CONSTANT DIAMETER SPHERICAL CONVEX BODIES AND WULFF SHAPES

HUHE HAN

1. Basic definitions

Throughout this note, let S^n denote the unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For any given point P of S^n, we denote by $H(P)$ the *hemisphere* whose center is P, namely,

$$H(P) = \{Q \in S^n \mid P \cdot Q \geq 0\}.$$

Here the dot in the center stands for the scalar product of P, Q in $\mathbb{R}^{n+1}$. A non-empty subset W of S^n is *hemispherical* if there exists a point P of S^n such that the intersection set $W \cap H(P)$ is the empty set. A hemispherical W of S^n is said to be *spherical convex* if the arc between any two points $P, Q \in W$ lies in the W. Equivalently, a hemispherical W of S^n is convex if PQ is a subset of W, for $P, Q \in W$, where PQ stands for the following arc

$$PQ = \left\{ \frac{tP + (1-t)Q}{|| \, tP + (1-t)Q \, ||} \in S^n \mid 0 \leq t \leq 1 \right\}.$$

Denote the *great-circle distance* between two points P, Q of S^n by $| \, PQ \, |$, namely, $| \, PQ \, | = \arccos^{-1}(P \cdot Q)$. Denote the boundary of W is denoted by ∂W. A spherical convex set W of S^n is said to be *spherical convex body* if W has an interior point and closed. For any subset W of S^n, the *spherical polar set* of W is the following set, denoted by W°,

$$\bigcap_{P \in W} H(P).$$

For any non-empty closed hemispherical subset $W \subset S^n$, the equality s-conv(W) = (s-conv$(W))^{\circ\circ}$ holds ([10]), where s-conv(W) is the *spherical convex hull of* W, namely,

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{|| \sum_{i=1}^k t_i P_i ||} \mid \sum_{i=1}^k t_i = 1, t_i \geq 0, k \in \mathbb{N} \text{ and } P_i \in W \right\},$$

The *diameter* of a spherical convex body W is defined by

$$\max\{| \, PQ \, | \mid P, Q \in W\}.$$

A spherical convex body W is said to be *constant diameter* τ, if the diameter of K is τ, and for every point $P \in \partial W$ there exists a point Q of ∂W such that $|PQ| = \tau$ ([7]). We say a hemisphere $H(Q)$ *supports* W *at* P if W is a subset of $H(Q)$ and P is a point of $\partial W \cap \partial H(Q)$. The hemisphere $H(Q)$ as defined above is called a

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supporting hemisphere of W at P. For any two points $P, Q (P \neq -Q)$ of S^n, the intersection

$$H(P) \cap H(Q)$$

is called a *lune*. The *thickness* of lune $H(P) \cap H(Q)$ is the real number $\pi - \mid PQ \mid$, denoted by $\Delta(H(P) \cap H(Q))$. It is clear that thickness of any lune is greater than 0 and less than π. Let $H(P)$ be a supporting hemisphere of a spherical convex body W. The *width* of W with respect to $H(P)$ is defined by ([6])

$$\text{width}_{H(P)}(K) = \min\{\Delta(H(P) \cap H(Q)) \mid W \subset H(Q)\}.$$

The minimum width of W is called *thickness of* W, denoted by ΔW. A spherical convex body W is said to be *of constant width*, if all widths of W with respect to any supporting hemispheres $H(P)$ are equal. A convex body W of S^n is said to be *reduced* if $\Delta(X) < \Delta(W)$ for every convex body X properly contained in W ([6]).

2. Some Known results

Lemma 2.1 ([10]). *Let X, Y be subsets of S^n. Suppose that the X is a subset of Y. Then, Y° is a subset of X°.*

Lemma 2.2 ([10]). *The subset W is a spherical polytope if and only if W° is a spherical polytope.*

Lemma 2.3 ([7]). *Every spherical convex body of constant width smaller than $\pi/2$ on S^n is strictly convex.*

Lemma 2.4 ([5]). *Let W be a spherical convex body in S^n, and $0 < \tau < \pi$. The following two assertions are equivalent:*

(1) *W is of constant width τ.*
(2) *W° is of constant width $\pi - \tau$.*

In the case of S^2, an alternative proof of Lemma 2.4 given in [9].

Lemma 2.5 ([6]). *Every smooth reduced body W of S^n is of constant width.*

Theorem 1 ([5]). *Let W be a spherical convex body in S^n, and $0 < \tau < \pi$. The following two are equivalent:*

(1) *W is of constant diameter τ.*
(2) *W is of constant width τ.*

For the cases of smoothness boundary and S^2, see [8]. The following corollary is an easy consequence of Theorem 1 and Lemma 2.4.

Corollary 2.1 ([5]). *Let W be a spherical convex body in S^n, and $0 < \tau < \pi$. The following two propositions are equivalent:*

(1) *W is of constant diameter τ.*
(2) *W° is of constant diameter $\pi - \tau$.*

The following corollary is an easy consequence of Theorem 1 and Lemma 2.5.

Corollary 2.2. *Every spherical convex body of constant diameter smaller than $\pi/2$ on S^n is strictly convex.*

Corollary 2.3. *Every smooth reduced body W of S^n is of constant diameter.*

3. Applications to Wulff shapes

Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function, where $\mathbb{R}_+$ is the set consisting of positive real numbers. Then the *Wulff shape* associated with the function γ, denoted by $\mathcal{W}_\gamma$, is defined by

$$\bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$

Here $\Gamma_{\gamma,\theta}$ is the half space determined by the given continuous function γ and $\theta \in S^n$,

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

By definition, Wulff shape is a convex body and contains the origin of $\mathbb{R}^{n+1}$ as an interior point. Conversely, for any convex body W contains the origin of $\mathbb{R}^{n+1}$ as an interior point, there exits a continuous function $\gamma : S^n \to \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. For more details in Wulff shapes, see for instance [1, 2, 3]. Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by

$$Id(x) = (x, 1).$$

Let $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ be the north pole of S^{n+1}, and let $S^{n+1}_{N,+}$ denote the north open hemisphere of S^{n+1},

$$S^{n+1}_{N,+} = S^{n+1} \setminus H(-N) = \{Q \in S^{n+1} \mid N \cdot Q > 0\}.$$

Let $\alpha_{N} : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N, defined by

$$\alpha_{N}\,(P_1, \ldots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

We call the spherical convex body $\widetilde{W}_\gamma = \alpha^{-1}(Id(\mathcal{W}_\gamma))$ is *the spherical Wulff shape of* $\mathcal{W}_\gamma$. The Wulff shape

$$Id^{-1} \circ \alpha_N\big((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ\big).$$

is called *dual Wulff shape of* $\mathcal{W}_\gamma$, denoted by $\mathcal{DW}_\gamma$. We call a Wulff shape $\mathcal{W}$ is a *self-dual* if $\mathcal{W} = \mathcal{DW}$, namely, $\mathcal{W}$ and its dual Wulff shape $\mathcal{DW}$ are exactly the same convex body. By Theorem 1, Lemma 2.4 and Corollary 2.1, we have the following.

Corollary 3.1 ([5])**.** *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Suppose that the spherical Wulff shape $\widetilde{W}_\gamma = \alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ of $\mathcal{W}_\gamma$ is of constant width. Then*

(1) $\Delta(\widetilde{W}_\gamma) + diam\,(\widetilde{W}^\circ_\gamma) = \pi$,

(2) $\Delta(\widetilde{W}_\gamma) + \Delta\,(\widetilde{W}^\circ_\gamma) = \pi$,

(3) $diam(\widetilde{W}_\gamma) + \Delta\,(\widetilde{W}^\circ_\gamma) = \pi$,

(4) $diam(\widetilde{W}_\gamma) + diam\,(\widetilde{W}^\circ_\gamma) = \pi$,

where $\Delta(C)$ and $diam(C)$ are the width and the diameter of spherical convex body C in S^n, respectively.

A characterization of self-dual Wulff shape is given as follows.

Proposition 3.1 ([4])**.** *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant width $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is of constant width $\pi/2$.*

By Theorem 1, we have the following:

Corollary 3.2 ([5]). *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant diameter $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is of constant diameter $\pi/2$.*

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College of Science, Northwest Agriculture and Forestry University, China
Email address: han-huhe@nwafu.edu.cn