Global existence of solutions of the compressible viscoelastic fluid around a parallel flow

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1 Introduction

This article is the summary of [4]. We consider the system for a motion of compressible viscoelastic fluids:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \operatorname{div} + \nabla p(\rho) = \beta^2 \operatorname{div}(\rho F^\top F) + \rho g, \tag{1.2}$$

$$\partial_t F + v \cdot \nabla F = \nabla v F \tag{1.3}$$

in a domain

$$\Omega = \{x = (x', x_3);\ x' = (x_1, x_2) \in \mathbb{T}_{\frac{2\pi}{\alpha_1}} \times \mathbb{T}_{\frac{2\pi}{\alpha_2}}, 0 < x_3 < 1\}.$$

Here $\mathbb{T}_{\frac{2\pi}{\alpha_j}} = \mathbb{R}/\frac{2\pi}{\alpha_j}\mathbb{Z}$, $\alpha_j > 0$; $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$ and $F = (F^{ij}(x,t))_{1 \le i,j \le 3}$ denote the unknown density, velocity field and deformation tensor, respectivity, at time $t \ge 0$ and position $x \in \Omega$. We note that ρ, v, F are $\frac{2\pi}{\alpha_j}$-periodic in x_j for $j = 1, 2$. $p(\rho)$ is the pressure that is a smooth function of ρ satisfying $p'(1) > 0$. We denote γ by $\gamma = \sqrt{p'(1)}$. ν, ν' and β are nondimensional parameters that are constants satisfying $\nu > 0$, $2\nu + 3\nu' \ge 0, \beta > 0$. Here ν and ν' are the viscosity coefficients; β is the strength of elasticity. If we set $\beta = 0$ formally, we obtain the usual compressible Navier-Stokes equations. $g = g^1(x_3, t)e_1$ is a given external force, where $e_1 = {}^\top(1, 0, 0)$ and g^1 is a smooth function of (x_3, t) converging to $g^1_\infty = g^1_\infty(x_3) \ne 0$ as t goes to infinity. Here and in what follows ${}^\top\cdot$ stands for the transposition.

We impose the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0),\ \operatorname{div}(\rho_0{}^\top F_0) = 0, \tag{1.4}$$

and the non-slip boundary condition

$$v|_{x_3=0,1} = 0. \tag{1.5}$$

Under the suitable assumption for g, the problem (1.1)–(1.5) has a nontrivial solution $(\bar{\rho}, \bar{v}, \bar{F}) = (1, \bar{v}^1(x_3,t)e_1, (\nabla(x - \bar{\psi}^1(x_3,t)e_1))^{-1})$ satisfying

$$\partial_t \bar{\psi}^1 = \bar{v}^1, \ \bar{\psi}^1|_{t=0} = 0, \ (\bar{v}^1, \bar{\psi}^1)|_{x_3=0,1} = 0.$$

Here $\bar{\psi}(x_3,t) = \bar{\psi}^1(x_3,t)e_1$ denotes the displacement vector. This solution represents a motion of parallel flow. Moreover, we will show that $\bar{v}$ and $\bar{F}$ satisfy the following estimates;

$$\|\bar{v}^1\|_{H^4(0,1)} \le C\|\bar{v}_0\|^2_{H^5} + O\left(\frac{1}{\nu}\right) e^{-c\kappa t},$$
$$\|\bar{F} - \bar{F}_\infty\|_{H^5(0,1)} \le \frac{C}{\nu}\|\bar{v}_0\|^2_{H^5} + O\left(\frac{1}{\beta^2}\right) e^{-c\kappa t},$$

where $\bar{F}_\infty = (\nabla(x - \bar{\psi}^1_\infty e_1))^{-1} = I + \nabla(\bar{\psi}^1_\infty e_1)$; $\bar{\psi}^1_\infty = \bar{\psi}^1_\infty(x_3)$ is the unique solution of the problem $-\beta^2\partial^2_{x_3}\bar{\psi}^1_\infty = g^1_\infty$, $\bar{\psi}^1_\infty|_{x_3=0,1} = 0$; I is the identity matrix.

The purpose of this article is to investigate the stability of the parallel flow under the large numbers of ν, γ and β.

The system (1.1)–(1.3) is one of the basic models describing motion of compressible viscoelastic fluids. The first equation (1.1) and the second equation (1.2) are the compressible Navier-Stokes equations including the elastic force $\beta^2\mathrm{div}(\rho F^\top F)$. The third equation (1.3) is the time evolution of the deformation tensor. We refer to [2, 11] for more details of the physical background.

In the hydrodynamic theory parallel flow is a good example of the simple shear flow and its stability has been widely investigated. Kagei [5] studied the stability of stationary parallel flows in the usual compressible case. It was proved [5] that if the Reynolds number and the Mach number are sufficiently small, the nonlinear stability of stationary parallel flow holds. Endo, Giga, Götz and Liu [1] discussed the stability of time-dependent parallel flow effected by the pressure gradient for the incompressible viscoelastic case. As for the compressible viscoelastic case, there are no mathematical results of the stability of nontrivial flow, while the stability of the motionless state $(1,0,I)$ has been studied by [3, 9, 10].

In this article we will show that if $\nu \gg 1$, $\gamma^2 \gg 1$, $\beta^2 \gg 1$ and $\|\bar{v}_0\|^2_{H^5(0,1)} \ll 1$, then the problem (1.1)–(1.5) has a unique global solution (ρ, v, F) such that $(\rho, v, F) \in C([0,\infty), H^2(\Omega))$ and $\|(\rho(t), v(t), F(t)) -$

$(1, \bar{v}(t), \bar{F}(t))\|_{H^2(\Omega)} \to 0$ exponentially as $t \to \infty$, provided that $(\rho_0 - 1, v_0 - \bar{v}_0, F_0 - \bar{F}_0) \in H^2(\Omega)$ is sufficiently small. Moreover, we prove that $(\rho(t), v(t), F(t)) \to (1, 0, \bar{F}_\infty)$ exponentially in $L^\infty(\Omega)$ as $t \to \infty$ by applying the decay estimate of the parallel flow immediately. Here, $\bar{u}_\infty = (1, 0, \bar{F}_\infty)$ is a stationary solution of the problem (1.1)–(1.5) in the case $g = g^1_\infty(x_3)e_1$.We call this the motionless state with nontrivial deformation given by $\bar{F}_\infty$, throughout this article.

The proof of the main result of this article is obtained by the Matsumura-Nishida energy method [8] to establish a priori estimate of exponential decay type. Let ψ be the displacement vector denoted by

$$\psi(x,t) = X(x,t) - x.$$

Here $X(x,t)$ is the material coordinate which has the inverse $x = x(X,t)$ solving the flow map

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v(x(X,t),t), \\ x(X,0) = X. \end{cases}$$

We set $\zeta(x,t) = \psi(x_3,t) - (-\bar{\psi}^1(x_3,t)e_1)$. We then see that F is written in terms of ζ as

$$F = \bar{F} - \bar{F}\nabla\zeta\bar{F} + h\left(\nabla\zeta\right),$$

where h satisfies $h(\nabla\zeta) = O(|\nabla\zeta|^2)$ for $|\nabla\zeta| \ll 1$, $\beta \gg 1$ and $\|\bar{v}_0\|_{H^5} \ll 1$. By using ψ, the problem for the perturbation is reduced to the one for $u(t) = (\phi(t), w(t), \zeta(t)) = \big(\rho(t) - 1, v(t) - \bar{v}(t), \psi(t) - (-\bar{\psi}^1(t)e_1)\big)$ which takes the following form:

$$\begin{cases} \partial_t\phi + \mathrm{div}w = f^1, \\ \partial_t w - \nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w + \gamma^2\nabla\phi + \beta^2(\Delta\zeta + K_\infty\zeta) = f^2, \\ \partial_t\zeta + w - w\cdot\nabla\bar{\psi}_\infty = f^3, \\ w|_{x_3=0,1} = 0, \quad \zeta|_{x_3=0,1} = 0, \quad (\phi, w, \zeta)|_{t=0} = (\phi_0, w_0, \zeta_0). \end{cases} \tag{1.6}$$

Here $\tilde{\nu} = \nu + \nu'$ and $\bar{\psi}_\infty = \bar{\psi}^1_\infty e_1$; $K_\infty\zeta$ is a linear term of ζ satisfying $\|K_\infty\zeta\|_{L^2} \le \frac{C}{\beta^2}\|\nabla\zeta\|_{H^1}$; f^i, $i = 1,2,3$ are written in a sum of nonlinear terms and linear terms with coefficients including $\bar{v}, \bar{\psi}^1 - \bar{\psi}^1_\infty$. Applying a variant of the Matsumura-Nishida energy method given in [9] to (1.6) and dealing with the interaction between the parallel flow and the perturbation, the following estimate holds:

$$\|u(t)\|^2_{H^2\times H^2\times H^3} + \int_0^t e^{-c_1(t-s)}\|u(s)\|^2_{H^2\times H^3\times H^3}\,\mathrm{d}s \le Ce^{-c_1 t}\|u_0\|^2_{H^2\times H^2\times H^3},$$

provided that $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, and the initial perturbation is sufficiently small.

We finally note a comparison between the case $\beta = 0$ and the case $\beta \gg 1$. When $\beta = 0$, we formally obtain the usual compressible Navier-Stokes equations as mentioned before. In this case, the system (1.1)–(1.3) with $\beta = 0$ has a stationary parallel flow $\bar{u}_s = (1, \bar{v}_s(x_3))$ with $\bar{v}_s(x_3) \neq 0$; and it was shown by Kagei [5] that $\bar{u}_s$ is asymptotically stable provided by $\nu \gg 1$ and $\gamma \gg 1$. On the other hand, the main result of this article imples that the time-dependent parallel flow $\bar{u}$ in the compressible viscoelastic fluid is asymptotically stable if $\nu \gg 1$, $\gamma \gg 1$ and $\beta \gg 1$. When $g(x_3, t) \equiv g^1_\infty(x_3)e_1$, $g^1_\infty \neq 0$, the parallel flow in this article is a stationary solution $\bar{u}_\infty = (1, 0, \bar{F}_\infty(x_3))$, which represents the motionless state with nontrivial deformation given by $\bar{F}_\infty$. Namely, the motionless state $\bar{u}_\infty$ with nontrivial deformation is stable if $\beta \gg 1$, while the parallel flow with non-zero velocity field is stable if $\beta = 0$. This offers an interesting question what happens when the elastic force weakens; it should occur some transition to nontrivial flows at some value of β. We are going to deal with this issue in the future work.

This article is organized as follows. In Section 2 we introduce the parallel flow and then state the main result of this paper on the stability of the parallel flow. In Section 3 we give the outline of the proof of the main result.

2 Main Result

In this section we summerize the results in [4]. For $1 \leq p \leq \infty$ and $D = (0,1), \Omega$, we denote by $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p}$. Let m be a nonnegative integer. We denote by $H^m(D)$ the m-th order L^2 Sobolev space on D with norm $\|\cdot\|_{H^m}$. We denote by $H^1_0(D)$ the completion of $C^\infty_c(D)$ in $H^1(D)$. Here $C^\infty_c(D)$ is the set of all C^∞ functions with compact support in D.

The inner product of $L^2(D)$ is denoted by

$$(f, g) = \int_D f(x)g(x)\ \mathrm{d}x,\ \ f, g \in L^2(D).$$

Divergence of a matrix-valued function $F = (F^{ij})_{1\leq i,j\leq 3}$ is denoted by

$$(\mathrm{div} F)^i = \sum_{j=1}^{3} \partial_{x_j} F^{ij}.$$

For matrix-valued functions $F = (F^{ij})_{1\le i,j\le n}$ and $G = (G^{ij})_{1\le i,j\le n}$, we denote $F\nabla G$ by

$$(F\nabla G)^i = (\mathrm{div}(G^\top F) - G(\mathrm{div}^\top F))^i = \sum_{k,l=1}^{3} F^{lk}\partial_{x_l} G^{ik}.$$

In this article, we assume the following conditions for ρ_0, F_0:

$$\mathrm{div}({\rho_0}^\top F_0) = 0, \tag{2.1}$$

$$\rho_0 \mathrm{det} F_0 = 1. \tag{2.2}$$

It then follows from (1.1)–(1.5) that these quantities are conserved. See [10, Proposition 1] for a proof of Lemma 2.1.

Lemma 2.1 *If (ρ, v, F) is solution of the problem (1.1)–(1.5), then the following identities hold for $t \ge 0$:*

$$\mathrm{div}(\rho^\top F) = 0, \tag{2.3}$$

$$\rho \mathrm{det} F = 1. \tag{2.4}$$

We set $\kappa = \min\left\{\nu, \frac{\beta^2}{\nu}\right\}$. We state the existence of a parallel flow of the problem as follows.

Proposition 2.2 *Let $g^1 \in H^1_{loc}([0,\infty); H^3(0,1))$ satisfy suitable compatibility conditions and $g^1_\infty \in H^3(0,1)$. If g^1 satisfies $e^{c_0\kappa t}\partial_t g^1 \in L^2((0,\infty); H^3(0,1))$ for some positive constant c_0, then the following assertion holds:*

If $\bar{v}|_{t=0} = \bar{v}_0 \in H^5(0,1)$, then there exist a parallel flow $(\bar\rho, \bar v, \bar F)$ of the problem (1.1)–(1.5) satisfying the following estimates uniformly for $t \ge 0$:

$$\|\bar v(t)\|^2_{H^5} \le Ce^{-c_0\kappa t}\left(\|\bar v_0\|^2_{H^5} + \frac{1}{\nu^2}\|g^1(0)\|^2_{H^3} + \frac{1}{\kappa\nu^2}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3)}\right),$$
$$\|\partial_t \bar v(t)\|^2_{H^3} \le Ce^{-c_0\kappa t}\left(\frac{\beta^4}{\nu^2}\|\bar v_0\|^2_{H^5} + \|g^1(0)\|^2_{H^3} + \frac{1}{\kappa}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3)}\right),$$
$$\|\bar\psi^1(t) - \bar\psi^1_\infty\|^2_{H^5} \le Ce^{-c_0\kappa t}\left(\frac{1}{\nu^2}\|\bar v_0\|^2_{H^5} + \frac{1}{\beta^4}\|g^1(0)\|^2_{H^3} + \frac{1}{\kappa\beta^4}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3)}\right).$$

Here $\bar\psi^1_\infty$ is the solution of the problem $-\beta^2\partial^2_{x_3}\bar\psi^1_\infty = g^1_\infty, \bar\psi^1_\infty|_{x_3=0,1} = 0$.

The proof can be found in [4, Appendix] . So we omit it.

We consider the stability of the parallel flow $(1, \bar{v}, \bar{F})$. We set $U(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t))$. Then U satisfies the following problem

$$\begin{cases} \partial_t \phi + \mathrm{div} w = \tilde{f}^1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G \\ \quad + \beta^2 \mathrm{div}(G^\top \bar{E}_\infty + \bar{E}_\infty{}^\top G) = \tilde{f}^2, \\ \partial_t G - \nabla w + w^3 \partial_{x_3} \bar{E}_\infty - (\nabla w) \bar{E}_\infty = \tilde{f}^3, \\ \nabla \phi = -\mathrm{div}^\top G + {}^\top \bar{E}_\infty \mathrm{div}^\top G + \tilde{f}_4, \\ w|_{x_3 = 0,1} = 0, \quad (\phi, w, G)|_{t=0} = (\phi_0, w_0, G_0). \end{cases} \tag{2.5}$$

Here $\tilde{\nu} = \nu + \nu'$, $\bar{\psi}_\infty = \bar{\psi}^1_\infty e_1$, $\bar{E}_\infty = \bar{F}_\infty - I = \nabla(\bar{\psi}^1_\infty e_1)$; and f^i, $i = 1, 2, 3, 4$, denote the sum of nonlinear terms and linear terms with coefficients including $\bar{v}$, $\bar{\psi}_{exp} = \bar{\psi} - \bar{\psi}_\infty$;

$$\begin{aligned} &\tilde{f}^1 = \tilde{f}^1_L + \tilde{f}^1_N; \\ &\tilde{f}^1_L = -\bar{v}^1 \partial_{x_1} \phi, \ \tilde{f}^1_N = -\mathrm{div}(\phi w), \\ &\tilde{f}^2 = \tilde{f}^2_L + \tilde{f}^2_N; \\ &\tilde{f}^2_L = -\bar{v}^1 \partial_{x_1} w - (w^3 \partial_{x_3} \bar{v}^1) e_1 - \nu(\phi \partial^2_{x_3} \bar{v}^1) e_1 - \beta^2 \mathrm{div}(G^\top \bar{E}_{exp} + \bar{E}_{exp}{}^\top G), \\ &\tilde{f}^2_N = -w \cdot \nabla w + \frac{\nu \phi}{1+\phi}(-\Delta w + (\phi \partial^2_{x_3} \bar{v}^1) e_1) - \frac{\tilde{\nu} \phi}{1+\phi} \nabla \mathrm{div} w - \frac{\gamma^2 \phi}{1+\phi} \nabla \phi - \frac{\gamma^2}{1+\phi} \nabla Q(\phi) \\ &\qquad + \frac{\beta^2 \phi}{1+\phi} \mathrm{div}(G^\top \bar{E} + \bar{E}^\top G) + \frac{\beta^2}{1+\phi} \mathrm{div}(G^\top G + \phi(\bar{F}^\top G + G^\top \bar{F} + G^\top G)), \\ &\tilde{f}^3 = \tilde{f}^3_L + \tilde{f}^3_N; \\ &\tilde{f}^3_L = -\bar{v}^1 \partial_{x_1} G + \nabla w \bar{E}_{exp} - w^3 \partial_{x_3} \bar{E}_{exp} + \nabla \bar{v} G, \ \tilde{f}^3_N = -w \cdot \nabla G + \nabla w G, \\ &\tilde{f}^4 = \tilde{f}^4_L + \tilde{f}^4_N; \\ &\tilde{f}^4_L = -{}^\top \bar{E}_{exp} \mathrm{div}^\top G, \ \tilde{f}^4_N = -{}^\top \bar{F}^{-1} \mathrm{div}(\phi^\top G), \end{aligned}$$

where

$$\begin{aligned} &\bar{E} = \bar{F} - I, \ \bar{E}_{exp} = \bar{F} - \bar{F}_\infty = \nabla(\bar{\psi}^1_{exp} e_1), \\ &Q(\phi) = \phi^2 \int_0^1 P''(1 + s\phi) \mathrm{d}s, \ \nabla Q = O(\phi) \nabla \phi \text{ for } |\phi| \ll 1. \end{aligned}$$

We are now in a position to state the main result of this article.

Theorem 2.3 ([4]) *Under the assumption of Proposition* 2.2*, there exists positive constants* ν_1*,* γ_1*,* β_1 *such that if* $\nu \geq \nu_1$*,* $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$*,* $\frac{\beta^2}{\gamma^2} \geq \beta_1^2$*, then the following assertion holds. There is a positive number* ϵ_0 *such that if* $(\rho_0, v_0, F_0) \in H^2(\Omega)$ *and* $\bar{v}_0 \in H^5(0,1)$ *satisfy* $\|(\rho_0 - 1, v_0 - \bar{v}_0, F_0 - I)\|_{H^2(\Omega)} + \|\bar{v}_0\|_{H^5(0,1)} \leq \epsilon_0$*,* $\int_\Omega (\rho_0 - 1)\mathrm{d}x = 0$*, then there exists a unique solution* $(\rho(t), v(t), F(t)) \in C([0,\infty); H^2(\Omega))$ *of the problem* (2.5)*, and the perturbation* $U(t) = (\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t))$ *satisfies*

$$\|U(t)\|_{H^2}^2 + \int_0^t e^{-c_1(t-s)} \|U(s)\|_{H^2 \times H^3 \times H^2}^2 \mathrm{d}s \leq C e^{-c_1 t} \|U(0)\|_{H^2}^2$$

for $t \geq 0$ *uniformly.*

3 Outline of the proof of the main result

In this section we explain the outline of the proof of the main result. Theorem 2.4 is proved by a standard manner from the local solvability theorem and a priori estimate. We first state the local time existence of the solution of the problem (2.5). By a similar argument to that in [6, 9, 12], the local solvability theorem is guaranteed.

Proposition 3.1 *If* $(\rho_0, v_0, F_0) \in H^2(D)$ *satisfies* (2.1)–(2.2) *and* $\rho_0 \geq \frac{1}{2}$*, then there exists positive numbers* T *and* C *such that the following assertion holds. The problem* (2.5) *has a unique solution* $(\rho, v, F) \in C([0,T]; H^2(D))$ *satisfying* $\partial_t \rho, \partial_t F \in C([0,T]; L^2(D))$*,* $v \in L^2([0,T]; H^3(D))$*,* $\partial_t v \in C([0,T]; L^2(D)) \cap L^2([0,T]; H^1(D))$*.*

A priori estimate for $U(t)$ is stated as follows.

Proposition 3.2 *There exist positive numbers* ν_0*,* γ_0 *and* β_0 *such that if* $\nu \geq \nu_0$*,* $\frac{\gamma^2}{\nu+\nu'} \geq \gamma_0^2$ *and* $\frac{\beta^2}{\gamma^2} \geq \beta_0^2$*, then the following assertion holds. Let* T *be an arbitrarily given positive number. Then there exists a positive constant* δ *such that if* $\|\bar{v}_0\|_{H^5(0,1)}^2 + \tilde{E}(t) \leq \delta$ *uniformly for* $t \in [0,T]$*, it holds the following estimate:*

$$\tilde{E}(t) + \int_0^t e^{-c_1(t-s)} \tilde{D}(s)\mathrm{d}s \leq C \left(e^{-c_1 t} \tilde{E}(0) + \int_0^t e^{-c_1(t-s)} \tilde{\mathcal{R}}(s)\mathrm{d}s \right)$$

uniformly for $t \in [0,T]$*, where* C *is a positive constant independent of* T*. Here* $\tilde{E}(t)$ *and* $\tilde{D}(t)$ *are some quantities equivalent to*

$$\|U(t)\|_{H^2}^2 + \|\partial_t U(t)\|_{L^2}^2$$

and

$$\|U(t)\|^2_{H^2\times H^3\times H^2}+\|\partial_t U(t)\|^2_{L^2\times H^1\times L^2},$$

respectivity; $\tilde{\mathcal{R}}(t)$ *is a function satisfying*

$$\tilde{\mathcal{R}}(t)\le C\left(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{\sqrt{\nu}}{\beta}+\frac{\gamma}{\beta}\right)\tilde{D}(t)+(\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t))\tilde{D}(t)$$

uniformly for $t\in[0,T]$ *with a positive constant* C *independent of* T *.*

Proposition 3.2,together with Proposition 3.1, implies Theorem 2.3 if $\|U_0\|^2_{H^2}+\|\bar{v}_0\|^2_{H^5(0,1)}$ is small enough and $\nu\ge\nu_0$, $\frac{\gamma^2}{\nu+\nu'}\ge\gamma_0^2$, $\frac{\beta^2}{\gamma^2}\ge\beta_0^2$ for some positive constants ν_0, γ_0 and β_0.

To prove Proposition 3.2, we introduce the displacement vector and rewrite the problem (2.5) for the perturbation U by using the displacement vector in place of G. We denote the displacement vector by ψ:

$$\psi(x,t)=x-X(x,t).$$

Here $X(x,t)$ is the inverse of the material coordinate $x(X,t)$ which is introduced in Section 1. Then ψ satisfies

$$\partial_t\psi+v=-v\cdot\nabla\psi,$$
$$\psi|_{x_3=0,1}=0.$$

We see from Lemma 2.1 that F has its inverse F^{-1} for $t\ge 0$. According to the continuum mechanic theory, the deformation tensor F is defined by $F=\frac{\partial x}{\partial X}$ in the material coordinate. Hence we expect that F^{-1} is written as

$$F^{-1}=\nabla X. \tag{3.1}$$

We next impose the following conditions for F^{-1} and X:

$$\begin{cases} F^{-1}(x,0)=\nabla X(x,0) \text{ for } x\in\Omega, \\ X=x \text{ on } \{x_3=0,1\} \text{ for } t\ge 0. \end{cases} \tag{3.2}$$

The formal relation (3.1) is justified by the following lemma.

Lemma 3.3 *If* $G=F^{-1}$ *satisfies the condition* (3.2)*, then the identity* (3.1) *holds for* $x\in\Omega$ *and* $t\ge 0$*.*

In terms of ψ, we see from (3.1) that F is written as

$$F=(I+\nabla\psi)^{-1}=I-\nabla\psi+h(\nabla\psi).$$

Here

$$h(\nabla\psi) = (I + \nabla\psi)^{-1} - I + \nabla\psi.$$

We set $\psi = -\bar{\psi}^1 e_1 + \zeta$. Since

$$G = -\bar{F}\nabla\zeta\bar{F} + h^1(\nabla\zeta).$$

with

$$h^1 = h(\bar{F}\nabla\zeta)\bar{F},$$
$$|h^1| = O(|\nabla\zeta|^2) \text{ for } |\nabla\zeta| \ll 1,$$

we see that $u = (\phi, w, \zeta)$ solves the following initial boundary problem:

$$\partial_t\phi + \mathrm{div}w = f^1, \tag{3.3}$$
$$\partial_t w - \nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w + \gamma^2\nabla\phi + \beta^2(\Delta\zeta + K_\infty\zeta) = f^2, \tag{3.4}$$
$$\partial_t\zeta + w - w\cdot\nabla\bar{\psi}_\infty = f^3, \tag{3.5}$$
$$\nabla\phi = \nabla\mathrm{div}\zeta + M_\infty\zeta + f^4, \tag{3.6}$$
$$w|_{x_3=0,1} = 0, \quad \zeta|_{x_3=0,1} = 0, \quad (\phi, w, \zeta)|_{t=0} = (\phi_0, w_0, \zeta_0). \tag{3.7}$$

Here $K_\infty\zeta$ and $M_\infty\zeta$ are given by

$$K_\infty\zeta = \mathrm{div}(\bar{E}_\infty\nabla\zeta + \bar{E}_\infty\nabla\zeta + \bar{E}_\infty\nabla\zeta\bar{E}_\infty) + (\bar{F}_\infty\nabla\zeta\bar{F}_\infty)\nabla\bar{E}_\infty + \bar{E}_\infty\nabla(\bar{F}_\infty\nabla\zeta\bar{F}_\infty),$$
$$M_\infty\zeta = \mathrm{div}({}^\top(\bar{E}_\infty\nabla\zeta + \nabla\zeta\bar{E}_\infty + \bar{E}_\infty\nabla\zeta\bar{E}_\infty)) - \bar{E}_\infty^\top\mathrm{div}({}^\top(\bar{F}_\infty\nabla\zeta\bar{F}_\infty)),$$

and f^i, $i = 1, 2, 3, 4$, denote;

$$f^1 = f^1_L + f^1_N;$$
$$f^1_L = -\bar{v}\cdot\nabla\phi,\ f^1_N = -\mathrm{div}(\phi w),$$
$$f^2 = f^2_L + f^2_N;$$
$$f^2_L = -\bar{v}\cdot\nabla w - w\cdot\nabla\bar{v} - \nu(\phi\partial^2_{x_3}\bar{v}^1)e_1 - \beta^2 K_{exp}\zeta,$$
$$f^2_N = -w\cdot\nabla w + \frac{\nu\phi}{1+\phi}(-\Delta w + (\phi\partial^2_{x_3}\bar{v}^1)e_1) - \frac{\tilde{\nu}\phi}{1+\phi}\nabla\mathrm{div}w - \frac{\gamma^2\phi}{1+\phi}\nabla\phi - \frac{\gamma^2}{1+\phi}\nabla Q(\phi) + \beta^2 h^2,$$
$$f^3 = f^3_L + f^3_N;$$
$$f^3_L = -w\cdot\nabla\bar{\psi}_{exp} - \bar{v}\cdot\nabla\zeta,\ f^3_N = -w\cdot\nabla\zeta,$$
$$f^4 = f^4_L + f^4_N;$$
$$f^4_L = M_{exp}\zeta,\ f^4_N = {}^\top\bar{F}^{-1}\mathrm{div}({}^\top(\phi(\bar{F}\nabla\zeta\bar{F}) - (1+\phi)h^1)),$$

where

$$
\begin{aligned}
h^2 &= \bar{F}\nabla h^1 + (\bar{F}\nabla\zeta\bar{F})\nabla(\bar{F}\nabla\zeta\bar{F} - h^1) + h^1\nabla(\bar{F} - \bar{F}\nabla\zeta\bar{F} + h^1),\\
K_{exp}\zeta &= (\bar{F}\nabla\zeta\bar{F})\nabla\bar{E}_{exp} + (\bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp})\nabla\bar{E}_\infty\\
&\quad + \bar{E}_{exp}\nabla(\bar{F}\nabla\zeta\bar{F}) + \bar{F}_\infty\nabla(\bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp}),\\
M_{exp}\zeta &= {}^\top\bar{F}_\infty^{-1}\mathrm{div}^\top(\bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp}) - {}^\top\bar{E}_{exp}\mathrm{div}({}^\top(\bar{F}\nabla\zeta\bar{F})).
\end{aligned}
$$

We then have the following a priori estimate for the perturbation $u = (\phi, w, \zeta)$.

Proposition 3.4 *Under the assumption of Proposition 3.2, the following assertion holds. There exists a positive constant δ with $\delta < 1$ such that if $\|\bar{v}_0\|^2_{H^5(0,1)} + E(t) \le \delta$ uniformly for $t \in [0, T]$, then it holds the following estimate:*

$$
E(t) + \int_0^t e^{-c_1(t-s)} D(s)\mathrm{d}s \le C\left(e^{-c_1 t}E(0) + \int_0^t e^{-c_1(t-s)}\mathcal{R}(s)\mathrm{d}s\right) \tag{3.8}
$$

uniformly for $t \in [0, T]$ with a positive constant C independent of T. Here $E(t)$ and $D(t)$ are equivalent to

$$
\|u(t)\|^2_{H^2\times H^2\times H^3} + \|\partial_t u(t)\|^2_{L^2\times L^2\times H^1},
$$

and

$$
\|u(t)\|^2_{H^2\times H^3\times H^3} + \|\partial_t u(t)\|^2_{L^2\times H^1\times H^1}
$$

respectivity; $\mathcal{R}(t)$ is a function satisfying

$$
\mathcal{R}(t) \le C\left(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\sqrt{\nu}}{\beta} + \frac{\gamma}{\beta}\right) D(t) + (E(t)^{\frac{1}{2}} + E(t))D(t)
$$

uniformly for $t \in [0, T]$ with a positive constant C independent of T .

Proposition 3.4 yields Proposition 3.2. We can verify the detail of the proof of Proposition 3.4 in [4, pp.2082-2099]. So we will derive the estimates for $\|\nabla w\|^2_{L^2}$ and $\|\nabla\zeta\|^2_{L^2}$ only in this article.

We prepare some propositions. We see from (3.3)-(3.5) that the estimate of $\|\nabla w\|^2_{L^2}$ is given as follows.

Proposition 3.5 *It holds the estimate:*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2\|\phi\|^2_{L^2} + \|w\|^2_{L^2} + \beta^2\|\nabla\zeta\|^2_{L^2}) + D_0(w)\\
&\le \beta^2\left(|(K_\infty\zeta, w)| + |(\nabla(w\cdot\nabla\bar{\psi}_\infty), \nabla\zeta)|\right) + N^1,
\end{aligned} \tag{3.9}
$$

where

$$D_0(w) = \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div} w\|_{L^2}^2,$$
$$N^1 = \gamma^2|(f^1, \phi)| + |(f^2, w)| + \beta^2|(\nabla f^3, \nabla\zeta)|.$$

To control $\|\nabla\zeta\|_{L^2}^2$, we need the following estimate.

Proposition 3.6 *It holds the estimate:*

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}(w,\zeta) + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\gamma^2}{2}\|\mathrm{div}\zeta\|_{L^2}^2 \\ &\le \left(1 + \frac{\nu^2}{2\beta^2}\right)\|\nabla w\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^2}\|\mathrm{div} w\|_{L^2}^2 \\ &\quad + \gamma^2|(M_\infty\zeta,\zeta)| + \beta^2|(K_\infty\zeta,\zeta)| + |(w\cdot\nabla\bar{\psi}_\infty, w)| + N^2, \end{aligned} \tag{3.10}$$

where

$$N^2 = |(f^2,\zeta)| + |(f^3, w)| + \gamma^2|(f^4,\zeta)|.$$

(3.10) is obtained by (3.4) and (3.5). The key point of the proof is using (3.6) to eliminate $\nabla\phi$. By using the Poincaré inequality and the Schwartz inequality, the right-hand side of inequality (3.9) is estimated by

$$\frac{\nu}{2}\|\nabla w\|_{L^2}^2 + \frac{C}{\nu}\|\nabla\zeta\|_{L^2}^2 + N^1.$$

We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0 + D_0 \le \frac{C}{\nu}\|\nabla\zeta\|_{L^2}^2 + 2N^1, \tag{3.11}$$

where

$$E_0 = \gamma^2\|\phi\|_{L^2}^2 + \|w\|_{L^2}^2 + \beta^2\|\nabla\zeta\|_{L^2}^2.$$

Similarly, the right-hand side of inequality (3.10) is estimated by

$$C\left(\frac{1}{\nu} + \frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{\nu\beta^2}\right)D_0 + C\left(1 + \frac{\gamma^2}{\beta^2}\right)\|\nabla\zeta\|_{L^2}^2 + N^2.$$

We thus obtain

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}(w,\zeta) + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\gamma^2}{2}\|\mathrm{div}\zeta\|_{L^2}^2 \\ &\le C\left(\frac{1}{\nu} + \frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{\nu\beta^2}\right)D_0 + C\left(1 + \frac{\gamma^2}{\beta^2}\right)\|\nabla\zeta\|_{L^2}^2 + N^2. \end{aligned} \tag{3.12}$$

By (3.11) + (3.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0 - (w,\zeta)\right) + \left(D_0 + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\gamma^2}{2}\|\mathrm{div}\zeta\|_{L^2}^2\right)$$
$$\leq C\left(\frac{1}{\nu} + \frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{\nu\beta^2}\right) D_0 + C\left(1 + \frac{\gamma^2}{\beta^2} + \frac{1}{\nu}\right)\|\nabla\zeta\|_{L^2}^2 \tag{3.13}$$
$$+2N^1 + N^2.$$

By taking $\nu,\ \tilde{\nu},\ \gamma^2, \beta^2$ large such that $C\left(\frac{1}{\nu} + \frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{\nu\beta^2}\right) \leq \frac{1}{2}$, $C\left(1 + \frac{\gamma^2}{\beta^2} + \frac{1}{\nu}\right) \leq \frac{\beta^2}{4}$, the following estimate is obtained by (3.13)

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1 + \frac{1}{4}D_1 \leq CR_1.$$

Here

$$E_1 = E_0 - (w,\zeta),$$
$$D_1 = D_0 + \beta^2\|\nabla\zeta\|_{L^2}^2 + \gamma^2\|\mathrm{div}\zeta\|_{L^2}^2,$$
$$R_1 = N^1 + N^2.$$

We note that E_1 is equivalent to E_0 for $\beta^2 \gg 1$.

This conclude the estimates of $\|\nabla w\|_{L^2}^2$ and $\|\nabla\zeta\|_{L^2}^2$.

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