Standard Basis for Mixed Module, Computational Algorithm and Application to Classification Problems in Singularity Theory

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Abstract

We review standard basis for mixed module introduced by Gatermann [5] and its parametric extension [19], their computational algorithms and application to Singularity theory.

1 Introduction

Mixed modules are sums of several modules over different rings. Mixed modules appear in various settings such as (extended) tangent spaces in singularity theory. In singularity theory, mixed modules appear in classification of map-germs relative to various equivalence relations such as $\mathcal{A}$ [13], $\mathcal{K}_{\mathcal{B}}$ [6, 7], and $\mathcal{A}[G]$-equivalence [9] for some Lie group G including equivalence among divergent diagrams. There, the concept of (extended) tangent space plays an important role and an (extended) tangent space is a mixed module relative to these equivalences. Compared to the conventional module over a single ring, the algebraic structure of a mixed module can be highly complicated, which makes classification of map-germs relative to these equivalences difficult. This is thought of as one of the motivations of Mather [14] to reduce classification of stable map-germs relative to $\mathcal{A}$ to that of those relative to $\mathcal{K}$ since, in the latter case, (extended) tangent spaces of map-germs are modules.

One of the pioneering works for automation of classification relative to these equivalences is done by Kirk [11, 10, 12] based on the complete transversal theorem [2, 16]. Unlike the conventional module where efficient computation can be done by using the standard basis, there was no such a concept for mixed module at that time. In their algorithm, they handle mixed modules in jet spaces as a huge vector space over $\mathbb{R}$. It seems that their software is no longer available and it is difficult to assess the efficiency of their algorithm but it can be made much more efficient if mixed module structures are taken into account.

Since then, a possible generalization of standard bases for mixed modules appearing in classification relative to $\mathcal{K}_{\mathcal{B}}$ is proposed by Gatermann et al.[5] where a mixed module is supposed to be a sum of two modules over two different rings. In [19], we extended it to parametric standard system for a mixed module (comprehensive standard system (CSS) for a mixed module), proposed a computational algorithm (Algorithm 2-4) for it, and applied the algorithm to solve classification problems involving complicated moduli structure.

In Sec. 2, we review standard basis for a mixed module by [5] and introduce a concrete computational algorithm (Algorithm 1) for it. In Sec. 3, we review CSS for a mixed module introduced in [19]. In Sec. 4, we provide our feature perspectives.

2 Standard Basis for Mixed Module [5]

Let K be a field and let $\lambda = (\lambda_1, \cdots, \lambda_{n_\lambda})$ and $x = (x_1, \cdots, x_{n_x})$ be variables such that they are disjoint with each other. Let $K[x, \lambda]$ be the polynomial ring with variables x and λ, $\langle x, \lambda \rangle$ be the ideal generated by x and λ, and $K[x, \lambda]_{\langle x, \lambda \rangle}$ be the localization of $K[x, \lambda]$ with respect to $\langle x, \lambda \rangle$.

Definition 2.1. *A (x, λ)-mixed module $M \subset \left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ is a $K[\lambda]_{\langle \lambda \rangle}$-module which may be written as a sum $M = N + Q$, where $N \subset \left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ is a $K[x, \lambda]_{\langle x, \lambda \rangle}$-module of finite codimension as a K-vector space in $\left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ and $Q \subset \left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ is a $K[\lambda]_{\langle \lambda \rangle}$-module.*

Let $\prec_{x,\lambda}$ be a local ordering in the set of monomials in x and λ. Let $\prec_{x,\lambda,m}$ be a module ordering in the monomials in $\left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ which is compatible with the ordering $\prec_{x,\lambda}$, i.e., a module ordering satisfying:

1. $x^\alpha \lambda^\beta e_i \prec_{x,\lambda,m} x^{\alpha'} \lambda^{\beta'} e_j \Rightarrow x^{\alpha+\alpha''} \lambda^{\beta+\beta''} e_i \prec_{x,\lambda,m} x^{\alpha'+\alpha''} \lambda^{\beta'+\beta''} e_j$,
2. $x^\alpha \lambda^\beta \prec_{x,\lambda} x^{\alpha'} \lambda^{\beta'} \Rightarrow x^\alpha \lambda^\beta e_i \prec_{x,\lambda,m} x^{\alpha'} \lambda^{\beta'} e_i$,

for all $\alpha, \alpha', \alpha'' \in \mathbb{Z}_{\geq 0}^{n_x}$, $\beta, \beta', \beta'' \in \mathbb{Z}_{\geq 0}^{n_\lambda}$, and $i, j \in \{1, \cdots, n\}$, where $e_i = (0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0) \in \left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$ for $i = 1, \cdots, n$.

Let $\mathrm{LM}_{\prec_{x,\lambda,m}}(f)$, $\mathrm{LT}_{\prec_{x,\lambda,m}}(f)$, and $\mathrm{LC}_{\prec_{x,\lambda,m}}(f)$ be the leading monomial, leading term and leading coefficient of $f \in \left(K[x, \lambda]_{\langle x, \lambda \rangle}\right)^n$, respectively.

Definition 2.2 (Initial Module)**.** *The initial module* $in_{\prec_{x,\lambda,m}}(M)$ *is defined as the* $K[\lambda]_{\langle\lambda\rangle}$*-module*

$$in_{\prec_{x,\lambda,m}}(M) = \langle \mathrm{LM}_{\prec_{x,\lambda,m}}(f) \,\Big|\, \forall g \in K[x,\lambda]_{\langle x,\lambda\rangle}, gf \in M \rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}} + \langle \mathrm{LM}_{\prec_{x,\lambda,m}}(f) \,|\, f \in M \rangle_{K[\lambda]_{\langle\lambda\rangle}}.$$

Definition 2.3 ((x,λ)-mixed standard basis)**.** *A* (x,λ)*-mixed standard basis of* M *is a pair* $\left(S^{(1)}, S^{(2)}\right)$ *of two finite sets* $S^{(1)}$ *and* $S^{(2)}$ *such that*

$$M = \langle S^{(1)} \rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}} + \langle S^{(2)} \rangle_{K[\lambda]_{\langle\lambda\rangle}}$$

and

$$in_{\prec_{x,\lambda,m}}(M) = \langle \mathrm{LM}_{\prec_{x,\lambda,m}}\left(S^{(1)}\right) \rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}} + \langle \mathrm{LM}_{\prec_{x,\lambda,m}}\left(S^{(2)}\right) \rangle_{K[\lambda]_{\langle\lambda\rangle}}.$$

Lemma 37 in [5] guarantees the existence of a (x,λ)-mixed standard basis with respect to an arbitrary local order $\prec_{x,\lambda}$. A brief sketch of the algorithm for computing standard basis for a given pair of finite number of generators in N and Q is given in [5]. Here, we provide a concrete algorithm for computing a pair $\left(S^{(1)}, S^{(2)}\right)$ for a given pair of finite number of generators in N and Q. We define the S-polynomial $\mathrm{spoly}(f,g)$ for non-zero $f,g \in K[x,\lambda]^n$ as follows: Suppose $\mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^\alpha \lambda^\beta e_i$ and $\mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'} \lambda^{\beta'} e_j$ ($\alpha, \alpha' \in \mathbb{Z}^{n_x}_{\geq 0}$, $\beta, \beta' \in \mathbb{Z}^{n_\lambda}_{\geq 0}$ and $i,j \in \{1,\cdots,n\}$). The S-polynomial $\mathrm{spoly}(f,g)$ is defined as

$$\mathrm{LCM}\left(x^\alpha \lambda^\beta, x^{\alpha'} \lambda^{\beta'}\right)\left(\frac{f}{\mathrm{LC}_{\prec_{x,\lambda}}(f)\, x^\alpha \lambda^\beta} - \frac{g}{\mathrm{LC}_{\prec_{x,\lambda}}(g)\, x^{\alpha'} \lambda^{\beta'}}\right) \tag{1}$$

if $i = j$ and 0 in the other cases, where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n_x}^{\alpha_{n_x}}$ and $\lambda^\beta = \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_{n_\lambda}^{\beta_{n_\lambda}}$.

Algorithm 1. *Compute Standard Basis for Mixed Module*

Input: $N, Q \subset K[x,\lambda]^n$: finite sets of generators of the mixed module $\langle N \rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}} + \langle Q \rangle_{K[\lambda]_{\langle\lambda\rangle}}$

Output: $\left(S^{(1)}, S^{(2)}\right)$: standard basis

1: $S^{(1)} \leftarrow$ the reduced standard basis of N;

2: $S^{(2)} \leftarrow$ the non-zero reduced normal forms of the elements of Q with respect to $S^{(1)}$;

3: $P_1 \leftarrow \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{c} f \in S^{(1)}, g \in S^{(2)}, i = j \text{ and } \alpha \leq \alpha' \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^\alpha \lambda^\beta e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'} \lambda^{\beta'} e_j \end{array} \right\}$;

4: $P_2 \leftarrow \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{c} f \in S^{(2)}, g \in S^{(2)}, i = j \text{ and } \alpha = \alpha' \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^\alpha \lambda^\beta e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'} \lambda^{\beta'} e_j \end{array} \right\}$;

$P = P_1 \cup P_2$;
while $P \neq \emptyset$ **do**
  $f \leftarrow$ one of the elements in P;
  $P \leftarrow P \setminus \{f\}$;
  $f \leftarrow$ the reduced normal form of f in **Algorithm 32** in [5] with respect to $\left(S^{(1)}, S^{(2)}\right)$;
  if $f \neq 0 \left(\mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^{\alpha}\lambda^{\beta}e_i\right)$ **then**
    $P \leftarrow P \cup \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{l} g \in S^{(1)}, i = j \text{ and } \alpha \geq \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'}\lambda^{\beta'}e_j \end{array} \right\}$;
    $P \leftarrow P \cup \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{l} g \in S^{(2)}, i = j \text{ and } \alpha = \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'}\lambda^{\beta'}e_j \end{array} \right\}$;
    $S^{(2)} \leftarrow S^{(2)} \cup \{f\}$;
  end if
end while

end

For Algorithm 1, the following theorem holds [19].

Theorem 2.1. *For a given finite set of generators $N, Q \subset K[x,\lambda]^n$, Algorithm 1 terminates in finite steps and outputs an (x,λ)-mixed standard basis $\left(S^{(1)}, S^{(2)}\right)$ of*

$$\langle N \rangle_{K[x,\lambda]_{\langle x,\lambda \rangle}} + \langle Q \rangle_{K[\lambda]_{\langle \lambda \rangle}}.$$

2.1 Example

In this example, we compute the $\mathcal{A}$-codimension of a map-germ $f\colon \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^2, 0\right)$, defined as

$$(x_1, x_2) \mapsto \left(y_1 = x_1, y_2 = x_1x_2 + x_2^5 + x_2^7\right),$$

which is Type 6 in [17], by using a mixed standard basis. In this example and in the forthcoming examples, we use the variables (x, y) instead of (x, λ) as in [5] since in this context y is supposed to be a coordinate in the target space of the map-germ and it is not common to use λ for that.

Let $\mathcal{E}_n$ be the set of function-germs $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$, $\mathcal{M}_n$ be its maximal ideal, and $\mathcal{M}_n^k$ for $k \in \mathbb{N}$ is recursively defined as: $\mathcal{M}_n^1 = \mathcal{M}_n$ and $\mathcal{M}_n^{k+1} = \mathcal{M}_n \cdot \mathcal{M}_n^k$. Let the tangent space of f relative to $\mathcal{A}$ be

$$T\mathcal{A}(f) = \mathcal{M}_2\langle(1, x_2), \left(0, x_1 + 5x_2^4 + 7x_2^6\right)\rangle_{\mathcal{E}_2} + f^*\langle \mathcal{M}_2\mathcal{E}_2^2 \rangle_{f^*\mathcal{E}_2},$$

where $f^*\mathcal{E}_2 = \{\eta \circ f \mid \eta \in \mathcal{E}_2\}$ and

$$f^*\langle (y_1, 0), (y_2, 0), (0, y_1), (0, y_2) \rangle = \langle (f_1(x), 0), (f_2(x), 0), (0, f_1(x)), (0, f_2(x)) \rangle.$$

Since f is 7-determined,

$$\frac{\mathcal{M}_2\mathcal{E}_2^2}{T\mathcal{A}(f)} \cong \frac{\mathcal{M}_2\mathcal{E}_2^2}{T\mathcal{A}(f) + \mathcal{M}_n^8\mathcal{E}_2^2}$$

which is isomorphic to $\dfrac{\left(\langle x_1, x_2\rangle \mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2}{M}$, as an $\mathbb{R}$-vector space where M is an (x, y)-mixed module with

$$N = \langle x_1, x_2\rangle \cdot \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}[x,y]_{\langle x,y\rangle}} + \langle y_1 - f_1\left(x\right), y_2 - f_2\left(x\right)\rangle \cdot \left(\mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2 + \langle x_1, x_2\rangle^8 \cdot \left(\mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2 \quad (2)$$

and

$$Q = \langle y_1, y_2\rangle \cdot \left(\mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2. \quad (3)$$

By computing an (x, y)-mixed standard basis of M, we can get the $\mathcal{A}$-codimension of f.

In this example, we use the following module ordering:

$$x_1^{\alpha_1} x_2^{\alpha_2} y_1^{\beta_1} y_2^{\beta_2} e_i \prec x_1^{\alpha_1'} x_2^{\alpha_2'} y_1^{\beta_1'} y_2^{\beta_2'} e_j$$

iff one of the following holds:

1. $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 > \alpha_1' + \alpha_2' + \beta_1' + \beta_2'$
2. $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \alpha_1' + \alpha_2' + \beta_1' + \beta_2'$ and $\beta_1 < \beta_1'$
3. $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \alpha_1' + \alpha_2' + \beta_1' + \beta_2'$ and $\beta_1 = \beta_1'$ and $\beta_2 < \beta_2'$
4. $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \alpha_1' + \alpha_2' + \beta_1' + \beta_2'$ and $\beta_1 = \beta_1'$ and $\beta_2 = \beta_2'$ and $\alpha_1 < \alpha_1'$
5. $\alpha = \alpha'$ and $\beta = \beta'$ and $i > j$.

For the module ordering, an (x, y)-mixed standard basis $\left(S^{(1)}, S^{(2)}\right)$ of $N+Q$ is computed by Algorithm 1 implemented in Singular [4]:

$$S^{(1)} = \left\{\left(0, y_2 + 4y_2^5 + 6x_2^7\right), \left(y_2, -4y_2^6 + 6x_2^8\right), \left(0, y_1 - x_2\right), \left(y_1, -5x_2^5 - 7x_2^7\right), \left(x_2, x_2^2\right), \left(x_1, -5x_2^5 - 7x_2^7\right), \left(0, x_1x_2 + 5x_2^5 + 7x_2^7\right), \left(0, x_1^2 - 25x_2^8\right), \left(x_2^9, 0\right)\right\}$$

and

$$S^{(2)} = \left\{\left(0, x_2^5 + 7/5x_2^7\right), \left(0, x_1\right), \left(0, x_2^6 + 3/2x_2^8\right), \left(0, x_2^5 + 3/2x_2^7\right), \left(0, x_2^7\right), \left(0, x_2^8\right)\right\}.$$

The quotient vector space $\dfrac{\left(\langle x_1, x_2\rangle \mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2}{M}$ is spanned by monomials in $\left(\langle x_1, x_2\rangle \mathbb{R}\left[x, y\right]_{\langle x,y\rangle}\right)^2$ that are neither multiples of $\mathrm{LM}_{\prec_{x,\lambda,m}}\left(f\right)$ for $f \in S^{(1)}$ nor involutive multiples of

$\mathrm{LM}_{\prec_{x,\lambda,m}}(f)$ for $f \in S^{(2)}$, where a monomial $x^{\alpha}y^{\beta}e_i$ is an involutive multiple of $x^{\alpha'}y^{\beta'}e_j$ if $i = j$, $\alpha = \alpha'$ and $\beta \geq \beta'$, i.e., $\beta_i \geq \beta'_i$ for all $i = 1, \cdots, n_y$. In this case,

$$\frac{\mathcal{M}_2\mathcal{E}_2^2}{T\mathcal{A}(f)} \cong \langle (0, x_2), \left(0, x_2^2\right), \left(0, x_2^3\right), \left(0, x_2^4\right) \rangle_{\mathbb{R}}$$

and the $\mathcal{A}$-codimension of f is 4, which coincides with the result in Table 1 in [17].

In the next section, we extend the result to that of mixed modules with parameters.

3 Comprehensive Standard System for Mixed Module

Let $\mathbb{K}$ be a field and let $\lambda = (\lambda_1, \cdots, \lambda_{n_\lambda})$, $a = (a_1, \cdots, a_{n_a})$, and $x = (x_1, \cdots, x_{n_x})$ be variables such that they are disjoint with each other. Let $\mathbb{K}[a][x, \lambda]$ be the polynomial ring with variables x, λ, and a. Let K be the algebraic closure of $\mathbb{K}$. Let $t = (t_1, \cdots, t_{n_a}) \in K^{n_a}$ and $\sigma_t \colon \mathbb{K}[a][x, \lambda] \to K[x, \lambda]$ be a specialization morphism defined as $\sigma_t(f) = f|_{a=t}$. Let $V(E) = \{t \in K^{n_a} \mid \forall h \in E, h(t) = 0\}$ be an affin algebraic set of an ideal $E \subset \mathbb{K}[a]$.

Under the setting, comprehensive standard basis for a mixed module is introduced in [19].

Definition 3.1 (Comprehensive Standard System for Mixed Module)**.** *Let $N, Q \subset \mathbb{K}[a][x, \lambda]^n$ be finite sets such that $\langle \sigma_t(N) \rangle_{K[x,\lambda]_{\langle x,\lambda \rangle}}$ has a finite codimension as a K-vector space in $\left(K[x, \lambda]_{\langle x,\lambda \rangle}\right)^n$ for all $t \in V$. Let $S_i^{(1)}, S_i^{(2)} \subset \mathbb{K}[a][x, \lambda]^n$ be a finite subset, and $(E_i, N_i) \subset \mathbb{K}[a] \times \mathbb{K}[a]$ for $i = 1, \cdots, \ell$. The triple set $G = \left\{\left(E_i, N_i, \left(S_i^{(1)}, S_i^{(2)}\right)\right)\right\}_{i=1,\cdots,\ell}$ is called Comprehensive Standard System (*CSS*) for N, Q with respect to $\prec_{x,\lambda,m}$ over $V \subset K^{n_a}$ if the following conditions hold:*

1. *$V \subset \bigcup_{i=1}^{\ell} V(E_i) \setminus V(N_i)$.*
2. *For any $t \in V$ and $i \in \{1, \cdots, \ell\}$ such that $t \in V(E_i) \setminus V(N_i)$ holds, the pair $\left(\sigma_t\left(S_i^{(1)}\right), \sigma_t\left(S_i^{(2)}\right)\right)$ is a (x, λ)-mixed standard basis of $\langle \sigma_t(N) \rangle_{K[x,\lambda]_{\langle x,\lambda \rangle}} + \langle \sigma_t(Q) \rangle_{K[\lambda]_{\langle \lambda \rangle}}$.*

Like in Algorithm 1, the first step is to compute the comprehensive standard basis of N. By Definition 3.1, for any $t \in K^{n_a}$, the specialization $\sigma_t(N)$ is of finite codimension in $\left(K[x, \lambda]_{\langle x,\lambda \rangle}\right)^n$, the comprehensive standard basis of N can be computed by using the algebraic local cohomology (ALC) [18, 15]. Another algorithm such as [8] can be used for that purpose but there is at least one benefit to using ALC in this part, that is, reduction by ALC does not require any division algorithm and can be made quite efficient. In Algorithm 3, reduction by a standard basis of N occurs many times and this part can be made quite efficient if ALC is used. In our implementation, we implemented ALC for finite-codimension modules with parameters in Singular.

In what follows, we provide our algorithm to compute CSS for given pairs of finite generators N and Q in $\mathbb{K}[a][x, \lambda]^n$.

Algorithm 2. *Compute* CSS

Input: $N, Q \subset \mathbb{K}[a][x,\lambda]^n,\ E_{in}, N_{in} \subset \mathbb{K}[a]$
Output: G : CSS on $V(E_{in}) \setminus V(N_{in})$

$G \leftarrow \emptyset$;
$\left\{\left(E_i, N_i, S_i^{(1)}\right)\right\}_{i=1,\cdots,\ell'} \leftarrow$ comprehensive standard system of N on $V(E_{in}) \setminus V(N_{in})$;
for $i \in \{1,\cdots,\ell'\}$ **do**
    $\left\{\left(E_{ij}, N_{ij}, S_i^{(1)}, S_{ij}^{(2)}\right)\right\}_{j=1,\cdots,\ell''} \leftarrow \mathrm{CSSMain}\left(E_i, N_i, S_i^{(1)}, Q\right)$;(See Algorithm 3)
    $G \leftarrow G \cup \left\{\left(E_{ij}, N_{ij}, S_i^{(1)}, S_{ij}^{(2)}\right)\right\}_{i=1,\cdots,\ell''}$;
end for

end

Algorithm 3. $\mathrm{CSSMain}\left(E_i, N_i, S_i^{(1)}, Q\right)$

Input: $E_i, N_i \subset \mathbb{K}[a],\ S_i^{(1)}, Q \subset \mathbb{K}[a][x,\lambda]^n$
Output: G : CSS on $V(E_i) \setminus V(N_i)$

$G \leftarrow \emptyset$;
$Q \leftarrow$ the reduced normal form of Q in terms of $S_i^{(1)}$ in $\left(\mathbb{K}(a)[x,\lambda]_{\langle x,\lambda\rangle}\right)^n$, keep non-zero elements only and multiply each non-zero element to a least commom multiple of the demominators of the coefficients of its terms in $\mathbb{K}[a]$;
$S^{(1)} \leftarrow S_i^{(1)}$;
$S^{(2)} \leftarrow$ the reduced normal form of Q in terms of $E_i\mathbb{K}[a][x,\lambda]^n$, keep non-zero elements only;
$h \leftarrow$ the square-free part of $\mathrm{LCM}\left(\mathrm{LC}_{\prec_{x,\lambda,m}}\left(S^{(2)}\right)\right)$;
$\left(h_1,\cdots,h_{n_f}\right) \leftarrow$ the irreducible factors of h;
$G \leftarrow G \cup \bigcup_{j=1}^{n_f} \mathrm{CSSMain}\left(E_i + \langle h_j\rangle, \left(\prod_{l=1}^{j-1} h_l\right) N_i, S^{(1)}, S^{(2)}\right)$; *
$$P_1 \leftarrow \left\{ \mathrm{spoly}(f,g) \;\middle|\; \begin{array}{c} f \in S^{(1)}, g \in S^{(2)}, i = j \text{ and } \alpha \le \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^\alpha \lambda^\beta e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'} \lambda^{\beta'} e_j \end{array} \right\};^{\dagger}$$
$$P_2 \leftarrow \left\{ \mathrm{spoly}(f,g) \;\middle|\; \begin{array}{c} f \in S^{(2)}, g \in S^{(2)}, i = j \text{ and } \alpha = \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^\alpha \lambda^\beta e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'} \lambda^{\beta'} e_j \end{array} \right\};$$
$P \leftarrow P_1 \cup P_2$;
$G \leftarrow G \cup \mathrm{CSSSub}\left(E_i, hN_i, S_i^{(1)}, S_i^{(2)}, P\right)$; (See Algorithm 4)

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*If $j = 1$, we suppose $\prod_{l=1}^{j-1} h_l = 1$.

†Here, we suppose $P_i \subset \mathbb{K}(a)[x,\lambda]^n$ and compute $\mathrm{spoly}(f,g)$ for f, g regarded as elements of $\mathbb{K}(a)[x,\lambda]^n$ by using Eq. (1).

return;

end

Algorithm 4. CSSSub $\left(E_i, N_i, S^{(1)}, S^{(2)}, P\right)$

Input: $E_i, N_i \subset \mathbb{K}[a]$, $S^{(1)}, S^{(2)} \subset \mathbb{K}[a][x,\lambda]^n$,
$P \subset \mathbb{K}(a)[x,\lambda]^n$
Output: G : CSS on $V(E_i) \setminus V(N_i)$

$G \leftarrow \emptyset$;
while $P \neq \emptyset$ and $N_i \not\subset \sqrt{E_i}$ **do**
  $f \leftarrow$ one of the elements in P;
  $P \leftarrow P \setminus \{f\}$;
  $f \leftarrow$ the reduced normal form of f in **Algorithm 32** in [5] with respect to $\left(S^{(1)}, S^{(2)}\right)$ in $\mathbb{K}(a)[x,\lambda]^n$ multiplied by a least common multiple of the denominators of the coefficients of all the terms of f so that $f \in \mathbb{K}[a][x,\lambda]$ holds;
  $f \leftarrow$ the reduced normal form of f in terms of $E_i\mathbb{K}[a][x,\lambda]^n$;
  while $f \neq 0$ **do**
    $P_1 \leftarrow \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{c} g \in S^{(1)}, i = j \text{ and } \alpha \geq \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^{\alpha}\lambda^{\beta}e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'}\lambda^{\beta'}e_j \end{array} \right\}$;
    $P_2 \leftarrow \left\{ \mathrm{spoly}(f,g) \,\middle|\, \begin{array}{c} g \in S^{(2)}, i = j \text{ and } \alpha = \alpha', \\ \mathrm{LM}_{\prec_{x,\lambda,m}}(f) = x^{\alpha}\lambda^{\beta}e_i, \\ \text{and } \mathrm{LM}_{\prec_{x,\lambda,m}}(g) = x^{\alpha'}\lambda^{\beta'}e_j \end{array} \right\}$;
    $P' \leftarrow P \cup P_1 \cup P_2$;
    $G \leftarrow G \cup \mathrm{CSSSub}\left(E_i, \mathrm{LC}_{\prec_{x,\lambda,m}}(f)N_i, S^{(1)}, S^{(2)} \cup \{f\}, P'\right)$;
    $E_i \leftarrow E_i + \langle \mathrm{LC}_{\prec_{x,\lambda,m}}(f) \rangle$;
    $f \leftarrow f - \mathrm{LT}_{\prec_{x,\lambda,m}}(f)$;
  end while
end while
if $N_i \not\subset \sqrt{E_i}$ **then**
  $G \leftarrow G \cup \left\{\left(E_i, N_i, S^{(1)}, S^{(2)}\right)\right\}$;
end if

end

For a given input $E_i, N_i \subset \mathbb{K}[a]$, $S_i^{(1)}, Q \subset \mathbb{K}[a][x,\lambda]^n$, CSSMain outputs a CSS for N, Q over $V(E_i) \setminus V(N_i)$. In Algorithm 2, in line 2, a comprehensive standard system of N over $V(E_{in}) \setminus V(N_{in})$ is computed. This computation can be done by using [18, 15]. By letting $\left\{\left(E_i, N_i, S_i^{(1)}\right)\right\}_{i=1,\cdots,\ell'}$ be a comprehensive standard system over $V(E_{in}) \setminus V(N_{in})$, the algorithm computes $S^{(2)}$ for each locally closed set $V(E_i) \setminus V(N_i)$ for $i \in \{1, \cdots, \ell'\}$ in line 4 and outputs a comprehensive standard system for a mixed module.

In Algorithm 3, initialization of the set $S^{(2)}$ and the set of S-polynomials for Algorithm 4 is done. In lines 5 and 6 in Algorithm 3, the irreducible factors $\left(h_1, \cdots, h_{n_f}\right)$ and their product h of the product of $\mathrm{LC}_{\prec_{x,\lambda,m}}\left(S^{(2)}\right)$ are computed. If $\sigma_t\left(h\right) \neq 0$ for $t \in K^{n_a}$, all the leading coefficients of $\sigma_t\left(S^{(2)}\right)$ are non-zero. Algorithm 3 decomposes the locally closed set $V\left(E_i\right) \setminus V\left(N_i\right)$ such as

$$V\left(E_i\right) \setminus V\left(N_i\right) = \left[V\left(E_i\right) \setminus V\left(hN_i\right)\right] \cup \bigcup_{j=1}^{n_f}\left[V\left(E_i + \langle h_j \rangle\right) \setminus V\left(\prod_{l=1}^{j-1} h_l N_i\right)\right],$$

and recursively call CSSMain for each locally closed set except to the first one $V\left(E_i\right) \setminus V\left(hN_i\right)$. On the locally closed set $V\left(E_i\right) \setminus V\left(hN_i\right)$, all the leading coefficients of the elements in $S^{(2)}$ are non-zero and thus the S-polynomials of the elements in between $S^{(1)}$ and $S^{(2)}$ or that of the elements among $S^{(2)}$ are well-defined on $V\left(E_i\right) \setminus V\left(hN_i\right)$. The set of the S-polynomials P is initiated in lines 8 - 10 of Algorithm 3 and forwarded to CSSSub in line 11.

In Algorithm 4, CSS on $V\left(E_i\right) \setminus V\left(N_i\right)$ (Put $E_i = E_i$ and $N_i = hN_i$ to match it with the previous context.) is computed. Note that all the leading coefficients of $S^{(2)}$ are supposed to be non-zero on $V\left(E_i\right) \setminus V\left(N_i\right)$. For each element f in the set of the S-polynomials P, its reduced normal form with respect to $\left(S^{(1)}, S^{(2)}\right)$ and $E_i\mathbb{K}\left[a\right]\left[x, \lambda\right]$ are computed in lines 5 and 6, respectively. If the reduced normal form of f is non-zero, Algorithm 4 enters into the while loop starting from line 7 to line 18. In the while loop, the locally closed set $V\left(E_i\right) \setminus V\left(N_i\right)$ is decomposed into

$$V\left(E_i\right) \setminus V\left(N_i\right) = \left[V\left(E_i\right) \setminus V\left(\mathrm{LC}_{\prec_{x,\lambda,m}}\left(f\right) N_i\right)\right] \cup \left[V\left(E_i + \langle \mathrm{LC}_{\prec_{x,\lambda,m}}\left(f\right) \rangle\right) \setminus V\left(N_i\right)\right].$$

For the first locally closed set $V\left(E_i\right) \setminus V\left(\mathrm{LC}_{\prec_{x,\lambda,m}}\left(f\right) N_i\right)$, the leading coefficient of f is non-zero. In this case, Algorithm 4 updates the set of the S-polynomials and $S^{(2)}$ and recursively call CSSSub. For the second locally closed set $V\left(E_i + \langle \mathrm{LC}_{\prec_{x,\lambda,m}}\left(f\right) \rangle\right) \setminus V\left(N_i\right)$, the leading coefficient of f is zero and thus subtracts $\mathrm{LT}_{\prec_{x,\lambda,a}}\left(f\right)$ from f, update E_i to $E_i + \langle \mathrm{LC}_{\prec_{x,\lambda,m}}\left(f\right) \rangle$ and continue while $P \neq \emptyset$ and $N_i \not\subset \sqrt{E_i}$. In the end, if $P = \emptyset$ but $N_i \not\subset \sqrt{E_i}$, Algorithm 4 adds the resulting $\left(E_i, N_i, S^{(1)}, S^{(2)}\right)$ to G. This is the flow of Algorithms 2-4. For Algorithms 2-4, the following holds [19].

Theorem 3.1 (Correctness and Termination in Finite Steps)**.** *For a given finite set of generators $N, Q \subset \mathbb{K}\left[a\right]\left[x, \lambda\right]^n$ such that $\langle \sigma_t\left(N\right) \rangle_{K[x,\lambda]_{\langle x,\lambda \rangle}}$ has a finite codimension as a K vector space in $\left(K\left[x, \lambda\right]_{\langle x,\lambda \rangle}\right)^n$ for all $t \in K^{n_a}$, Algorithms 2-4 terminate in finite steps and output a Comprehensive Standard System* (CSS) *for N, Q with respect to $\prec_{x,\lambda,m}$ over $V\left(E_{in}\right) \setminus V\left(N_{in}\right)$.*

3.1 Example

Consider

$$f\colon \left(x_1, x_2\right) \mapsto \left(y_1 = x_1, y_2 = x_1^2 x_2 + x_1 x_2^3 + \alpha x_2^5 + x_2^6 + \beta x_2^7\right)$$

(Type 18 in Table 1 in [17]). Its $\mathcal{A}$-codimension depends on the moduli parameters $\alpha, \beta \in \mathbb{R}$. We would like to detect exceptional values of the moduli parameters (In the generic case, it has $\mathcal{A}$-codimension 8 [17].). In this example, the degree of determinacy also depends on the moduli parameters. By applying a result of du Plessis [3], **Lemma 2.6**, f is k-$\mathcal{A}$-determined if

$$\mathcal{M}_2^{k+1}\mathcal{E}_2^2 \subset T\mathcal{A}_1(f) + \langle x_1 \frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}} + f^*\langle y_2 e_1\rangle_{\mathbb{R}} + \mathcal{M}_2^{k+1} f^*(\mathcal{M}_2)\mathcal{E}_2^2 + \mathcal{M}_2^{2k+2}\mathcal{E}_2^2$$

holds. This condition is equivalent to $\langle x_1, x_2\rangle^{k+1}\left(\mathbb{R}[x,y]_{\langle x,y\rangle}\right)^2$ is contained in the (x,y)-mixed module $M = N + Q$ where

$$\begin{aligned} N = \langle x_1, x_2\rangle \cdot \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}[x,y]_{\langle x,y\rangle}} + \langle x_1 \frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}[x,y]_{\langle x,y\rangle}} \\ + \langle y_1 - f_1(x), y_2 - f_2(x)\rangle \cdot \left(\mathbb{R}[x,y]_{\langle x,y\rangle}\right)^2 \\ + \langle x_1, x_2\rangle^{k+1} \cdot \langle y_1, y_2\rangle \cdot \left(\mathbb{R}[x,y]_{\langle x,y\rangle}\right)^2 + \langle x_1, x_2\rangle^{2k+2} \cdot \left(\mathbb{R}[x,y]_{\langle x,y\rangle}\right)^2 \end{aligned}$$

and

$$Q = \langle y_1, y_2\rangle^2 \cdot \left(\mathbb{R}[y]_{\langle y\rangle}\right)^2 + \langle y_2 e_1\rangle_{\mathbb{R}[x,y]_{\langle y\rangle}}$$

By computing CSS for the (x,y)-mixed module for $k = 7$ by using the same module ordering in Example 2.1, the parameter space $\mathbb{C}^2$ is decomposed into the 12 locally closed sets in Table 1. Note that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$ and thus Algorithms 2-4 provide a decomposition of $\mathbb{C}^2$ instead of $\mathbb{R}^2$. However, Algorithms 2-4 are based upon arithmetic operations in the ground field only. This means that if the scalars in the input data are contained in $\mathbb{R}$, then all scalars in the output also lie in $\mathbb{R}$. This guarantees that the decomposition in Table 1 restricted to $\mathbb{R}^2$ provides a semi-algebraic decomposition of $\mathbb{R}^2$ such that the pair $\left(S^{(1)}, S^{(2)}\right)$ corresponding to each semi-algebraic set specialized to any element in the semi-algebraic set is a (x,y)-mixed standard basis of $\langle \sigma_t(N)\rangle_{\mathbb{R}[x,y]_{\langle x,y\rangle}} + \langle \sigma_t(Q)\rangle_{\mathbb{R}[y]_{\langle y\rangle}}$.

The corresponding comprehensive standard basis is too large to be shown in this paper and thus we only show the ones corresponding to the first three strata:

1. $V(\langle \alpha, \beta + 4\rangle) \setminus V(\langle 2\beta + 1\rangle)$:

$$\begin{aligned} S^{(1)} = \{ &\left(0, y_2 - x_1^2 x_2 - x_1 x_2^3 - \alpha x_2^5 - x_2^6 - \beta x_2^7\right), \\ &\left(y_2, (-8\alpha + 10) x_1 x_2^6 + 16\alpha x_2^8 - 10 x_1 x_2^7 + 19 x_2^9 + 22\beta x_2^{10}\right), (0, y_1 - x_1), \\ &(y_1 - x_1, 0), \left(x_2^2, 2x_1 x_2^3 + x_2^5\right), \left(x_1 x_2, -5x_1 x_2^4 - 10\alpha x_2^6 - 12 x_2^7 - 14\beta x_2^8\right), \\ &\left(x_1^2, (-10\alpha + 15) x_1 x_2^5 + 25\alpha x_2^7 - 12 x_1 x_2^6 + 30 x_2^8 - 14\beta x_1 x_2^7 + 35\beta x_2^9\right), \\ &\left(0, x_1^3 + (5\alpha - 9) x_1 x_2^4 - 15\alpha x_2^6 + 6 x_1 x_2^5 - 18 x_2^7 + 7\beta x_1 x_2^6 - 21\beta x_2^8\right), \\ &\left(0, x_1^2 x_2^2 + 3x_1 x_2^4 + 5\alpha x_2^6 + 6 x_2^7 + 7\beta x_2^8\right), \left(0, x_1 x_2^8\right), \left(0, x_2^{12}\right) \}, \end{aligned}$$

and

$$
\begin{gathered}
S^{(2)} = \left\{\left(0, -3/2x_1x_2^5 + 6/5x_1x_2^6 - 3x_2^8 + (7\beta)/5x_1x_2^7 - 7\beta/2x_2^9\right),\right. \\
\left(0, x_1^2\right), \left(0, -3/2x_2^{11}\right), \left(0, -3/2x_1x_2^5 + 5/4x_1x_2^6 - 3x_2^8 + 3\beta/2x_1x_2^7 - 7\beta/2x_2^9\right), \\
\left(0, -3/2x_2^{11}\right), \left(0, -5/4x_1x_2^6 + 5/4x_1x_2^7 - 19/8x_2^9 - 11\beta/4x_2^{10}\right), \\
\left(0, (2\beta+1)/2x_1x_2^7 - 19/20x_2^9 - 11\beta/10x_2^{10}\right), \left(0, x_2^9 + (62\beta+20)/19x_2^{10}\right), \\
\left.\left(0, x_1x_2^4 + 2x_2^7 + (7\beta+4)/3x_2^8 + (-1400\beta^2 - 432\beta)/171x_2^{10}\right)\right\}
\end{gathered}
$$

2. $V\left(\langle\alpha\rangle\right) \setminus V\left(\langle 2\beta^2 + 9\beta + 4\rangle\right)$:

$$
\begin{gathered}
S^{(1)} = \left\{\left(0, y_2 - x_1^2x_2 - x_1x_2^3 - \alpha x_2^5 - x_2^6 - \beta x_2^7\right),\right. \\
\left(y_2, (-8\alpha+10)x_1x_2^6 + 16\alpha x_2^8 - 10x_1x_2^7 + 19x_2^9 + 22\beta x_2^{10}\right), (0, y_1 - x_1), \\
(y_1 - x_1, 0), \left(x_2^2, 2x_1x_2^3 + x_2^5\right), \left(x_1x_2, -5x_1x_2^4 - 10\alpha x_2^6 - 12x_2^7 - 14\beta x_2^8\right), \\
\left(x_1^2, (-10\alpha+15)x_1x_2^5 + 25\alpha x_2^7 - 12x_1x_2^6 + 30x_2^8 - 14\beta x_1x_2^7 + 35\beta x_2^9\right), \\
\left(0, x_1^3 + (5\alpha-9)x_1x_2^4 - 15\alpha x_2^6 + 6x_1x_2^5 - 18x_2^7 + 7\beta x_1x_2^6 - 21\beta x_2^8\right), \\
\left.\left(x_1^2x_2^2 + 3x_1x_2^4 + 5\alpha x_2^6 + 6x_2^7 + 7\beta x_2^8\right), \left(0, x_1x_2^8\right), \left(0, x_2^{12}\right)\right\},
\end{gathered}
$$

and

$$
\begin{gathered}
S^{(2)} = \left\{\left(0, -3/2x_1x_2^5 + 6/5x_1x_2^6 - 3x_2^8 + 7\beta/5x_1x_2^7 - 7\beta/2x_2^9\right), \left(0, x_1^2\right),\right. \\
\left(-3/2x_1x_2^5 + 5/4x_1x_2^6 - 3x_2^8 + 3\beta/2x_1x_2^7 - 7\beta/2x_2^9\right), \left(0, -3/2x_2^{11}\right), \\
\left(0, -5/4x_1x_2^6 + 5/4x_1x_2^7 - 19/8x_2^9 - 11\beta/4x_2^{10}\right), \\
\left(0, (2\beta+1)/2x_1x_2^7 - 19/20x_2^9 - 11\beta/10x_2^{10}\right), \\
\left(0, x_2^9 + (62\beta+20)/19x_2^{10}\right), \left(0, x_2^{10}\right), \left(0, -3/2x_2^{11}\right), \\
\left.\left(0, x_1x_2^4 + 2x_2^7 + (7\beta+4)/3x_2^8 + (-1400\beta^2 - 432\beta)/171x_2^{10}\right)\right\}.
\end{gathered}
$$

3. $V\left(\langle\alpha, 2\beta+1\rangle\right) \setminus V\left(\langle 1\rangle\right)$:

$$
\begin{gathered}
S^{(1)} = \left\{\left(0, y_2 - x_1^2x_2 - x_1x_2^3 + (-\alpha)x_2^5 - x_2^6 + (-\beta)x_2^7\right),\right. \\
\left(y_2, (-8\alpha+10)x_1x_2^6 + (16\alpha)x_2^8 - 10x_1x_2^7 + 19x_2^9 + (22\beta)x_2^{10}\right), \\
(0, y_1 - x_1), (y_1 - x_1, 0), \left(x_2^2, 2x_1x_2^3 + x_2^5\right), \\
\left(x_1x_2, -5x_1x_2^4 + (-10\alpha)x_2^6 - 12x_2^7 + (-14\beta)x_2^8\right), \\
\left(x_1^2, +(-10\alpha+15)x_1x_2^5 + (25\alpha)x_2^7 - 12x_1x_2^6 + 30x_2^8 + (-14\beta)x_1x_2^7 + (35\beta)x_2^9\right), \\
\left(0, x_1^3 + (5\alpha-9)x_1x_2^4 + (-15\alpha)x_2^6 + 6x_1x_2^5 - 18x_2^7 + (7\beta)x_1x_2^6 + (-21\beta)x_2^8\right), \\
\left.\left(0, x_1^2x_2^2 + 3x_1x_2^4 + (5\alpha)x_2^6 + 6x_2^7 + (7\beta)x_2^8\right), \left(0, x_1x_2^8\right), \left(0, x_2^{12}\right)\right\}
\end{gathered}
$$

and

$$S^{(2)} = \{ (0, -3/2x_1x_2^5 + 6/5x_1x_2^6 - 3x_2^8 + (7\beta)/5x_1x_2^7 + (-7\beta)/2x_2^9),$$
$$(0, x_1^2), (0, -3/2x_2^{11}), (0, -3/2x_1x_2^5 + 5/4x_1x_2^6 - 3x_2^8 + (3\beta)/2x_1x_2^7 + (-7\beta)/2x_2^9),$$
$$(0, -3/2x_2^{11}), (0, -5/4x_1x_2^6 + 5/4x_1x_2^7 - 19/8x_2^9 + (-11\beta)/4x_2^{10}),$$
$$(0, x_2^9 + (22\beta)/19x_2^{10}), (0, x_1x_2^7 + 2x_2^{10}),$$
$$(0, x_1x_2^4 + 2x_2^7 + 1/6x_2^8 - 134/171x_2^{10}), (0, x_2^{10}) \}$$

By reducing the generators of $\langle x_1, x_2\rangle^{k+1} \left(\mathbb{R}[x,y]_{\langle x,y\rangle}\right)^2$ by the mixed standard basis for each locally closed set, $M \subset N + Q$ holds for parameter values in the locally closed set $V(\langle 0\rangle)\backslash V\left(\langle \alpha\left(4000\alpha^4\beta - 8600\alpha^3\beta - 2500\alpha^3 + 4260\alpha^2\beta + 7825\alpha^2 - 540\alpha\beta - 2574\alpha + 81\right)\rangle\right)$ and thus f is 7-$\mathcal{A}$-determined for the parameter values $\alpha \neq 0$ and $4000\alpha^4\beta - 8600\alpha^3\beta - 2500\alpha^3 + 4260\alpha^2\beta + 7825\alpha^2 - 540\alpha\beta - 2574\alpha + 81 \neq 0$. If the parameters $\alpha = 0$ or $4000\alpha^4\beta - 8600\alpha^3\beta - 2500\alpha^3 + 4260\alpha^2\beta + 7825\alpha^2 - 540\alpha\beta - 2574\alpha + 81 = 0$, higher jets need to be investigated. If the parameters are in the locally closed set $V(\langle 0\rangle) \setminus V\left(\langle \alpha\left(4000\alpha^4\beta - 8600\alpha^3\beta - 2500\alpha^3 + 4260\alpha^2\beta + 7825\alpha^2 - 540\alpha\beta - 2574\alpha + 81\right)\rangle\right)$, the $\mathcal{A}$-codimension of f can be calculated by computing CSS of finite sets of generators of N and Q in Eq. (2) and Eq. (3). The resulting CSS is as follows:

1.

$$V(\langle 4\alpha - 5\rangle)$$
$$\backslash V\left(\langle 4000\alpha^5\beta - 8600\alpha^4\beta - 2500\alpha^4 + 4260\alpha^3\beta + 7825\alpha^3 - 540\alpha^2\beta - 2574\alpha^2 + 81\alpha\rangle\right):$$

$$S^{(1)} = \{ (0, y_2 + 2x_1x_2^3 + 4\alpha x_2^5 + 5x_2^6 + 6\beta x_2^7), (y_2, (-8\alpha + 10)x_1x_2^6),$$
$$(0, y_1 - x_1), (y_1, -5x_1x_2^3 + (-10\alpha)x_2^5 - 12x_2^6 + (-14\beta)x_2^7),$$
$$(x_2, 2x_1x_2^2 + x_2^4), (x_1, -5x_1x_2^3 + (-10\alpha)x_2^5 - 12x_2^6 + (-14\beta)x_2^7),$$
$$(0, x_1^2x_2 + 3x_1x_2^3 + (5\alpha)x_2^5 + 6x_2^6 + (7\beta)x_2^7), (0, x_2^8), (0, x_1x_2^7)$$
$$(0, x_1^3 + (5\alpha - 9)x_1x_2^4 + (-15\alpha)x_2^6 + 6x_1x_2^5 - 18x_2^7 + (7\beta)x_1x_2^6) \}$$

and

$$S^{(2)} = \{ (0, x_1x_2^3 + 5/2x_2^5 + 12/5x_2^6 + 14\beta/5x_2^7),$$
$$(0, x_1), (0, x_1x_2^3 + 5/2x_2^5 + 5/2x_2^6 + 3\beta x_2^7),$$
$$(0, x_2^6 + 2\beta x_2^7), (0, x_1x_2^6), (0, x_1x_2^5 + 25/2x_2^7),$$
$$(0, x_1^2), (0, x_1x_2^4 + (-150\beta + 372)/11x_2^7) \}.$$

2.

$$V\left(\langle 320000\alpha^9\beta - 2624000\alpha^8\beta - 200000\alpha^8 + 8871200\alpha^7\beta + 1836000\alpha^7 - 15852240\alpha^6\beta - 6723220\alpha^6 + 15898680\alpha^5\beta + 12519696\alpha^5 - 8711820\alpha^4\beta -12383937\alpha^4 + 2313360\alpha^3\beta + 6060663\alpha^3 - 218700\alpha^2\beta - 1130679\alpha^2 + 32805\alpha\rangle\right) \backslash V\left(\langle 4000\alpha^5\beta - 8600\alpha^4\beta - 2500\alpha^4 + 4260\alpha^3\beta + 7825\alpha^3 - 540\alpha^2\beta - 2574\alpha^2 + 81\alpha\rangle\right):$$

$$S^{(1)} = \left\{\left(0, y_2 + 2x_1x_2^3 + 4\alpha x_2^5 + 5x_2^6 + 6\beta x_2^7\right), \left(y_2, (-8\alpha+10)x_1x_2^6\right), (0, y_1 - x_1), \left(y_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7\right), \left(x_2, 2x_1x_2^2 + x_2^4\right), \left(x_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7\right), \left(0, x_2^8\right), \left(0, x_1^2x_2 + 3x_1x_2^3 + 5\alpha x_2^5 + 6x_2^6 + 7\beta x_2^7\right), \left(0, x_1x_2^7\right), \left(0, x_1^3 + (5\alpha - 9)x_1x_2^4 - 15\alpha x_2^6 + 6x_1x_2^5 - 18x_2^7 + 7\beta x_1x_2^6\right)\right\}$$

and

$$S^{(2)} = \left\{\left(0, x_1x_2^3 + 2\alpha x_2^5 + 12/5x_2^6 + 14\beta/5x_2^7\right), (0, x_1), \left(0, (4\alpha-5)/4x_1x_2^6\right), \left(0, x_1x_2^3 + 2\alpha x_2^5 + 5/2x_2^6 + 3\beta x_2^7\right), \left(0, x_2^6 + 2\beta x_2^7\right), \left(0, (2\alpha-3)/2x_1x_2^5 - 5\alpha/2x_2^7\right), \left(0, x_1^2\right), \left(0, (10\alpha^2 - 33\alpha + 27)/10x_1x_2^4 + (30\alpha^2\beta - 45\alpha\beta - 3\alpha + 27)/5x_2^7\right)\right\}.$$

3.

$$V\left(\langle 10\alpha^2 - 33\alpha + 27\rangle\right) \setminus V\left(\langle 960000\alpha^9\beta^2 - 6144000\alpha^8\beta^2 - 696000\alpha^8\beta + 15554400\alpha^7\beta^2 + 5762400\alpha^7\beta + 60000\alpha^7 - 19558800\alpha^6\beta^2 - 17282700\alpha^6\beta - 892800\alpha^6 + 12490200\alpha^5\beta^2 + 23930100\alpha^5\beta + 3865926\alpha^5 - 3653100\alpha^4\beta^2 - 15428475\alpha^4\beta - 6740487\alpha^4 + 364500\alpha^3\beta^2 + 3924450\alpha^3\beta + 4836753\alpha^3 - 273375\alpha^2\beta -1094229\alpha^2 + 32805\alpha\rangle\right):$$

$$S^{(1)} = \left\{\left(0, y_2 + 2x_1x_2^3 + 4\alpha x_2^5 + 5x_2^6 + 6\beta x_2^7\right), \left(y_2, (-8\alpha+10)x_1x_2^6\right), (0, y_1 - x_1), \left(y_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7\right), \left(x_2, 2x_1x_2^2 + x_2^4\right), \left(x_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7\right), \left(0, x_1^2x_2 + 3x_1x_2^3 + 5\alpha x_2^5 + 6x_2^6 + 7\beta x_2^7\right), \left(0, x_2^8\right), \left(0, x_1x_2^7\right), \left(0, x_1^3 + (5\alpha - 9)x_1x_2^4 - 15\alpha x_2^6 + 6x_1x_2^5 - 18x_2^7 + 7\beta x_1x_2^6\right)\right\}$$

and

$$S^{(2)} = \{(0, x_1x_2^3 + 2\alpha x_2^5 + 12/5x_2^6 + 14\beta/5x_2^7), (0, x_1), (0, (4\alpha-5)/4x_1x_2^6), (0, x_1x_2^3 + 2\alpha x_2^5 + 5/2x_2^6 + 3\beta x_2^7), (0, x_2^6 + 2\beta x_2^7), (0, (2\alpha-3)/2x_1x_2^5 + (-5\alpha)/2x_2^7), (0, x_1^2), (0, x_2^7)\}.$$

4.

$$V(\langle 2\alpha-3\rangle) \setminus V(\langle -80000\alpha^7\beta + 272000\alpha^6\beta + 50000\alpha^6 - 300200\alpha^5\beta - 219000\alpha^5 + 117300\alpha^4\beta + 247105\alpha^4 - 13500\alpha^3\beta - 65970\alpha^3 + 2025\alpha^2\rangle):$$

$$S^{(1)} = \{(0, y_2 + 2x_1x_2^3 + 4\alpha x_2^5 + 5x_2^6 + 6\beta x_2^7), (y_2, (-8\alpha+10)x_1x_2^6), (0, y_1 - x_1), (y_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7), (x_2, 2x_1x_2^2 + x_2^4), (x_1, -5x_1x_2^3 - 10\alpha x_2^5 - 12x_2^6 - 14\beta x_2^7), (0, x_2^8), (0, x_1^2x_2 + 3x_1x_2^3 + 5\alpha x_2^5 + 6x_2^6 + 7\beta x_2^7), (0, x_1x_2^7), (0, x_1^3 + (5\alpha-9)x_1x_2^4 - 15\alpha x_2^6 + 6x_1x_2^5 - 18x_2^7 + 7\beta x_1x_2^6)\}$$

and

$$S^{(2)} = \{(0, x_1x_2^3 + 2\alpha x_2^5 + 12/5x_2^6 + 14\beta/5x_2^7), (0, x_1), (0, (4\alpha-5)/4x_1x_2^6), (0, x_2^7), (0, x_1x_2^3 + 2\alpha x_2^5 + 5/2x_2^6 + 3\beta x_2^7), (0, x_2^6 + 2\beta x_2^7), (0, x_1^2), (0, x_1x_2^4 - 4x_1x_2^5)\}.$$

In this case, for all the parameter values in

$$V(\langle 0\rangle)\setminus V(\langle \alpha(4000\alpha^4\beta - 8600\alpha^3\beta - 2500\alpha^3 + 4260\alpha^2\beta + 7825\alpha^2 - 540\alpha\beta - 2574\alpha + 81)\rangle),$$

the $\mathcal{A}$-codimension of f is 8, which is consistent with the results in [17]. Further applications to singularity theory is reported in [19].

4 Feature Perspectives

In this paper, we have considered mixed-modules that are sums of two modules over two different rings, which can be applicable to classification of map-germs relative to $\mathcal{A}$ and $\mathcal{K}_{\mathcal{B}}$ equivalences. However, if we consider classification of divergent diagram (See Section 6 in [1] for summary of a classification result.) whose target dimension is more than 1 or $\mathcal{K}_{\mathcal{B}}$ equivalences with more than 1 types of external parameters, (extended) tangent space relative to them become sums of more than 2 modules over different rings. We are going to report the generalization of our current algorithm to cover these cases in the forthcoming paper.

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#	locally closed set
1	$V\left(\langle \alpha, \beta+4\rangle\right)$
2	$V\left(\langle \alpha\rangle\right) \setminus V\left(\langle \alpha, 2\beta^2+9\beta+4\rangle\right)$
3	$V\left(\langle \alpha, 2\beta+1\rangle\right) \setminus V\left(\langle 1\rangle\right)$
4	$V\left(\langle 4\alpha-5, 4200\beta-16829\rangle\right) \setminus V\left(\langle \alpha\rangle\right)$
5	$V\left(\langle 4\alpha-5\rangle\right) \setminus V\left(\langle 4200\alpha\beta-16829\alpha\rangle\right)$
6	$V\left(\langle 2\alpha-3\rangle\right) \setminus V\left(\langle 4\alpha^3-5\alpha^2\rangle\right)$
7	$V\left(\langle 8000\alpha^6\beta-29200\alpha^5\beta-5000\alpha^5+34320\alpha^4\beta+23150\alpha^4-13860\alpha^3\beta -28623\alpha^3+1620\alpha^2\beta+7884\alpha^2-243\alpha\rangle\right) \setminus V\left(\langle 9680\alpha^{10}-74404\alpha^9 +234444\alpha^8-387189\alpha^7+353079\alpha^6-168399\alpha^5+32805\alpha^4\rangle\right)$
8	$V\left(\langle 0\rangle\right) \setminus V\left(\langle 77440000\alpha^{16}\beta-877888000\alpha^{15}\beta-48400000\alpha^{15}+4380366400\alpha^{14}\beta + 596112000\alpha^{14}-12630986880\alpha^{13}\beta-3171743240\alpha^{13}+23223589920\alpha^{12}\beta + 9569306412\alpha^{12}-28315353600\alpha^{11}\beta-18028264338\alpha^{11}+23043600900\alpha^{10}\beta + 21964721265\alpha^{10}-12258280800\alpha^9\beta-17278210035\alpha^9+4031865720\alpha^8\beta + 8457282090\alpha^8-727483680\alpha^7\beta-2352433428\alpha^7+53144100\alpha^6\beta +299555577\alpha^6-7971615\alpha^5\rangle\right)$
9	$V\left(\langle 110\alpha^4-453\alpha^3+594\alpha^2-243\alpha, 652\alpha^3\beta+286\alpha^3-1716\alpha^2\beta-363\alpha^2 +1107\alpha\beta-99\alpha\rangle\right) \setminus V\left(\langle -5808\alpha^{10}+34188\alpha^9-79128\alpha^8+89883\alpha^7 -50058\alpha^6+10935\alpha^5\rangle\right)$
10	$V\left(\langle 110\alpha^4-453\alpha^3+594\alpha^2-243\alpha\rangle\right) \setminus V\left(\langle -3786816\alpha^{13}\beta-1661088\alpha^{13} + 32257104\alpha^{12}\beta+11886072\alpha^{12}-116687520\alpha^{11}\beta-34465860\alpha^{11} + 232233480\alpha^{10}\beta+51045390\alpha^{10}-274471740\alpha^9\beta-39110445\alpha^9+192529629\alpha^8\beta +12400047\alpha^8-74178666\alpha^7\beta+986337\alpha^7+12105045\alpha^6\beta-1082565\alpha^6\rangle\right)$
11	$V\left(\langle 11\alpha-9, 306\beta-1111\rangle\right) \setminus V\left(\langle 80\alpha^7-340\alpha^6+480\alpha^5-225\alpha^4\rangle\right)$
12	$V\left(\langle 11\alpha-9\rangle\right) \setminus V\left(24480\alpha^7\beta-88880\alpha^7-104040\alpha^6\beta+377740\alpha^6+146880\alpha^5\beta -533280\alpha^5-68850\alpha^4\beta+249975\alpha^4\rangle\right)$

Table 1: Decomposition of the parameter space $\mathbb{C}^2$ into the locally closed sets