On the asymptotic stability of rarefaction waves for a hyperbolic system of balance laws*

Kenta Nakamura†

†Mathematical Institute Tohoku University, Sendai 980-8578, Japan
kenta.nakamura.d3@tohoku.ac.jp

Abstract

We study the rarefaction waves for a model system of hyperbolic balance laws in the whole space. We prove the asymptotic stability of rarefaction waves under smallness assumptions on the initial perturbation and on the amplitude of the waves. The proof is based on the classical L^2 energy method.

1 Introduction

We consider the following model system of hyperbolic balance laws:

$$\begin{cases} u_t + f(u)_x + q_x = 0, \\ q_t + q + u_x = 0. \end{cases} \tag{1.1}$$

Here u and q are unknown real valued functions of $x \in \mathbb{R}$ and $t > 0$, and the flux f is a given smooth function of u. We assume that f is strictly convex with respect to u, that is, $f''(u) > 0$ for any u under consideration. In this talk, we shall treat the system (1.1) in the whole space $\mathbb{R}$ for brevity. Further, we prescribe the initial condition

$$(u, q)(x, 0) = (u_0, q_0)(x), \quad x \in \mathbb{R}. \tag{1.2}$$

Our system (1.1) can be regarded as a model system in kinetic theory. Indeed, u and q are considered as the variables describing macroscopic and microscopic states, respectively. In this case, by applying the Chapman-Enskog expansion to (1.1), we have $q = 0$ and

$$u_t + f(u)_x = 0 \tag{1.3}$$

*July 5, 2019 at RIMS Workshop on Mathematical Analysis in Fluid and Gas Dynamics

as the first order approximation. Note that (1.3) is regarded as a model of the compressible Euler equation. Also, as the second order approximation, we have $q = -u_x$ and

$$u_t + f(u)_x = u_{xx}, \tag{1.4}$$

which is considered as a model of the compressible Navier-Stokes equation. Our rarefaction wave of (1.1) is the function of the form $(u^r, 0)$, where u^r is the centered rarefaction wave of (1.3) which connects the constant states $u_\pm$ with $u_- < u_+$. Namely, u^r is the continuous weak solution of the Riemann problem for (1.3) with the Riemann data

$$u(x,0) = u_0^r(x) := \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.5}$$

Notice that u^r is given explicitly as

$$u^r(x,t) = \begin{cases} u_-, & x/t \leqq f'(u_-), \\ (f')^{-1}(x/t), & f'(u_-) \leqq x/t \leqq f'(u_+), \\ u_+, & f'(u_+) \leqq x/t. \end{cases} \tag{1.6}$$

We assume that the initial data (u_0, q_0) is close to $(u_0^r, 0)$ in a suitable sense and the amplitude $\delta = |u_+ - u_-|$ of the rarefaction wave $(u^r, 0)$ is small. Then it will be shown that a unique solution (u, q) to the problem (1.1)-(1.2) exists globally in time and approaches the rarefaction wave $(u^r, 0)$ uniformly in x as $t \to \infty$. Namely, we have

$$\|(u - u^r, q)(t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty.$$

See Theorem 2.1 below for details.

There are a lot of works concerning the asymptotic stability of rarefaction waves for physically interesting systems. The pioneering work was done by Il'in and Oleinik [3] in 1960 for the scalar viscous conservation law (1.4). The rate of convergence toward the rarefaction waves for (1.4) was investigate in [1, 2]. For the half space problem for (1.4), Liu, Matsumura and Nishihara [10] first proved the asymptotic stability of rarefaction waves.

The asymptotic stability of rarefaction waves for the compressible Navier-Stokes equation (barotropic model) was first proved by Matsumura and Nishihara [12] in 1986. This stability result was improved in [13] for large data and in [9] for the half space problem. A similar asymptotic stability result is known also for the full system of the compressible Navier-Stokes equation. See [5].

The asymptotic stability of rarefaction waves was shown also for related systems. We refer the reader to [11, 6] for the Broadwell model in the discrete kinetic theory and to [7] for a model system of radiating gas.

2 Main result

We state our main result concerning the asymptotic stability of rarefaction waves for (1.1). Let $(u^r, 0)$ be the rarefaction wave for (1.1), where u^r is the centered rarefaction wave of (1.3) and connects the constant state $u_\pm$ with $u_- < u_+$. Note that u^r is the continuous weak solution of the Riemann problem (1.3), (1.5) and is given explicitly by (1.6). Then our stability result in the whole space is stated as follows.

Theorem 2.1 (Stability in the whole space) *Let $u_- < u_+$ and put $\delta = |u_+ - u_-|$. Assume that the initial data (u_0, q_0) satisfies $u_0 - u_0^r \in L^2$, $(u_0)_x \in H^1$ and $q_0 \in H^2$, and put*

$$I_0 = \|u_0 - u_0^r\|_{L^2} + \|(u_0)_x\|_{H^1} + \|q_0\|_{H^2},$$

where u_0^r is the Riemann data in (1.5). If $I_0 + \delta$ is suitably small, then the initial value problem (1.1), (1.2) has a unique global solution (u, q) in an appropriate sense. Moreover, this solution approaches the rarefaction wave $(u^r, 0)$ specified above uniformly in $x \in \mathbb{R}$ as $t \to \infty$:

$$\|(u - u^r, q)(t)\|_{L^\infty} \to 0 \quad as \quad t \to \infty. \tag{2.1}$$

3 Preliminaries

We now recall the following basic propositions related to Sobolev space (see [14]).

Proposition 3.1 (Convergence at infinity [14]) *Let $E(t)$ be an absolutely continuous function on $[0, \infty)$. Assume that $E \in L^1(0, \infty)$ and $\frac{dE}{dt} \in L^1(0, \infty)$. Then*

$$\lim_{t \to \infty} E(t) = 0.$$

We next recall Sobolev's inequality in one-dimensional space, which is crucial later.

Proposition 3.2 (Sobolev's inequality in one-dimensional space [14]) *Let $u \in H^1(\mathbb{R})$. Then the following statements hold true:*

$$u \in C^0 \cap L^\infty(\mathbb{R}), \qquad \lim_{t \to \pm\infty} u(t) = 0$$

and

$$\|u\|_{L^\infty(\mathbb{R})} \leqq C\|u\|_{L^2(\mathbb{R})}^{1/2}\|u_t\|_{L^2(\mathbb{R})}^{1/2}.$$

Smooth approximation of rarefaction waves In this chapter we make a smooth approximation of the rarefaction wave u^r.

Our rarefaction wave u^r is the continuous weak solution of (1.3) and is not smooth. Following to [12, 14], we construct a smooth approximation of the rarefaction wave u^r.

To this end, we recall that if u is a solution of (1.3), then $w := f'(u)$ satisfies the inviscid Burgers equation

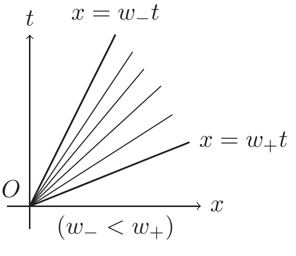
$$w_t + \left(\frac{1}{2}w^2\right)_x = 0. \tag{3.1}$$

Consequently, we find that for our rarefaction wave u^r, the function $w^r := f'(u^r)$ becomes a weak solution of the Riemann problem for (3.1) with the Riemann data

$$w(x,0) = w_0^r(x) := \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \tag{3.2}$$

where $w_\pm := f'(u_\pm)$ with $u_- < u_+$, i.e., $w_- < w_+$. We know that this w^r is given explicitly as

$$w^r(x,t) = \begin{cases} w_-, & x/t \leqq w_-, \\ x/t, & w_- \leqq x/t \leqq w_+, \\ w_+, & w_+ \leqq w/t. \end{cases}$$



This is the centered rarefaction wave of (3.1) which connects the constant states $w_\pm$ with $w_- < w_+$.

For our purpose we first construct a smooth approximation of the rarefaction wave w^r. To this end, following to [12, 14], we consider (3.1) with the following smooth initial data:

$$w(x,0) = w_0(x) := \frac{1}{2}(w_+ + w_-) + \frac{1}{2}(w_+ - w_-)\tanh(\varepsilon x), \tag{3.3}$$

where $\varepsilon \in (0,1]$ is a parameter. (In this paper we only use the case $\varepsilon = 1$.) It is known ([12, 14]) that the problem (3.1), (3.3) has a unique smooth solution w. We state this result in the following lemma.

Lemma 3.3 ([12, 14]) *Let $w_- < w_+$ and put $\delta = |w_+ - w_-|$. Let $\delta_0 > 0$ be any fixed constant and assume that $\delta \leqq \delta_0$. Then the problem (3.1), (3.3) has a unique smooth solution w with the following properties:*

(i) *$w_- < w(x,t) < w_+$ and $w_x(x,t) > 0$ for $x \in \mathbb{R}$ and $t \geqq 0$.*

(ii) *$\|w_x(t)\|_{L^p} \leqq \min\left\{C\varepsilon^{1-1/p}\delta,\ C\delta^{1/p}(1+t)^{-(1-1/p)}\right\}$ for $t \geqq 0$, where $1 \leqq p \leqq \infty$.*

(iii) *$\|\partial_x^k w(t)\|_{L^p} \leqq \min\left\{C\varepsilon^{k-1/p}\delta,\ C\varepsilon^{(k-1)-1/p}(1+t)^{-1}\right\}$ for $t \geqq 0$, where $1 \leqq p \leqq \infty$ and $k = 2, 3, 4$.*

(iv) *$\|(w - w^r)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, where w^r is the rarefaction wave of (3.1).*

Now, following [12, 13], we define a smooth approximation U^R of our rarefaction wave u^r for (1.3) by the formula

$$U^R(x,t) := (f')^{-1}((w(x,t)), \quad i.e., \quad f'(U^R(x,t)) = w(x,t), \tag{3.4}$$

where w is the smooth solution of (3.1), (3.3) with $w_\pm = f'(u_\pm)$ and $\varepsilon = 1$. Note that this U^R satisfies

$$\begin{cases} U^R_t + f(U^R)_x = 0, \\ U^R(x,0) = U^R_0(x) := (f')^{-1}(w_0(x)), \end{cases} \tag{3.5}$$

where w_0 is given by (3.3) with $w_\pm = f'(u_\pm)$ and $\varepsilon = 1$. As an easy consequence of Lemma 3.3 with $\varepsilon = 1$, we have the following result for our smooth approximation U^R.

Lemma 3.4 (Smooth approximation in the whole space [12, 14]) *Let $u_- < u_+$ and put $\delta = |u_+ - u_-|$. Let $\delta_0 > 0$ be any fixed constant and assume that $\delta \leqq \delta_0$. Then the smooth approximation U^R defined by (3.4) with $\varepsilon = 1$ satisfies the following properties:*

(i) *$u_- < U^R(x,t) < u_+$ and $U^R_x(x,t) > 0$ for $x \in \mathbb{R}$ and $t \geqq 0$.*

(ii) *$\|U^R_x(t)\|_{L^p} \leqq \min\{C\delta,\ C\delta^{1/p}(1+t)^{-(1-1/p)}\}$ for $t \geqq 0$, where $1 \leqq p \leqq \infty$.*

(iii) *Let $\theta \in [0,1]$. Then $\|\partial_x^k U^R(t)\|_{L^p} \leqq C\delta^\theta(1+t)^{-(1-\theta)}$ for $t \geqq 0$, where $1 \leqq p \leqq \infty$ and $k = 2,3,4$. Here the constant C is independent of θ.*

(iv) *$\|(U^R - u^r)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, where u^r is the rarefaction wave of (1.3).*

4 Reformulation of the problems

This section is devoted to reformulate the given problems.

We consider the initial value problem (1.1), (1.2). Let u^r be the centered rarefaction wave for (1.3) which is given in (1.6), and let U^R be the smooth approximation of u^r. This U^R is constructed in Lemma 3.4 and satisfies (3.5). We regard $(U^R, -U^R_x)$ as a smooth approximation of the rarefaction wave $(u^r, 0)$ for (1.1), and look for solutions (u, q) of the problem (1.1), (1.2) in the form

$$u = U^R + \phi, \qquad q = -U^R_x + r. \tag{4.1}$$

Our problem (1.1), (1.2) is then rewritten in the following form for the perturbation (ϕ, r):

$$\phi_t + (f(U^R + \phi) - f(U^R))_x + r_x = U^R_{xx}, \tag{4.2a}$$

$$r_t + r + \phi_x = -f(U^R)_{xx}, \tag{4.2b}$$

$$(\phi, r)(x,0) = (\phi_0, r_0)(x), \qquad x \in \mathbb{R}, \tag{4.3}$$

where

$$\phi_0 = u_0 - U^R_0, \qquad r_0 = q_0 + (U^R_0)_x.$$

For this reformulated problem (4.2), (4.3), we obtain the following result on the global existence and asymptotic stability.

Theorem 4.1 (Global existence and stability in the whole space [15]) *Let $u_- < u_+$ and put $\delta = |u_+ - u_-|$. Asuume that $(\phi_0, r_0) \in H^2$ and put $E_0 = \|(\phi_0, r_0)\|_{H^2}$. Then there is a positive constant δ_1 such that if $E_0 + \delta \leqq \delta_1$, then the problem (4.2), (4.3) has a unique global solution (ϕ, r) satisfying*

$$(\phi, r) \in C^0([0,\infty);\, H^2) \cap C^1([0,\infty);\, H^1),$$
$$\phi_x \in L^2(0,\infty;\, H^1), \qquad r \in L^2(0,\infty;\, H^2).$$

Moreover, the solution (ϕ, r) decays to $(0,0)$ uniformly in $x \in \mathbb{R}$ as $t \to \infty$:

$$\|(\phi, r)(t)\|_{W^{1,\infty}} \to 0 \quad as \quad t \to \infty. \tag{4.4}$$

The key to the proof of Theorem 4.1 is to show the desired a priori estimate of solutions to the problem (4.2), (4.3). To state our a priori estimate, we introduce the energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ as follows:

$$\begin{aligned} E(t) &:= \sup_{0 \leqq \tau \leqq t} \|(\phi, r)(\tau)\|_{H^2}, \\ D(t)^2 &:= \int_0^t \|\sqrt{U_x^R}\phi(\tau)\|_{L^2}^2 + \|\phi_x(\tau)\|_{H^1}^2 + \|r(\tau)\|_{H^2}^2 \, d\tau. \end{aligned} \tag{4.5}$$

Then the result on our a priori estimate is stated as follows.

Proposition 4.2 (A priori estimate in the whole space [15]) *Let $T > 0$ and let (ϕ, r) be a solution to the problem (4.2), (4.3) such that*

$$(\phi, r) \in C^0([0,T];\, H^2) \cap C^1([0,T];\, H^1).$$

Then there is a positive constant δ_2 not depending on T such that if $E(T) + \delta \leqq \delta_2$, then the solution (ϕ, r) verifies the a priori estimate

$$E(t)^2 + D(t)^2 \leqq C(E_0^2 + \delta^{2\theta}) \tag{4.6}$$

for $t \in [0,T]$, where $\theta \in (0, 1/4)$ is a fixed number and C is a positive constant independent of T.

We will give the proof of Proposition 4.2 in Section 5.

Proof of Theorem 4.1. The global existence result in Theorem 4.1 can be proved by the standard method which is based on the local existence result combined with the a priori estimate stated in Proposition 4.2. Here we omit the details on the proof of the global existence result and only give the proof of the convergence (4.4).

To this end, we first note that our global solution (ϕ, r) satisfies the energy estimate (4.6) for any $t \geqq 0$. This together with (4.2) yields the estimate for the time derivatives:

$$\int_0^t \|\phi_{tx}(\tau)\|_{L^2}^2 + \|r_t\|_{H^1}^2 \, d\tau \leqq C(E_0^2 + \delta^{2\theta}) \tag{4.7}$$

for any $t \geqq 0$. To prove (4.4) for ϕ, we put $\Phi(t) := \|\phi_x(t)\|_{L^2}^2$. We see that $\Phi \in L^1(0,\infty)$ by (4.6). Also we observe that $|\Phi'| \leqq 2\|\phi_x\|_{L^2}\|\phi_{tx}\|_{L^2}$. Therefore we find that $\Phi' \in L^1(0,\infty)$ by (4.6) and (4.7). By Proposition 3.1 we thus have $\Phi \in W^{1,1}(0,\infty)$, which shows the convergence $\Phi(t) = \|\phi_x(t)\|_{L^2}^2 \to 0$ as $t \to \infty$. This together with Sobolev's inequality (Proposition 3.2) yields

$$\|\phi\|_{L^\infty} \leqq C\|\phi\|_{L^2}^{1/2}\|\phi_x\|_{L^2}^{1/2} \to 0,$$
$$\|\phi_x\|_{L^\infty} \leqq C\|\phi_x\|_{L^2}^{1/2}\|\phi_{xx}\|_{L^2}^{1/2} \to 0$$

as $t \to \infty$, where we also used (4.6). Thus we have proved $\|\phi(t)\|_{W^{1,\infty}} \to 0$ as $t \to 0$.

We can prove (4.4) for r in a similar way. We put $R(t) := \|r(t)\|_{H^1}^2$. Then, using (4.6) and (4.7), we see that $R \in W^{1,1}(0,\infty)$, which shows $R(t) = \|r(t)\|_{H^1}^2 \to 0$ as $t \to \infty$. This together with Sobolev's inequality and (4.6) gives the convergence $\|r(t)\|_{W^{1,\infty}} \to 0$ as $t \to 0$. Thus the proof of Theorem 4.1 is completed. □

Finally in this section, we give the proof of Theorem 2.1 by using Theorem 4.1.

Proof of Theorem 2.1. We assume the smallness condition in Theorem 2.1. Namely, we assume that $I_0 + \delta$ is suitably small, where $I_0 = \|u_0 - u_0^r\|_{L^2} + \|(u_0)_x\|_{H^1} + \|q_0\|_{H^2}$. For the initial data (ϕ_0, r_0) in Theorem 4.1, we see that

$$\|(\phi_0, r_0)\|_{L^2} \leqq \|(u_0 - u_0^r, q_0)\|_{L^2} + \|(u_0^r - U_0^R, (U_0^R)_x)\|_{L^2} \leqq I_0 + C\delta,$$
$$\|\partial_x(\phi_0, r_0)\|_{H^1} \leqq \|\partial_x(u_0, q_0)\|_{H^1} + \|\partial_x(-U_0^R, (U_0^R)_x)\|_{H^1} \leqq I_0 + C\delta.$$

Therefore we have $E_0 \leqq I_0 + C\delta$. Since $I_0 + \delta$ is assumed to be small, we know that $E_0 + \delta$ is also small. Consequently, by applying Theorem 4.1, we obtain a unique global solution (ϕ, r) to the problem (4.2), (4.3). Then the function (u, q) defined by (4.1) becomes the desired global solution to the original problem (1.1), (1.2).

Finally, we show the convergence (2.1) by using (4.4). We see that

$$\|(u - u^r, q)(t)\|_{L^\infty} \leqq \|(U^R - u^r, -U_x^R)(t)\|_{L^\infty} + \|(\phi, r)(t)\|_{L^\infty} \to 0$$

as $t \to \infty$, where we also used Lemma 3.4. This completes the proof of Theorem 2.1. □

5 A priori estimate in the whole space

The aim of this section is to prove Proposition 4.2 on the a priori estimate of solutions to the problem (4.2), (4.3) in the whole space. In this section we assume that the solution (ϕ, r) satisfies the additional regularity

$$(\phi, r) \in C^0([0,T];\, H^3) \cap C^1([0,T];\, H^2).$$

This can be realized by using the mollifier with respect to $x \in \mathbb{R}$. Also we assume that

$$E(T) + \delta \leqq \delta_0, \tag{5.1}$$

where $\delta_0 > 0$ is a fixed constant. In this section θ denotes a fixed number satisfying $\theta \in (0, 1/4)$.

First we show the basic energy estimate.

Lemma 5.1 ([15]) *We have*

$$\|(\phi, r)(t)\|_{L^2}^2 + \int_0^t \|\sqrt{U_x^R}\phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{L^2}^2 \, d\tau \leqq CE_0^2 + C\delta^\theta E(t)^{1/2} D(t)^{1/2}. \tag{5.2}$$

Proof. We multiply (4.2a) and (4.2b) by ϕ and r, respectively, and add these two equalities. After a technical computation we obtain

$$\begin{aligned} &\Big\{\frac{1}{2}(\phi^2 + r^2)\Big\}_t + \Big\{(f(U^R+\phi) - f(U^R))\phi - \int_0^\phi (f(U^R+\eta) - f(U^R)) d\eta + \phi r\Big\}_x \\ &\quad + \{f(U^R+\phi) - f(U^R) - f'(U^R)\phi\} U_x^R + r^2 = \phi U_{xx}^R - r f(U^R)_{xx}. \end{aligned} \tag{5.3}$$

We integrate (5.3) over $\mathbb{R} \times (0, t)$. Then, using $\{f(U^R+\phi) - f(U^R) - f'(U^R)\phi\}U_x^R \geqq cU_x^R\phi^2$, we obtain

$$\|(\phi, r)(t)\|_{L^2}^2 + \int_0^t \|\sqrt{U_x^R}\phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{L^2}^2 \, d\tau \leqq CE_0^2 + C\int_0^t R^{(0)}(\tau)\, d\tau, \tag{5.4}$$

where

$$R^{(0)} = \int_{\mathbb{R}} |\phi||U_{xx}^R| + |r||f(U^R)_{xx}| \, dx \tag{5.5}$$

Here we can show that

$$\int_0^t R^{(0)}(\tau)\, d\tau \leqq C\delta^\theta E(t)^{1/2} D(t)^{1/2}. \tag{5.6}$$

Once this is verified, the desired estimate (5.2) follows from (5.4) and (5.6).

We verify the estimate (5.6). Applying Sobolev's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}} |\phi||U_{xx}^R| \, dx \leqq \|\phi\|_{L^\infty} \|U_{xx}^R\|_{L^1} &\leqq C\|\phi\|_{L^2}^{1/2} \|\phi_x\|_{L^2}^{1/2} \|U_{xx}^R\|_{L^1} \\ &\leqq C\delta^\theta E(t)^{1/2} \|\phi_x\|_{L^2}^{1/2} (1+\tau)^{-(1-\theta)}, \end{aligned}$$

where we have used Lemma 3.4. Therefore we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |\phi||U_{xx}^R| \, dx d\tau &\leqq C\delta^\theta E(t)^{1/2} \int_0^t \|\phi_x(\tau)\|_{L^2}^{1/2} (1+\tau)^{-(1-\theta)} \, d\tau \\ &\leqq C\delta^\theta E(t)^{1/2} \Big(\int_0^t \|\phi_x(\tau)\|_{L^2}^2 \, d\tau \Big)^{1/4} \Big(\int_0^t (1+\tau)^{-\frac{4}{3}(1-\theta)} \, d\tau \Big)^{3/4} \\ &\leqq C\delta^\theta E(t)^{1/2} D(t)^{1/2}, \end{aligned}$$

where we used the Hölder inequality and the fact that $\frac{4}{3}(1-\theta) > 1$ for $\theta \in (0, 1/4)$. Another term in $R^{(0)}$ (see (5.5)) is estimated similarly and we obtain (5.6). This completes the proof of Lemma 5.1. □

Next we show the energy estimate for the derivatives.

Lemma 5.2 (**[15]**) *We have*

$$\|\partial_x(\phi,r)(t)\|_{H^1}^2+\int_0^t\|r_x(\tau)\|_{H^1}^2\,d\tau\leqq CE_0^2+C(\delta+E(t))D(t)^2+C\delta^\theta D(t). \tag{5.7}$$

Proof. We rewrite (4.2) slightly as follows.

$$\phi_t+f'(U^R+\phi)\phi_x+r_x=g+U^R_{xx}, \tag{5.8a}$$

$$r_t+r+\phi_x=-f(U^R)_{xx}, \tag{5.8b}$$

where

$$g=-(f'(U^R+\phi)-f'(U^R))U^R_x. \tag{5.9}$$

We apply ∂_x^k $(k\leqq 2)$ to (5.8) and obtain

$$\partial_x^k\phi_t+f'(U^R+\phi)\partial_x^k\phi_x+\partial_x^k r_x=g^k+\partial_x^k U^R_{xx}, \tag{5.10a}$$

$$\partial_x^k r_t+\partial_x^k r+\partial_x^k\phi_x=-\partial_x^k f(U^R)_{xx}, \tag{5.10b}$$

where

$$g^k=-[\partial_x^k,f'(U^R+\phi)]\phi_x+\partial_x^k g. \tag{5.11}$$

Here $[\cdot,\cdot]$ denotes the commutator. We multiply (5.10a) and (5.10b) by $\partial_x^k\phi$ and $\partial_x^k r$, respectively, and add the resulting equalities. This yields

$$\begin{aligned}&\left\{\frac12\left((\partial_x^k\phi)^2+(\partial_x^k r)^2\right)\right\}_t+\left\{\frac12 f'(U^R+\phi)(\partial_x^k\phi)^2+\partial_x^k\phi\cdot\partial_x^k r\right\}_x+(\partial_x^k r)^2\\&=\frac12 f'(U^R+\phi)_x(\partial_x^k\phi)^2+\partial_x^k\phi\,(g^k+\partial_x^k U^R_{xx})-\partial_x^k r\cdot\partial_x^k f(U^R)_{xx}.\end{aligned}$$

Integrating this equality over $\mathbb{R}\times(0,t)$, we obtain

$$\|\partial_x^k(\phi,r)(t)\|_{L^2}^2+\int_0^t\|\partial_x^k r(\tau)\|_{L^2}^2\,d\tau\leqq CE_0^2+C\int_0^t R^{(k)}(\tau)\,d\tau \tag{5.12}$$

for $k=1,2$, where

$$R^{(k)}=\int_{\mathbb{R}}(|U^R_x|+|\phi_x|)|\partial_x^k\phi|^2+|g^k||\partial_x^k\phi|+|\partial_x^k\phi||\partial_x^k U^R_{xx}|+|\partial_x^k r||\partial_x^k f(U^R)_{xx}|\,dx. \tag{5.13}$$

We will show that

$$\int_0^t R^{(k)}(\tau)\,d\tau\leqq C(\delta+E(t))D(t)^2+C\delta^\theta D(t) \tag{5.14}$$

for $k=1,2$. Once this is done, we substitute (5.14) into (5.12) and add for $k=1,2$. This yields the desired estimate (5.7).

It remains to prove (5.14). Concerning the first term in (5.13), we observe that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(|U_x^R|+|\phi_x|)|\partial_x^k\phi|^2\,dxd\tau\\&\leqq C(\delta+E(t))\int_0^t\|\partial_x^k\phi(\tau)\|_{L^2}^2\,d\tau\leqq C(\delta+E(t))D(t)^2\end{aligned}$$

for $k=1,2$. Next we estimate the second term in (5.13). Recalling (5.11), we see that $\|[\partial_x^k,f'(U^R+\phi)]\phi_x\|_{L^2}\leqq C(\delta+E(t))\|\phi_x\|_{H^1}$ for $k=1,2$. Therefore we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}|[\partial_x^k,f'(U^R+\phi)]\phi_x||\partial_x^k\phi|\,dxd\tau\\&\leqq C(\delta+E(t))\int_0^t\|\phi_x(\tau)\|_{H^1}^2\,d\tau\leqq C(\delta+E(t))D(t)^2\end{aligned}$$

for $k=1,2$. Also we apply a direct computation to g in (5.9) and find that

$$\|\partial_x^k g\|_{L^2}\leqq C\delta^\theta E(t)(1+\tau)^{-(1-\theta)}+C\delta\|\phi_x\|_{H^1}$$

for $k=1,2$, where we used Lemma 3.4. Therefore we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}|\partial_x^k g||\partial_x^k\phi|\,dxd\tau&\leqq C\delta^\theta E(t)\int_0^t\|\partial_x^k\phi(\tau)\|_{L^2}(1+\tau)^{-(1-\theta)}\,d\tau+C\delta\int_0^t\|\phi_x(\tau)\|_{H^1}^2\,d\tau\\&\leqq C\delta^\theta E(t)D(t)+C\delta D(t)^2\end{aligned}$$

for $k=1,2$. Moreover, we see easily that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}|\partial_x^k\phi||\partial_x^k U_{xx}^R|+|\partial_x^k r||\partial_x^k f(U^R)_{xx}|\,dxd\tau\\&\qquad\leqq C\delta^\theta\int_0^t\|\partial_x^k(\phi,r)(\tau)\|_{L^2}(1+\tau)^{-(1-\theta)}\,d\tau\leqq C\delta^\theta D(t)\end{aligned}$$

for $k=1,2$. All these estimates give the desired estimate (5.14). Thus the proof of Lemma 5.2 is completed. □

Finally, we show the dissipative estimate for ϕ_x.

Lemma 5.3 ([15]) *We have*

$$\begin{aligned}\int_0^t\|\phi_x(\tau)\|_{H^1}^2\,d\tau\leqq CE_0^2+C\|(\phi,r)(t)\|_{H^2}^2+C\int_0^t\|r(\tau)\|_{H^2}^2\,d\tau\\+C(\delta+E(t))D(t)^2+C\delta^\theta D(t).\end{aligned}\tag{5.15}$$

Proof. We use (5.10) for $k=0,1$. (Note that (5.10) with $k=0$ coincides with (5.8).) To create the dissipative estimate for $\partial_x^k\phi_x$, we multiply (5.10b) and (5.10a) by $\partial_x^k\phi_x$ and $-\partial_x^k r_x$, respectively, and add these two equalities. This gives

$$\begin{aligned}&(\partial_x^k\phi_x\,\partial_x^k r)_t-(\partial_x^k\phi_t\,\partial_x^k r)_x+(\partial_x^k\phi_x)^2+\partial_x^k\phi_x\,\partial_x^k r-\big(f'(U^R+\phi)\partial_x^k\phi_x+\partial_x^k r_x\big)\,\partial_x^k r_x\\&\qquad=-\partial_x^k r_x(g^k+\partial_x^k U_{xx}^R)-\partial_x^k\phi_x\,\partial_x^k f(U^R)_{xx}.\end{aligned}$$

Integrating over $\mathbb{R}\times(0,t)$, we obtain

$$\begin{aligned}\int_0^t \|\partial_x^k\phi_x(\tau)\|_{L^2}^2\,d\tau &\leqq CE_0^2 + C\|\partial_x^k(\phi,r)(t)\|_{H^1}^2 \\ &\quad + C\int_0^t \|\partial_x^k r(\tau)\|_{H^1}^2\,d\tau + C\int_0^t S^{(k)}(\tau)\,d\tau \end{aligned} \tag{5.16}$$

for $k=0,1$, where

$$S^{(k)} = \int_{\mathbb{R}} |g^k||\partial_x^k r_x| + |\partial_x^k r_x||\partial_x^k U_{xx}^R| + |\partial_x^k\phi_x||\partial_x^k f(U^R)_{xx}|\,dx. \tag{5.17}$$

We will show that

$$\int_0^t S^{(k)}(\tau)\,d\tau \leqq C(\delta+E(t))D(t)^2 + C\delta^\theta D(t) \tag{5.18}$$

for $k=0,1$. Once this is verified, we substitute (5.18) into (5.16) and add for $k=0,1$. Then we obtain the desired estimate (5.15).

To complete the proof, we need to show (5.18). Recalling (5.11), we observe that $\|[\partial_x^k, f'(U^R+\phi)]\phi_x\|_{L^2} \leqq C(\delta+E(t))\|\phi_x\|_{L^2}$ for $k=0,1$. (The commutator vanishes for $k=0$.) Therefore we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}} |[\partial_x^k, f'(U^R+\phi)]\phi_x||\partial_x^k r_x|\,dxd\tau \\ &\leqq C(\delta+E(t))\int_0^t \|\phi_x(\tau)\|_{L^2}\|\partial_x^k r_x(\tau)\|_{L^2}\,d\tau \leqq C(\delta+E(t))D(t)^2\end{aligned}$$

for $k=0,1$. Concerning the term g in (5.9), we see that $|g|\leqq CU_x^R|\phi|$. Therefore we find that $\|g\|_{L^2}\leqq C\delta^{1/2}\|\sqrt{U_x^R}\phi\|_{L^2}$. Thus we have

$$\int_0^t\int_{\mathbb{R}} |g||r_x|\,dxd\tau \leqq C\delta^{1/2}\int_0^t \|\sqrt{U_x^R}\phi(\tau)\|_{L^2}\|r_x(\tau)\|_{L^2}\,d\tau \leqq C\delta^{1/2}D(t)^2.$$

Also we see that $\|\partial_x g\|_{L^2}\leqq C\delta^\theta E(t)(1+\tau)^{-(1-\theta)} + C\delta\|\phi_x\|_{L^2}$. Therefore we obtain

$$\int_0^t\int_{\mathbb{R}} |\partial_x g||\partial_x r_x|\,dxd\tau \leqq C\delta^\theta E(t)D(t) + C\delta D(t)^2.$$

The remaining terms in (5.17) are estimated easily and we have

$$\int_0^t\int_{\mathbb{R}} |\partial_x^k r_x||\partial_x^k U_{xx}^R| + |\partial_x^k\phi_x||\partial_x^k f(U^R)_{xx}|\,dxd\tau \leqq C\delta^\theta D(t)$$

for $k=0,1$. Gathering these estimates, we obtain the desired estimate (5.18) and thus, completes the proof of Lemma 5.3. □

Proof of Proposition 4.2. We add (5.2) and (5.7) to obtain that

$$\begin{aligned} &\|(\phi, r)(t)\|_{H^2}^2 + \int_0^t \|\sqrt{U_x^R}\phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{H^2}^2 \, d\tau \\ &\leqq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^\theta(E(t) + D(t)). \end{aligned} \tag{5.19}$$

We substitute (5.19) into (5.15). This gives

$$\int_0^t \|\phi_x(\tau)\|_{H^1}^2 \, d\tau \leqq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^\theta(E(t) + D(t)). \tag{5.20}$$

Adding (5.19) and (5.20), we arrive at the inequality

$$E(t)^2 + D(t)^2 \leqq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^\theta(E(t) + D(t)).$$

This inequality is reduced to $E(t)^2 + D(t)^2 \leqq C(E_0^2 + \delta^{2\theta}) + C(\delta + E(t))D(t)^2$, which yields the desired estimate (4.6), provided that $E(T) + \delta$ is suitably small. Thus the proof of Proposition 4.2 is done. □

Acknowledgement

The author is supported by Foundation of Research Fellows, The Mathematical Society of Japan. Also, I wish to express his sincere gratitude to Assistant Professor Yoshihiro Ueda (Kobe Univ.) and Assistant Professor Masashiro Suzuki (Nagoya Inst. Tech.) for supporting and giving him an opportunity to talk at RIMS.

References

[1] E. Harabetian, Rarefactions and large time behavior for parabolic equations and monotone schemes, *Comm. Math. Phys.*, **114** (1988), 527–536.

[2] Y. Hattori and K. Nishihara, A note on the stability of the rarefaction wave of the Burgers equation, *Japan J. Indust. Appl. Math.*, **8** (1991), 85–96.

[3] A.M. Il'in and O.A. Oleinik, Asymptotic behavior of the solutions of Cauchy problem for certain quasilinear equations for large time, (in Russian), *Mat. Sb.*, **51** (1960), 191–216.

[4] S. Kawashima and A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, *Comm. Math. Phys.*, **101** (1985), 97–127.

[5] S. Kawashima, A. Matsumura and K. Nishihara, Asymptotic behavior of solutions for the equations of a viscous heat conductive gas, *Proc. Japan Acad.*, **62**, Ser. A (1986), 249–252.

[6] S. Kawashima and Y. Nikkuni, Stability of rarefaction waves of for the discrete Boltzmann equations, *Adv. Math. Sci. Appl.*, **12** (2002), 327–353.

[7] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, *Kyushu J, Math.*, **58** (2004), 211–250.

[8] S. Kawashima, T. Yanagisawa and Y. Shizuta, Mixed problems for quasi-linear symmetric hyperbolic systems, *Proc. Japan Acad.*, **63** (1987), 243–246.

[9] S. Kawashima and P. Zhu, Asymptotic stability of rarefaction wave for the Navier-Stokes equations for a compressible fluid wave in the half space, *Arch. Rat. Mech. Anal.*, **194** (2009), 105–132.

[10] T.-P. Liu, A. Matsumura and K. Nishihara, Behavior of solutions for the Burgers equations with boundary corresponding to rarefaction waves, *SIAM. J. Math. Anal.*, **29** (1998), 293–308.

[11] A. Matsumura, Asymptotic toward rarefaction wave of solutions of the Broadwell model of a discrete velocity gas, *Japan J. Appl. Math.*, **4** (1987), 489–502.

[12] A. Matsumura and K. Nishihara, Asymptotic toward the rarefaction waves of solutions of a one-dimensional model system for compressible viscous gas, *Japan. J. Appl. Math.*, **3** (1986), 1–13.

[13] A. Matsumura and K. Nishihara, Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas, *Comm. Math. Phys.*, **144** (1992), 325–335.

[14] A. Matsumura and K. Nishihara, *Global solutions to nonlinear differential equation - mathematical analysis of compressible viscous flow (in Japanese)*, Amazon (POD), 2015.

[15] K. Nakamura, T. Nakamura and S. Kawashima, *Asymptotic stability of rarefaction waves for a hyperbolic system of balance laws*, Kinetic Related Models Vol.**12**. No.4 (2019), 923–944.

[16] T. Nakamura and S. Kawashima, Viscous shock profile and singular limit for hyperbolic systems with Cattaneo's law, *Kinet. Relat. Models*, **11** (2018), 795–819.

[17] K. Nishihara, T. Yang and H. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **35** (2004), 1561–1597.

[18] S. Schochet, The compressible Euler equations in a bounded domain: Existence of solutions and the incompressible limit, *Comm. Math. Phys.*, **104** (1986), 49–75.