(x) = -\partial_x v_0(x)$. In this paper, we consider (1.2) with the flux function $f(u) \equiv au + \frac{b}{2}u^2 + \frac{c}{3!}u^3$, where $|a| < 1$, $b \neq 0$ and $c \in \mathbb{R}$. In addition, for the initial data, we assume that

$$\begin{aligned} &\exists \alpha > 1, \quad \exists C > 0 \ \ s.t. \ \ |u_0(x)| \le C(1+|x|)^{-\alpha}, \quad x \in \mathbb{R}, \\ &\exists \beta > 1, \quad \exists C > 0 \ \ s.t. \ \ |u_1(x)| \le C(1+|x|)^{-\beta}, \quad x \in \mathbb{R}. \end{aligned} \tag{1.3}$$

The purpose of this study is to obtain an asymptotic profile of the solution $u(x,t)$ and to examine the optimality of its asymptotic rate to the asymptotic function.

First of all, let us recall the known results about the asymptotic behavior of the solutions to (1.2). Orive and Zuazua [20] studied the global existence and the asymptotic behavior of the solutions to (1.2) with $a = 0$ when $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. In [21], Ueda and Kawashima generalized the results in [20] to the case $f(u)$ satisfies the so called sub-characteristic condition $|f'(0)| = |a| < 1$. In addition,

This paper is a summary of the original paper [5] by the author.

they constructed the solutions to (1.2), provided the initial data $u_0 \in W^{1,p}(\mathbb{R}) \cap L^1(\mathbb{R})$ and $u_1 \in L^p(\mathbb{R}) \cap L^1(\mathbb{R})$ for $1 \leq p \leq \infty$. Moreover, they studied the detailed asymptotic behavior of the solution for $|a| < 1$. To discuss the asymptotic behavior, we apply the Chapman-Enskog expansion (cf. [1, 17]) to (1.1) and derive a viscous conservation law

$$w_t + (f(w))_x = (\mu(w)w_x)_x \tag{1.4}$$

as the second order approximation of the expansion, where $\mu(w) = 1 - (f'(w))^2$. Here, we note that the sub-characteristic condition $|f'(w)| < 1$ implies the parabolicity of (1.4). Therefore, one can expect that the solution to (1.2) is approximated by the solution to (1.4) or its simpler version (Burgers equation):

$$w_t + \left(aw + \frac{b}{2}w^2\right)_x = \mu w_{xx}, \tag{1.5}$$

where $\mu = 1 - a^2$. Actually, under the additional condition $u_0, u_1 \in L^1_1(\mathbb{R})$, it was shown in [21] that the solution of (1.2) converges to the nonlinear diffusion wave which is a modification of the self-similar solution of the Burgers equation (1.5) and is defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}}\chi_*\left(\frac{x - a(1+t)}{\sqrt{1+t}}\right), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.6}$$

where

$$\chi_*(x) \equiv \frac{\sqrt{\mu}}{b}\frac{(e^{\frac{bM}{2\mu}} - 1)e^{-\frac{x^2}{4\mu}}}{\sqrt{\pi} + (e^{\frac{bM}{2\mu}} - 1)\int_{x/\sqrt{4\mu}}^{\infty} e^{-y^2}dy}, \quad M \equiv \int_{\mathbb{R}} (u_0(x) + u_1(x))dx, \quad \mu \equiv 1 - a^2. \tag{1.7}$$

More precisely, if $u_0 \in W^{1,p}(\mathbb{R}) \cap L^1_1(\mathbb{R})$, $u_1 \in L^p(\mathbb{R}) \cap L^1_1(\mathbb{R})$ and $\|u_0\|_{W^{1,p}} + \|u_0\|_{L^1} + \|u_1\|_{L^p} + \|u_1\|_{L^1}$ is sufficiently small, then, for any $\varepsilon > 0$, we have

$$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t))\|_{L^p} \leq C(1+t)^{-1+\frac{1}{2p}-\frac{l}{2}+\varepsilon}, \quad t \geq 0, \quad l = 0, 1. \tag{1.8}$$

Here the weighted Lebesgue space $L^1_1(\mathbb{R})$ is defined by

$$L^1_1(\mathbb{R}) \equiv \left\{f \in L^1(\mathbb{R});\ \|f\|_{L^1_1} \equiv \int_{\mathbb{R}} |f(x)|(1+|x|)dx < \infty\right\}.$$

Also, by the Hopf-Cole transformation (cf. [2, 8]), we can see that $\chi(x,t)$ satisfies the following Burgers equation and the conservation law:

$$\chi_t + \left(a\chi + \frac{b}{2}\chi^2\right)_x = \mu\chi_{xx}, \quad \int_{\mathbb{R}} \chi(x,t)dx = M. \tag{1.9}$$

Moreover, the optimality of the asymptotic rate to the nonlinear diffusion wave given in (1.8) was obtained by Kato and Ueda [14] by constructing the second asymptotic profile of the solution which is the leading term of $u - \chi$. Indeed, if $u_0 \in W^{s,p}(\mathbb{R}) \cap W^{2,1}(\mathbb{R}) \cap L^1_1(\mathbb{R})$, $u_1 \in W^{s-1,p}(\mathbb{R}) \cap W^{1,1}(\mathbb{R}) \cap L^1_1(\mathbb{R})$ for $s \geq 2$, $1 \leq p \leq \infty$, and $\|u_0\|_{W^{s,p}} + \|u_0\|_{W^{2,1}} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{W^{1,1}}$ is sufficiently small, then we have

$$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t))\|_{L^p} \leq C(1+t)^{-1+\frac{1}{2p}-\frac{l}{2}}, \quad t \geq 1 \tag{1.10}$$

for $0 \le l \le s-2$, where

$$V(x,t) \equiv -\kappa d V_*\left(\frac{x-a(1+t)}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \tag{1.11}$$

and

$$V_*(x) \equiv \frac{1}{\sqrt{4\pi\mu}}\frac{d}{dx}(\eta_*(x)e^{-\frac{x^2}{4\mu}}), \quad \eta_*(x) \equiv \exp\left(\frac{b}{2\mu}\int_{-\infty}^{x}\chi_*(y)dy\right), \tag{1.12}$$

$$d \equiv \int_{\mathbb{R}}(\eta_*(y))^{-1}(\chi_*(y))^3dy, \quad \kappa \equiv \frac{ab^2}{4\mu}+\frac{c}{3!}. \tag{1.13}$$

Furthermore, in view of the second asymptotic profile, from (1.10), the triangle inequality and (1.11), one can obtain the following improved optimal decay estimate:

$$\|\partial_x^l(u(\cdot,t)-\chi(\cdot,t))\|_{L^p} = (\tilde{C}+o(1))(1+t)^{-1+\frac{1}{2p}-\frac{l}{2}}\log(1+t), \quad 0\le l\le s-2 \tag{1.14}$$

as $t\to\infty$, where $\tilde{C} \equiv |\kappa d|\|\partial_x^l V_*\|_{L^p}$. Therefore, we see that the solution $u(x,t)$ of (1.2) tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1+\frac{1}{2p}}\log t$ in L^p if $M\neq 0$ and $\kappa\neq 0$, i.e. we cannot take $\varepsilon = 0$ in the estimate (1.8). The similar results for (1.8) and (1.10) were obtained for Burgers type equations such as generalized Burgers equation, KdV-Burgers equation and BBM-Burgers equation (cf. [3, 6, 7, 12, 13, 18]).

The above results [21, 14] are corresponding to the case where the decay rate of the initial data u_0 and u_1 are rapid because $u_0, u_1 \in L^1_1(\mathbb{R})$ are realized when $\alpha,\beta > 2$ in (1.3). However for (1.2) in the case of $1<\alpha\le 2$ or $1<\beta\le 2$ in (1.3), it is not known that the optimal asymptotic rate to the nonlinear diffusion wave, up to the author's knowledge. On the other hand, it was studied that the asymptotic profile for the solution to the damped wave equation with power type nonlinearity for slowly decaying data. Actually, Narazaki and Nishihara [19] studied the following initial value problem when the initial data are not in L^1:

$$\begin{aligned} &u_{tt}-u_{xx}+u_t = |u|^{p-1}u, \ x\in\mathbb{R}, \quad t>0, \\ &u(x,0)=u_0(x), \ u_t(x,0)=u_1(x), \ x\in\mathbb{R}. \end{aligned} \tag{1.15}$$

They assumed that the initial data satisfy the condition (1.3) with $\alpha = \beta =: k$ and $0<k\le 1$ and showed that if $p>1+2/k$ (supercritical case) and the initial data $u_0\in B^{1,k}(\mathbb{R})$, $u_1\in B^{0,k}(\mathbb{R})$ are small, then the asymptotic profile is given by

$$\Psi(x,t) \equiv c_k\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}(1+|y|)^{-k}dy, \tag{1.16}$$

provided that the data satisfy $\lim_{|x|\to\infty}(1+|x|)^k(u_0+u_1)(x) = c_k$. Here, we set

$$B^{m,k} \equiv \{f\in C^m(\mathbb{R}); (1+|x|)^k|\partial_x^l f| \in L^\infty(\mathbb{R}) \ (0\le l\le m)\}.$$

More precisely, they proved

$$\lim_{t\to\infty} a_k(t)\|u(\cdot,t)-\Psi(\cdot,t)\|_{L^\infty} = 0, \ a_k(t) = \begin{cases} (1+t)^{k/2}, & 0<k<1, \\ \frac{(1+t)^{1/2}}{\log(1+t)}, & k=1. \end{cases} \tag{1.17}$$

Moreover in [19], the damped wave equation (1.15) in two and three space dimensional cases were also studied. For the related results concerning (1.15), we also refer to [9, 10]. However, as we mentioned in the above, the asymptotic profile of the solution to (1.2) with slowly decaying data is not well known even if the data are in L^1. For this reason, we would like to analyze the asymptotic behavior of the solution to (1.2) in the case of $1 < \alpha \leq 2$ or $1 < \beta \leq 2$ in (1.3).

Now, we state our first main result which generalizes the result given in [21]:

Theorem 1.1. *Assume the condition* (1.3) *holds with* $1 < \min\{\alpha, \beta\} \leq 2$. *Let* s *be a positive integer and* $1 \leq p \leq \infty$. *Suppose that* $u_0 \in W^{s,p}(\mathbb{R})$, $u_1 \in W^{s-1,p}(\mathbb{R})$ *and* $\|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1}$ *is sufficiently small. Then* (1.2) *has a unique global solution* $u(x,t)$ *with*

$$u \in \begin{cases} \bigcap_{k=0}^{\sigma} C^k([0,\infty); W^{s-k,p}) \cap C([0,\infty); L^1), & 1 \leq p < \infty, \\ \bigcap_{k=0}^{\sigma} W^{k,\infty}(0,\infty; W^{s-k,\infty}) \cap C([0,\infty); L^1), & p = \infty, \end{cases}$$

where $\sigma = \min\{2, s\}$. *Moreover, for any* $\varepsilon > 0$, *the estimate*

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^q} \leq C \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2} + \frac{1}{2q}}, & t \geq 0,\ 1 < \min\{\alpha,\beta\} < 2, \\ (1+t)^{-1+\frac{1}{2q}+\varepsilon}, & t \geq 0,\ \min\{\alpha,\beta\} = 2 \end{cases} \tag{1.18}$$

holds for any q *with* $1 \leq q \leq \infty$, *and the estimate*

$$\|\partial_t^k \partial_x^l (u(\cdot,t) - \chi(\cdot,t))\|_{L^p} \leq C \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2} + \frac{1}{2p} - \frac{k+l}{2}}, & t \geq 0,\ 1 < \min\{\alpha,\beta\} < 2, \\ (1+t)^{-1+\frac{1}{2p}-\frac{k+l}{2}+\varepsilon}, & t \geq 0,\ \min\{\alpha,\beta\} = 2 \end{cases} \tag{1.19}$$

holds for $0 \leq k \leq 2$ *and* $l \geq 0$ *with* $0 \leq k + l \leq s$, *where* $\chi(x,t)$ *is defined by* (1.6).

Furthermore, we can show that the above asymptotic rate given in (1.18) is optimal with respect to the time decaying order in the L^∞ sense by constructing the second asymptotic profile for the solution to (1.2). Indeed, we have the following result:

Theorem 1.2. *Assume the condition* (1.3) *holds with* $1 < \min\{\alpha, \beta\} \leq 2$. *Suppose that* $u_0 \in H^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$, $u_1 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ *and* $\|u_0\|_{H^2} + \|u_0\|_{W^{2,1}} + \|u_1\|_{H^1} + \|u_1\|_{W^{1,1}}$ *is sufficiently small. We set* $\chi_0(x) \equiv \chi(x,0)$, $\eta_0(x) \equiv \eta(x,0)$ *and*

$$z_0(x) \equiv \eta_0(x)^{-1} \int_{-\infty}^{x} (u_0(y) + u_1(y) - \chi_0(y))dy. \tag{1.20}$$

If there exists $\lim_{x\to\pm\infty}(1+|x|)^{\min\{\alpha,\beta\}-1} z_0(x) \equiv c^{\pm}_{\alpha,\beta}$, *then the solution to* (1.2) *satisfies*

$$\lim_{t\to\infty} (1+t)^{\frac{\min\{\alpha,\beta\}}{2}} \|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t)\|_{L^\infty} = 0,\ 1 < \min\{\alpha,\beta\} < 2, \tag{1.21}$$

$$\lim_{t\to\infty} \frac{(1+t)}{\log(1+t)} \|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t) - V(\cdot,t)\|_{L^\infty} = 0,\ \min\{\alpha,\beta\} = 2, \tag{1.22}$$

where $\chi(x,t)$ *and* $V(x,t)$ *are defined by* (1.6) *and* (1.11)*, respectively, while* $Z(x,t)$ *and* $\eta(x,t)$ *are defined by*

$$Z(x,t) \equiv \int_{\mathbb{R}} \frac{c_{\alpha,\beta}(y)\partial_x(G_0(x-y,t)\eta(x,t))}{(1+|y|)^{\min\{\alpha,\beta\}-1}}dy, \quad c_{\alpha,\beta}(y) \equiv \begin{cases} c^+_{\alpha,\beta}, & y \geq 0, \\ c^-_{\alpha,\beta}, & y < 0, \end{cases} \tag{1.23}$$

$$G_0(x,t) \equiv \frac{1}{\sqrt{4\pi\mu t}}e^{-\frac{(x-at)^2}{4\mu t}}, \quad \eta(x,t) \equiv \eta_*\left(\frac{x-a(1+t)}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2\mu}\int_{-\infty}^{x}\chi(y,t)dy\right) \tag{1.24}$$

with $\eta_*(x)$ *being defined by* (1.12)*. Moreover, if* $M \neq 0$*, there exist* $\nu_0 > 0$ *and* $\nu_1 > 0$ *independent of* x *and* t *such that*

$$\|Z(\cdot,t)\|_{L^\infty} \begin{cases} \leq C\max\{|c^+_{\alpha,\beta}|, |c^-_{\alpha,\beta}|\}(1+t)^{-\frac{\min\{\alpha,\beta\}}{2}}, \\ \geq \nu_0|\tilde{\nu_0}|(1+t)^{-\frac{\min\{\alpha,\beta\}}{2}} \end{cases} \tag{1.25}$$

holds for sufficiently large t *with* $1 < \min\{\alpha,\beta\} < 2$ *and*

$$\|Z(\cdot,t)+V(\cdot,t)\|_{L^\infty} \begin{cases} \leq C(\max\{|c^+_{\alpha,\beta}|, |c^-_{\alpha,\beta}|\} + |\kappa d|\|V_*(\cdot)\|_{L^\infty})(1+t)^{-1}\log(1+t), \\ \geq \nu_1|\tilde{\nu_1}|(1+t)^{-1}\log(1+t) \end{cases} \tag{1.26}$$

holds for sufficiently large t *with* $\min\{\alpha,\beta\} = 2$*, where*

$$\begin{aligned} \tilde{\nu_0} &\equiv \sqrt{\mu}(c^+_{\alpha,\beta} - c^-_{\alpha,\beta})\Gamma\left(\frac{3-\min\{\alpha,\beta\}}{2}\right) + \frac{b\chi_*(0)(c^+_{\alpha,\beta}+c^-_{\alpha,\beta})}{2-\min\{\alpha,\beta\}}\Gamma\left(2-\frac{\min\{\alpha,\beta\}}{2}\right), \\ \tilde{\nu_1} &\equiv \frac{c^+_{\alpha,\beta}+c^-_{\alpha,\beta}}{2} - \kappa d, \quad \Gamma(s) \equiv \int_0^\infty e^{-x}x^{s-1}dx, \quad s > 0, \end{aligned} \tag{1.27}$$

while M*,* d *and* κ *are defined by* (1.7) *and* (1.13)*, respectively.*

By virtue of Theorem 1.2, the optimality of the estimate (1.18) can be examined from the estimates (1.21), (1.22), (1.25) and (1.26). Now, we denote $f(t) \sim g(t)$ if there exist positive constants c_0 and C_0 independent of t such that $c_0 g(t) \leq f(t) \leq C_0 g(t)$ holds. Then, we have the following optimal estimates of $u - \chi$:

Corollary 1.3. *Under the same assumptions in Theorem 1.2, if* $\tilde{\nu_0} \neq 0$ *and* $\tilde{\nu_1} \neq 0$*, then the following estimates*

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^\infty} \sim \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2}}, & 1 < \min\{\alpha,\beta\} < 2, \\ (1+t)^{-1}\log(1+t), & \min\{\alpha,\beta\} = 2 \end{cases} \tag{1.28}$$

hold for sufficiently large t*.*

Remark 1.4. The similar result for Theorem 1.1 is obtained by Kitagawa [16] for the generalized Burgers equation. For Theorem 1.2, recently, the author in [4] obtained the similar result for the generalized KdV-Burgers equation.

2 Basic Estimates and Auxiliary Problem

In this section, we introduce a couple of lemmas to prove the main theorems.

First, we shall mention the global existence and the decay estimates for the solutions to (1.2). Now, we consider the initial value problem for the following linear damped wave equation:

$$\begin{aligned} &u_{tt} - u_{xx} + u_t + au_x = 0,\ x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

By taking the Fourier transform for (2.1), it follows that

$$\hat{u}(\xi,t) = \hat{G}(\xi,t)(\hat{u}_0(\xi) + \hat{u}_1(\xi)) + \partial_t \hat{G}(\xi,t)\hat{u}_0(\xi),$$

where

$$\hat{G}(\xi,t) \equiv \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)}(e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t}), \tag{2.2}$$

$$\lambda_1(\xi) \equiv \frac{1}{2}(-1 + \sqrt{1 - 4(\xi^2 + ai\xi)}),\quad \lambda_2(\xi) \equiv \frac{1}{2}(-1 - \sqrt{1 - 4(\xi^2 + ai\xi)}).$$

Therefore, the solution of (2.1) can be expressed as follows:

$$u(t) = G(t) * (u_0 + u_1) + \partial_t G(t) * u_0,$$

where we set

$$G(x,t) \equiv \mathcal{F}^{-1}[\hat{G}(\cdot,t)](x). \tag{2.3}$$

For this function $G(x,t)$, we can show the following decay estimates (for the proof, see Corollary 3.3 in [21] and Corollary 2.5 in [14]).

Lemma 2.1. *Let $1 \le q \le p \le \infty$. Then the following $L^p - L^q$ estimates hold:*

$$\|G(t) * \phi\|_{L^p} \le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\phi\|_{L^q},\quad t \ge 0, \tag{2.4}$$

$$\|\partial_t^k \partial_x^l G(t) * \phi\|_{L^p} \le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k+l}{2}}\|\phi\|_{L^q} + Ce^{-c_0 t}\|\phi\|_{W^{k+l-1,p}},\quad t \ge 0, \tag{2.5}$$

for $m + l \ge 1$, where $G(x,t)$ and $G_0(x,t)$ are defined by (2.3) and (1.24), respectively. Moreover, the solution operator $G(t)*$ is approximated by $G_0(t)*$ in the following sense:

$$\|(G - G_0)(t) * \phi\|_{L^p} \le Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{-\frac{1}{2}}\|\phi\|_{L^q},\quad t > 0, \tag{2.6}$$

$$\|\partial_t^k \partial_x^l (G - G_0)(t) * \phi\|_{L^p} \le Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k+l}{2}}(1+t)^{-\frac{1}{2}}\|\phi\|_{L^q} + Ce^{-c_0 t}\|\phi\|_{W^{k+l-1,p}},\quad t > 0, \tag{2.7}$$

for $k + l \ge 1$. Here c_0 is a positive constant.

Applying the Duhamel principle to (1.2), we obtain

$$u(t) = G(t) * (u_0 + u_1) + \partial_t G(t) * u_0 - \int_0^t G(t-\tau) * (g(u)_x)(\tau)d\tau, \tag{2.8}$$

where $g(u) \equiv \frac{b}{2}u^2 + \frac{c}{3!}u^3$. Therefore, by using Lemma 2.1, we obtain the global existence and the decay estimates of the solutions to (1.2) as in the following proposition. The proof of this proposition is given by a standard argument which is based on the contraction mapping principle (for the proof, see Theorem 2.1 in [21] and Proposition 3.1 in [14]):

Proposition 2.2. *Let s be a positive integer and $1 \le p \le \infty$. Suppose that $u_0 \in W^{s,p}(\mathbb{R}) \cap L^1(\mathbb{R})$, $u_1 \in W^{s-1}(\mathbb{R}) \cap L^1(\mathbb{R})$ and $E_0^{(s,p)} \equiv \|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1}$ is sufficiently small. Then (1.2) has a unique global solution $u(x,t)$ with*

$$u \in \begin{cases} \bigcap_{k=0}^{\sigma} C^k([0,\infty); W^{s-k,p}) \cap C([0,\infty); L^1), & 1 \le p < \infty, \\ \bigcap_{k=0}^{\sigma} W^{k,\infty}(0,\infty; W^{s-k,\infty}) \cap C([0,\infty); L^1), & p = \infty, \end{cases}$$

where $\sigma = \min\{2, s\}$. Moreover, the estimate

$$\|u(\cdot,t)\|_{L^q} \le CE_0^{(1,p)}(1+t)^{-\frac{1}{2}+\frac{1}{2q}} \tag{2.9}$$

holds for any q with $1 \le q \le \infty$, and the estimate

$$\|\partial_t^k \partial_x^l u(\cdot,t)\|_{L^p} \le CE_0^{(s,p)}(1+t)^{-\frac{1}{2}+\frac{1}{2p}-\frac{k+l}{2}} \tag{2.10}$$

holds for $0 \le k \le 2$ and $l \ge 0$ with $0 \le k+l \le s$.

Next, we treat the nonlinear diffusion wave $\chi(x,t)$ defined by (1.6), and the heat kernel $G_0(x,t)$ defined by (1.24). For $\chi(x,t)$, it is easy to see that

$$|\chi(x,t)| \le C|M|(1+t)^{-\frac{1}{2}} e^{-\frac{(x-at)^2}{4\mu(1+t)}}, \ x \in \mathbb{R}, \ t \ge 0. \tag{2.11}$$

Moreover, $\chi(x,t)$ satisfies the following estimate (for the proof, see Lemma 4.3 in [14]):

Lemma 2.3. *Let k, l and m be non-negative integers. Then, for $|M| \le 1$ and $p \in [1,\infty]$, we have*

$$\|\partial_t^k \partial_x^l (\partial_t + a\partial_x)^m \chi(\cdot,t)\|_{L^p} \le C|M|(1+t)^{-\frac{1}{2}+\frac{1}{2p}-\frac{k+l+2m}{2}}, \quad t \ge 0. \tag{2.12}$$

On the other hand, we have the following estimates for the heat kernel $G_0(x,t)$ (for the proof, see Lemma 2.4 in [5]):

Lemma 2.4. *Let k and l be non-negative integers. Then, for $p \in [1,\infty]$, we have*

$$\|\partial_t^k \partial_x^l G_0(\cdot,t)\|_{L^p} \le Ct^{-\frac{1}{2}+\frac{1}{2p}-\frac{k+l}{2}}, \quad t > 0. \tag{2.13}$$

Moreover, if $\int_{\mathbb{R}} \phi(x)dx = 0$ and

$$\exists \gamma > 1, \quad \exists C > 0 \quad s.t. \quad |\phi(x)| \le C(1+|x|)^{-\gamma}, \quad x \in \mathbb{R}, \tag{2.14}$$

then we have

$$\|\partial_t^k \partial_x^l G_0(t) * \phi\|_{L^p} \le C \begin{cases} t^{-\frac{\gamma}{2}+\frac{1}{2p}-\frac{k+l}{2}}, & t > 0, \ 1 < \gamma < 2, \\ t^{-1+\frac{1}{2p}-\frac{k+l}{2}} \log(2+t), & t > 0, \ \gamma = 2. \end{cases} \tag{2.15}$$

In the rest of this section, let us prepare the ingredients to prove Theorem 1.2. First, we consider the function $\eta(x,t)$ defined by (1.24). For this function, we can easily obtain that

$$\min\{1, e^{\frac{bM}{2\mu}}\} \le \eta(x,t) \le \max\{1, e^{\frac{bM}{2\mu}}\}, \tag{2.16}$$

$$\min\{1, e^{-\frac{bM}{2\mu}}\} \le \eta(x,t)^{-1} \le \max\{1, e^{-\frac{bM}{2\mu}}\}. \tag{2.17}$$

Moreover, by using Lemma 2.3, we have the following L^p-decay estimate (for the proof, see Corollary 2.3 in [13] or Lemma 5.4 in [14]).

Lemma 2.5. *Let l be a positive integer and $p \in [1, \infty]$. If $|M| \leq 1$, then we have*

$$\|\partial_x^l \eta(\cdot, t)\|_{L^p} + \|\partial_x^l (\eta(\cdot, t)^{-1})\|_{L^p} \leq C|M|(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{l}{2}+\frac{1}{2}}, \quad t \geq 0. \tag{2.18}$$

In the proof of Theorem 1.2, we examine the second asymptotic profile of the solution to (1.2). To analyze the second asymptotic profile, we prepare the following auxiliary problem:

$$\begin{aligned} z_t + az_x + (b\chi z)_x - \mu z_{xx} = \partial_x \lambda(x, t), \ x \in \mathbb{R}, \ t > 0, \\ z(x, 0) = z_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2.19}$$

where $\lambda(x, t)$ is a given regular function decaying at spatial infinity. If we set

$$\begin{aligned} U[h](x, t, \tau) \equiv \int_{\mathbb{R}} \partial_x (G_0(x-y, t-\tau)\eta(x, t))(\eta(y, \tau))^{-1} \left(\int_{-\infty}^{y} h(\xi) d\xi \right) dy, \\ x \in \mathbb{R}, \ 0 \leq \tau < t, \end{aligned} \tag{2.20}$$

then, applying Lemma 2.6 in [5] or Lemma 5.1 in [14] to (2.19), we have the following representation formula:

Lemma 2.6. *Let $z_0(x)$ be a sufficiently regular function decaying at spatial infinity. Then we can get the smooth solution of* (2.19) *which satisfies the following formula:*

$$z(x, t) = U[z_0](x, t, 0) + \int_0^t U[\partial_x \lambda(\tau)](x, t, \tau) d\tau, \ x \in \mathbb{R}, \ t > 0. \tag{2.21}$$

This explicit representation formula (2.21) plays an important role in the proof of Theorem 1.2. Also, by using Young's inequality, Lemma 2.5, (2.16), (2.13) and (2.17), we can easily obtain the following estimate:

Lemma 2.7. *Assume that $|M| \leq 1$. Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following estimate*

$$\|U[\partial_x \lambda(\tau)](\cdot, t, \tau)\|_{L^\infty} \leq C \sum_{n=0}^{1} (1+t)^{-\frac{1}{2}+\frac{n}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}-\frac{n}{2}} \|\lambda(\cdot, \tau)\|_{L^q} \tag{2.22}$$

holds for $t > \tau$.

3 Asymptotic Behavior

In order to obtain the upper bound of $u - \chi$, we rewrite the differential equations (1.2) and (1.9) to the integral equations as follows:

$$u(t) = G(t) * (u_0 + u_1) + \partial_t G(t) * u_0 - \int_0^t G(t-\tau) * (g(u)_x)(\tau) d\tau, \tag{3.1}$$

$$\chi(t) = G_0(t) * \chi_0 - \frac{b}{2} \int_0^t G_0(t-\tau) * ((\chi^2)_x)(\tau) d\tau, \tag{3.2}$$

where $g(u) = \frac{b}{2}u^2 + \frac{c}{3!}u^3$ and $\chi_0(x) = \chi(x, 0)$. Therefore, if we set

$$\phi(x,t) \equiv u(x,t) - \chi(x,t), \tag{3.3}$$

then $\phi(x,t)$ satisfies the following relation:

$$\begin{aligned}\phi(t) = &(G - G_0)(t) * (u_0 + u_1) + G_0(t) * (u_0 + u_1 - \chi_0) \\ &+ \partial_t G(t) * u_0 - \frac{c}{3!}\int_0^t G(t-\tau) * ((u^3)_x)(\tau)d\tau \\ &- \frac{b}{2}\int_0^t (G - G_0)(t-\tau) * ((u^2)_x)(\tau)d\tau - \frac{b}{2}\int_0^t G_0(t-\tau) * ((u^2 - \chi^2)_x)(\tau)d\tau.\end{aligned} \tag{3.4}$$

Then, applying the decay estimates stated previous section to (3.4), we can derive the following two propositions. These propositions were proved in the original paper [5].

Proposition 3.1. *Assume the same conditions on u_0 and u_1 in Theorem 1.1 are valid. Then, for any $\varepsilon > 0$, we have*

$$\|\phi(\cdot,t)\|_{L^q} \le C \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2} + \frac{1}{2q}}, & t \ge 0,\ 1 < \min\{\alpha,\beta\} < 2, \\ (1+t)^{-1+\frac{1}{2q}+\varepsilon}, & t \ge 0,\ \min\{\alpha,\beta\} = 2 \end{cases} \tag{3.5}$$

for any q with $1 \le q \le \infty$, where $\phi(x,t)$ is defined by (3.3).

Proposition 3.2. *Assume the same conditions on u_0 and u_1 in Theorem 1.1 are valid. Then, for any $\varepsilon > 0$, we have*

$$\|\partial_x^l \phi(\cdot,t)\|_{L^p} \le C \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2} + \frac{1}{2p} - \frac{l}{2}}, & t \ge 0,\ 1 < \min\{\alpha,\beta\} < 2, \\ (1+t)^{-1+\frac{1}{2p}-\frac{l}{2}+\varepsilon}, & t \ge 0,\ \min\{\alpha,\beta\} = 2 \end{cases} \tag{3.6}$$

for $0 \le l \le s$, where $\phi(x,t)$ is defined by (3.3).

Idea of the proof of Theorem 1.1. We shall explain only for the proof of (1.19) with $k = 1, 2$, since we have already mentioned (1.18) and (1.19) with $k = 0$ (Proposition 3.1 and Proposition 3.2). First, differentiating (3.1) with respect to t, then we have

$$\partial_t u(t) = \partial_t G(t) * (u_0 + u_1) + \partial_t^2 G(t) * u_0 - \int_0^t \partial_t G(t-\tau) * \partial_x(g(u))(\tau)d\tau, \tag{3.7}$$

where $g(u) = \frac{b}{2}u^2 + \frac{c}{3!}u^3$. Here we have used $G(0) * \rho = 0$ for any function ρ. On the other hand, we have from (3.2) that

$$\partial_t \chi(t) = \partial_t G_0(t) * \chi_0 - \frac{b}{2}\int_0^t \partial_t G_0(t-\tau) * \partial_x(\chi^2)(\tau)d\tau - \frac{b}{2}\partial_x(\chi^2)(t), \tag{3.8}$$

where $\chi_0(x) = \chi(x,0)$. Thus, combining (3.7) and (3.8), it follows that

$$\begin{aligned}\partial_t(u(t) - \chi(t)) = &\partial_t(G - G_0)(t) * (u_0 + u_1) + \partial_t G_0(t) * (u_0 + u_1 - \chi_0) \\ &+ \partial_t^2 G(t) * u_0 - \int_0^t \partial_t G(t-\tau) * \partial_x\left(g(u) - \frac{b}{2}\chi^2\right)(\tau)d\tau \\ &- \frac{b}{2}\int_0^t \partial_t(G - G_0)(t-\tau) * \partial_x(\chi^2)(\tau)d\tau + \frac{b}{2}\partial_x(\chi^2)(t).\end{aligned} \tag{3.9}$$

By using the decay estimates stated previous section and the above propositions, we can evaluate the all terms of the right hand side of (3.9). Therefore, we can obtain (1.19) with $k=1$ and $0 \le l \le s-1$.

Next, we shall treat (1.19) with $k=2$. By using the integration by parts, in the same way to get (3.7) and (3.8), we obtain

$$\begin{aligned} \partial_t^2 u(t) &= \partial_t^2 G(t) * (u_0 + u_1) + \partial_t^3 G(t) * u_0 - \int_0^{t/2} \partial_t^2 \partial_x G(t-\tau) * g(u)(\tau) d\tau \\ &\quad - \int_{t/2}^t \partial_t G(t-\tau) * \partial_t \partial_x (g(u))(\tau) d\tau - \partial_t \partial_x G\left(\frac{t}{2}\right) * (g(u))\left(\frac{t}{2}\right) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \partial_t^2 \chi(t) &= \partial_t^2 G_0(t) * \chi_0 - \frac{b}{2}\int_0^{t/2} \partial_t^2 \partial_x G_0(t-\tau) * (\chi^2)(\tau) d\tau - \frac{b}{2}\partial_t \partial_x (\chi^2)(t) \\ &\quad - \frac{b}{2}\int_{t/2}^t \partial_t G_0(t-\tau) * \partial_t \partial_x (\chi^2)(\tau) d\tau - \frac{b}{2}\partial_t \partial_x G_0\left(\frac{t}{2}\right) * (\chi^2)\left(\frac{t}{2}\right). \end{aligned} \tag{3.11}$$

Thus, from (3.10) and (3.11), we have

$$\begin{aligned} &\partial_t^2 (u(t) - \chi(t)) \\ &= \partial_t^2 (G - G_0)(t) * (u_0 + u_1) + \partial_t^2 G_0(t) * (u_0 + u_1 - \chi_0) + \partial_t^3 G(t) * u_0 \\ &\quad - \int_0^{t/2} \partial_t^2 \partial_x G(t-\tau) * \left(g(u) - \frac{b}{2}\chi^2\right)(\tau) d\tau \\ &\quad - \int_{t/2}^t \partial_t G(t-\tau) * \partial_t \partial_x \left(g(u) - \frac{b}{2}\chi^2\right)(\tau) d\tau \\ &\quad - \frac{b}{2}\int_0^{t/2} \partial_t^2 \partial_x (G - G_0)(t-\tau) * (\chi^2)(\tau) d\tau - \frac{b}{2}\int_{t/2}^t \partial_t (G - G_0)(t-\tau) * \partial_t \partial_x (\chi^2)(\tau) d\tau \\ &\quad + \frac{b}{2}\partial_t \partial_x (\chi^2) - \partial_t \partial_x G\left(\frac{t}{2}\right) * \left(g(u) - \frac{b}{2}\chi^2\right)\left(\frac{t}{2}\right) - \frac{b}{2}\partial_t \partial_x (G - G_0)\left(\frac{t}{2}\right) * (\chi^2)\left(\frac{t}{2}\right). \end{aligned} \tag{3.12}$$

Therefore, by using the same argument given in the above paragraph, we can prove (1.19) with $k=2$ and $0 \le l \le s-2$. □

In the rest of this section, we introduce the additional decay property for $u - \chi$. From the original equations (1.2) and (1.9), we see that

$$(\partial_t + a\partial_x)(u - \chi) = (-\partial_t^2 + \partial_x^2)(u - \chi) - \frac{b}{2}\partial_x (u^2 - \chi^2) - \frac{c}{3!}\partial_x (u^3) - (\partial_t - a\partial_x)(\partial_t + a\partial_x)\chi.$$

By virtue of this relation, we have the following estimate:

Corollary 3.3. *Assume the same conditions on u_0 and u_1 in Theorem 1.1 are valid. Then, for any $\varepsilon > 0$, the estimate*

$$\|\partial_x^l (\partial_t + a\partial_x)((u - \chi)(\cdot, t))\|_{L^p} \le C \begin{cases} (1+t)^{-\frac{\min\{\alpha,\beta\}}{2} + \frac{1}{2p} - \frac{l}{2} - 1}, & t \ge 0,\ 1 < \min\{\alpha, \beta\} < 2, \\ (1+t)^{-2 + \frac{1}{2p} - \frac{l}{2} + \varepsilon}, & t \ge 0,\ \min\{\alpha, \beta\} = 2 \end{cases} \tag{3.13}$$

holds for $0 \le l \le s-2$.

4 Second Asymptotic Profile

Finally in this section, we would like to prove Theorem 1.2. Especially, we shall prove only (1.22), since (1.21) can be shown by the same way. On the other hand, for the lower bound (1.25) and (1.26), we can derive these estimates by a direct calculation (for details, see the original paper [5]).

First, let us recall the following fact derived in [13]. We consider

$$\begin{aligned} v_t + av_x + (b\chi v)_x - \mu v_{xx} = -\kappa \partial_x(\chi^3),\ x \in \mathbb{R},\ t > 0, \\ v(x, 0) = 0,\ x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where κ is defined by (1.13). The leading term of the solution $v(x, t)$ to (4.1) is given by $V(x, t)$ defined by (1.11). More precisely, the following asymptotic formula can be shown (for the proof, see Proposition 4.3 in [13]):

Proposition 4.1. *Assume that* $|M| \le 1$. *Then the estimate*

$$\|v(\cdot, t) - V(\cdot, t)\|_{L^\infty} \le C|M|(1 + t)^{-1},\ t \ge 1 \tag{4.2}$$

holds. Here $v(x, t)$ *is the solution to* (4.1) *and* $V(x, t)$ *is defined by* (1.11).

Now, let us prove (1.22). We set

$$\phi(x, t) \equiv u(x, t) + u_t(x, t) - \chi(x, t) - v(x, t), \quad \phi_0(x) \equiv u_0(x) + u_1(x) - \chi_0(x). \tag{4.3}$$

Then, from (1.2), (1.9) and (4.1), we have the following initial value problem:

$$\begin{aligned} \phi_t + a\phi_x + (b\chi\phi)_x - \mu\phi_{xx} = \partial_x N_0(\chi) + \partial_x N_1(u, \chi),\ x \in \mathbb{R},\ t > 0, \\ \phi(x, 0) = \phi_0(x) = u_0(x) + u_1(x) - \chi_0(x),\ x \in \mathbb{R}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} N_1(\chi) &\equiv 2a\mu\chi_{xx} - 2ab\chi\chi_x + \frac{ab^2}{4\mu}\chi^2, \\ N_2(u, \chi) &\equiv a(\partial_t + a\partial_x)(u - \chi) - \mu\partial_t\partial_x(u - \chi) + b\chi\partial_t(u - \chi) \\ &\quad - \mu\partial_x(\partial_t + a\partial_x)\chi + b\chi(\partial_t + a\partial_x)\chi - \frac{b}{2}(u - \chi)^2 - \frac{c}{3!}(u - \chi)^3 - \frac{c}{2}u\chi(u - \chi). \end{aligned} \tag{4.5}$$

Therefore, from Lemma 2.6, we obtain

$$\phi(x, t) = U[\phi_0](x, t, 0) + \int_0^t U[\partial_x N_1(\chi)(\tau)](x, t, \tau)d\tau + \int_0^t U[\partial_x N_2(u, \chi)(\tau)](x, t, \tau)d\tau. \tag{4.6}$$

For the first term of the right hand side in the above equation (4.6), we have the following asymptotic formula. This formula is a key of the proof of Theorem 1.2.

Proposition 4.2. *Assume the same conditions on u_0 and u_1 in Theorem 1.2 are valid. Then we have*

$$\lim_{t\to\infty}(1+t)^{\frac{\min\{\alpha,\beta\}}{2}}\|U[\phi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty}=0,\ 1<\min\{\alpha,\beta\}<2, \tag{4.7}$$

$$\lim_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|U[\phi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty}=0,\ \min\{\alpha,\beta\}=2, \tag{4.8}$$

where $Z(x,t)$ is defined by (1.23)*.*

Proof. From the definition of U given by (2.20) and $\eta_0(x)=\eta(x,0)$, we have

$$\begin{aligned}U[\phi_0](x,t,0)&=\int_{\mathbb{R}}\partial_x(G_0(x-y,t)\eta(x,t))\eta_0(y)^{-1}\left(\int_{-\infty}^{y}(u_0(\xi)+u_1(\xi)-\chi_0(\xi))d\xi\right)dy\\&=\int_{\mathbb{R}}\partial_x(G_0(x-y,t)\eta(x,t))z_0(y)dy,\end{aligned} \tag{4.9}$$

where $z_0(y)$ is defined by (1.20). Since $\int_{\mathbb{R}}(u_0(x)+u_1(x))dx=\int_{\mathbb{R}}\chi_0(x)dx=M$, by a direct calculation, we have the following estimate:

$$|z_0(x)|\le C(1+|x|)^{-(\min\{\alpha,\beta\}-1)},\quad x\in\mathbb{R}. \tag{4.10}$$

Moreover, from the assumption on $z_0(y)$, for any $\varepsilon>0$ there is a constant $R=R(\varepsilon)>0$ such that

$$\begin{aligned}&|z_0(y)-c^+_{\alpha,\beta}(1+|y|)^{-(\min\{\alpha,\beta\}-1)}|\le\varepsilon(1+|y|)^{-(\min\{\alpha,\beta\}-1)},\ y\ge R,\\&|z_0(y)-c^-_{\alpha,\beta}(1+|y|)^{-(\min\{\alpha,\beta\}-1)}|\le\varepsilon(1+|y|)^{-(\min\{\alpha,\beta\}-1)},\ y\le -R.\end{aligned}$$

Therefore, from (1.23) and (4.9), we have the following estimate

$$\begin{aligned}&|U[\phi_0](x,t,0)-Z(x,t)|\\&\le\int_{\mathbb{R}}|\partial_x(G_0(x-y,t)\eta(x,t))||z_0(y)-c_{\alpha,\beta}(y)(1+|y|)^{-(\min\{\alpha,\beta\}-1)}|dy\\&\le\int_{|y|\le R}|\partial_x(G_0(x-y,t)\eta(x,t))||z_0(y)-c_{\alpha,\beta}(y)(1+|y|)^{-(\min\{\alpha,\beta\}-1)}|dy\\&\quad+\varepsilon\int_{|y|\ge R}|\partial_x(G_0(x-y,t)\eta(x,t))|(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy\\&\le C\sum_{n=0}^{1}\|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty}\|\partial_x^nG_0(\cdot,t)\|_{L^\infty}\int_{|y|\le R}|z_0(y)-c_{\alpha,\beta}(y)(1+|y|)^{-(\min\{\alpha,\beta\}-1)}|dy\\&\quad+\varepsilon C\sum_{n=0}^{1}\|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty}\int_{\mathbb{R}}|\partial_x^nG_0(x-y,t)|(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy.\end{aligned} \tag{4.11}$$

For the integral in the last term of the right hand side of (4.11), we can estimate it as follows

$$
\begin{aligned}
&\int_{\mathbb{R}} |\partial_x^n G_0(x-y,t)|(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy \\
&= \left(\int_{|y|\geq\sqrt{1+t}-1} + \int_{|y|\leq\sqrt{1+t}-1}\right)|\partial_x^n G_0(x-y,t)|(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy \\
&\leq \left(\sup_{|y|\geq\sqrt{1+t}-1}(1+|y|)^{-(\min\{\alpha,\beta\}-1)}\right)\int_{|y|\geq\sqrt{1+t}-1}|\partial_x^n G_0(x-y,t)|dy \\
&\quad + \left(\sup_{|y|\leq\sqrt{1+t}-1}|\partial_x^n G_0(x-y,t)|\right)\int_{|y|\leq\sqrt{1+t}-1}(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy \\
&\leq (1+t)^{-\frac{\min\{\alpha,\beta\}-1}{2}}\|\partial_x^n G_0(\cdot,t)\|_{L^1} + \|\partial_x^n G_0(\cdot,t)\|_{L^\infty}\int_{|y|\leq\sqrt{1+t}-1}(1+|y|)^{-(\min\{\alpha,\beta\}-1)}dy \\
&\leq C(1+t)^{-\frac{\min\{\alpha,\beta\}-1}{2}-\frac{n}{2}} + Ct^{-\frac{1+n}{2}}\int_0^{\sqrt{1+t}-1}(1+y)^{-(\min\{\alpha,\beta\}-1)}dy \\
&\leq C(1+t)^{-\frac{\min\{\alpha,\beta\}-1}{2}-\frac{n}{2}} + Ct^{-\frac{1+n}{2}}\begin{cases}(1+t)^{-\frac{\min\{\alpha,\beta\}-1}{2}+\frac{1}{2}}, & 1<\min\{\alpha,\beta\}<2, \\ \log(1+t), & \min\{\alpha,\beta\}=2\end{cases} \\
&\leq C\begin{cases}(1+t)^{-\frac{\min\{\alpha,\beta\}-1}{2}-\frac{n}{2}}, & t\geq 1,\ 1<\min\{\alpha,\beta\}<2, \\ (1+t)^{-\frac{1+n}{2}}\log(1+t), & t\geq 1,\ \min\{\alpha,\beta\}=2.\end{cases}
\end{aligned}
\tag{4.12}
$$

Here, we have used (2.13). Therefore, by using (4.11), Lemma 2.5, Lemma 2.4 and (4.12), we get

$$
\begin{aligned}
&\|U[\psi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty} \\
&\leq C(1+t)^{-1} + \varepsilon C\begin{cases}(1+t)^{-\frac{\min\{\alpha,\beta\}}{2}}, & t\geq 1,\ 1<\min\{\alpha,\beta\}<2, \\ (1+t)^{-1}\log(1+t), & t\geq 1,\ \min\{\alpha,\beta\}=2.\end{cases}
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}(1+t)^{\frac{\min\{\alpha,\beta\}}{2}}\|U[\psi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty} \leq \varepsilon C,\ 1<\min\{\alpha,\beta\}<2, \\
&\limsup_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|U[\psi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty} \leq \varepsilon C,\ \min\{\alpha,\beta\}=2.
\end{aligned}
$$

Therefore, we get (4.7) and (4.8), because $\varepsilon>0$ can be chosen arbitrarily small. □

End of the Proof of Theorem 1.2. From (4.3) and (4.6), we have

$$
\begin{aligned}
&u(x,t)-\chi(x,t)-Z(x,t)-V(x,t) \\
&= U[\phi_0](x,t,0)-Z(x,t)-u_t(x,t)+v(x,t)-V(x,t) \\
&\quad + \int_0^t U[\partial_x N_1(\chi)(\tau)](x,t,\tau)d\tau + \int_0^t U[\partial_x N_2(u,\chi)(\tau)](x,t,\tau)d\tau \\
&\equiv U[\phi_0](x,t,0)-Z(x,t)-u_t(t,t)+v(x,t)-V(x,t)+K_1+K_2,
\end{aligned}
\tag{4.13}
$$

where $Z(x,t)$ and $V(x,t)$ are defined by (1.23) and (1.11), respectively. To prove (1.22), now we only need to evaluate the last two terms in the right hand side of (4.13). First, we evaluate K_1. To estimate it, we introduce the useful property of $N_1(\chi)$. Actually, if we set $N_0(\chi) \equiv 2\mu\chi_x - \frac{b}{2}\chi^2$, from $N_1(\chi) = a(\partial_x N_0(\chi) - \frac{b}{2\mu}\chi N_0(\chi))$, we get $N_1(\chi) = \eta\partial_x(\eta^{-1}N_0(\chi))$. Therefore, from the definition of K_1 and (2.20), and by making the integration by parts, we have

$$\begin{aligned}
K_1(x,t) &= \int_0^t \int_{\mathbb{R}} \partial_x(G_0(x-y,t-\tau)\eta(x,t))\partial_y((\eta(y,\tau))^{-1}N_0(y,\tau))dyd\tau \\
&= \sum_{n=0}^{1} \partial_x^{1-n}\eta(x,t)\Big(\int_0^{t/2} + \int_{t/2}^{t}\Big)\int_{\mathbb{R}} \partial_x^n G_0(x-y,t-\tau)\partial_y((\eta(y,\tau))^{-1}N_0(y,\tau))dyd\tau \\
&= \sum_{n=0}^{1} \partial_x^{1-n}\eta(x,t)\Big(\int_0^{t/2}\int_{\mathbb{R}} \partial_x^{n+1}G_0(x-y,t-\tau)(\eta(y,\tau))^{-1}N_0(y,\tau)dyd\tau \\
&\quad + \int_{t/2}^{t}\int_{\mathbb{R}} \partial_x^n G_0(x-y,t-\tau)\partial_y((\eta(y,\tau))^{-1}N_0(y,\tau))dyd\tau\Big).
\end{aligned} \tag{4.14}$$

Also from Lemma 2.3 and Lemma 2.5, for any non-negative integer l and $1 \le q \le \infty$, it is easy to see that

$$\|\partial_x^l(\eta^{-1}N_0(\chi))(\cdot,t)\|_{L^q} \le C\sum_{j=0}^{l}(1+t)^{-\frac{1}{2}(l-j)}\|\partial_x^j N_0(\cdot,t)\|_{L^q} \le C(1+t)^{-1+\frac{1}{2q}-\frac{l}{2}}. \tag{4.15}$$

Hence, from (4.14), Young's inequality, Lemma 2.5, (2.13) and (4.15), we have

$$\begin{aligned}
&\|K_1(\cdot,t)\|_{L^\infty} \\
&\le C\sum_{n=0}^{1}\|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty}\Big(\int_0^{t/2}\|\partial_x^{n+1}G_0(\cdot,t-\tau)\|_{L^\infty}\|(\eta^{-1}N_0(\chi))(\cdot,\tau)\|_{L^1}d\tau \\
&\quad + \int_{t/2}^{t}\|\partial_x^n G_0(\cdot,t-\tau)\|_{L^1}\|\partial_x(\eta^{-1}N_0(\chi))(\cdot,\tau)\|_{L^\infty}d\tau\Big) \\
&\le C\sum_{n=0}^{1}(1+t)^{-\frac{1}{2}+\frac{n}{2}}\Big(\int_0^{t/2}(t-\tau)^{-1-\frac{n}{2}}(1+\tau)^{-\frac{1}{2}}d\tau + \int_{t/2}^{t}(t-\tau)^{-\frac{n}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\Big) \\
&\le C(1+t)^{-1},\ t \ge 1.
\end{aligned} \tag{4.16}$$

Next, we estimate K_2. For $0 < \varepsilon < \frac{1}{2}$, from (3.13), (1.19), (2.12), (1.18) and (2.9), we have the following estimates:

$$\|N_2(\cdot,t)\|_{L^1} \le C(1+t)^{-\frac{3}{2}+\varepsilon}, \tag{4.17}$$

$$\|N_2(\cdot,t)\|_{L^2} \le C(1+t)^{-\frac{7}{4}+\varepsilon}. \tag{4.18}$$

Therefore, by using Lemma 2.7, (4.17) and (4.18), we obtain

$$\begin{aligned}
&\|K_2(\cdot,t)\|_{L^\infty}\\
&\le C\sum_{n=0}^{1}(1+t)^{-\frac{1}{2}+\frac{n}{2}}\left(\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}}\|N_2(\cdot,\tau)\|_{L^1}d\tau+\int_{t/2}^{t}(t-\tau)^{-\frac{1}{4}-\frac{n}{2}}\|N_2(\cdot,\tau)\|_{L^2}d\tau\right)\\
&\le C\sum_{n=0}^{1}(1+t)^{-\frac{1}{2}+\frac{n}{2}}\left(\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}}(1+\tau)^{-\frac{3}{2}+\varepsilon}d\tau+\int_{t/2}^{t}(t-\tau)^{-\frac{1}{4}-\frac{n}{2}}(1+\tau)^{-\frac{7}{4}+\varepsilon}d\tau\right)\\
&\le C(1+t)^{-1},\ t\ge 1.
\end{aligned} \tag{4.19}$$

Thus, from (4.13), (2.10), Gagliardo-Nirenberg inequality, (4.2), (4.16) and (4.19), we obtain

$$\|u(\cdot,t)-\chi(\cdot,t)-Z(\cdot,t)-V(\cdot,t)\|_{L^\infty}\le\|U[\psi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty}+C(1+t)^{-1},\quad t\ge 1.$$

Therefore, from (4.8), we finally arrive at

$$\limsup_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|u(\cdot,t)-\chi(\cdot,t)-Z(\cdot,t)-V(\cdot,t)\|_{L^\infty}=0.$$

This completes the proof of (1.22). □

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