REEB GRAPHS OF SMOOTH FUNCTIONS ON MANIFOLDS

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Abstract. In this article we announce three theorems on Reeb spaces of smooth functions on closed manifolds with finitely many critical values.

1. Introduction

In this article, we announce three theorems on Reeb spaces of smooth functions on compact manifolds of dimension $n \geq 2$ without boundary that have finitely many critical values. The first theorem states that the Reeb spaces of such functions always have the structure of a finite (multi-)graph without loops. Many of the results in the literature concern Morse functions or smooth functions with finitely many critical points, and the theorems to be announced in this article generalize some of them.

The proofs of the theorems announced in this article will appear elsewhere.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds.

2. Reeb space

Let $f : X \to Y$ be a continuous map between topological spaces. For two points $x, x' \in X$, we write $x \sim x'$ if $f(x) = f(x')$ and they lie on the same connected component of $f^{-1}(f(x)) = f^{-1}(f(x'))$. Let $W_f = X/\sim$ be the quotient space with respect to this equivalence relation: i.e. W_f is a topological space endowed with the quotient topology. Let $q_f : X \to W_f$ denote the quotient map. Then, there exists a unique map $\bar{f} : W_f \to Y$ that is continuous and makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q_f"'] & & Y \\ & W_f. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The space W_f is called the *Reeb space* of f, and the map $\bar{f} : W_f \to Y$ is called the *Reeb map* of f. The decomposition of f as $\bar{f} \circ q_f$ as in the above commutative diagram is called the *Stein factorization* of f [16]. For a schematic example, see Figure 1.

In this article, a smooth real-valued function on a manifold is called a *Morse function* if its critical points are all non-degenerate. Such a Morse function is *simple* if its restriction to the set of critical points is injective.

2010 *Mathematics Subject Classification.* Primary 58K05; Secondary 58K30, 57R45, 57R70.
Key words and phrases. Reeb graph, finite critical values.

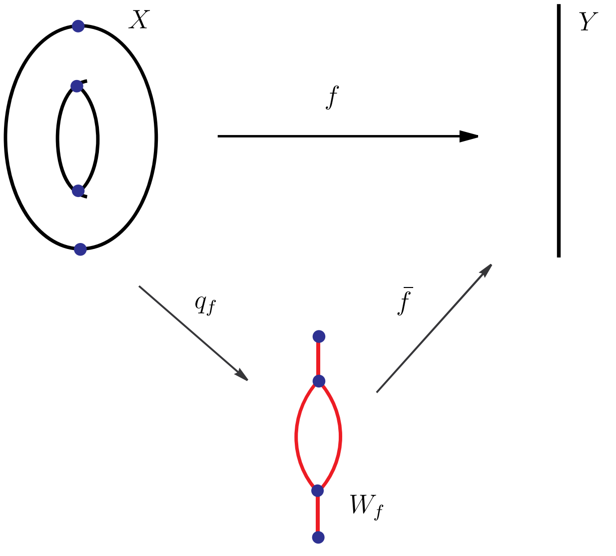

FIGURE 1. Example of a Reeb space

Furthermore, in the following, a *graph* means a compact 1–dimensional polyhedron: in other words, a graph is a finite "multi-graph" which may contain multi-edges or loops.

It is known that the Reeb space of a Morse function on a smooth closed manifold has the structure of a graph, which is often called a *Reeb graph* [24] or is sometimes called a *Kronrod–Reeb graph* [26]. This fact has been first stated in [24] without proof. A proof for simple Morse functions can be found in [9, Teorema 2.1] (see also [7]–[11]).

If we use several known results, this fact for Morse functions can also be proved as follows, for example. First, Morse functions on closed manifolds are triangulable (for example, by a result of Shiota [26]). Then, by [6], its Stein factorization is triangulable and consequently the quotient space, i.e. the Reeb space, is a 1–dimensional polyhedron. Therefore, it has the structure of a graph.

Furthermore, the following properties are known for Morse functions.

(1) The vertices of a Reeb graph correspond to the components of level sets that contain critical points.
(2) The restriction of the Reeb map on each edge is an embedding into $\mathbb{R}$. In particular, no edge is a loop.

Then, the following natural problems arise.

Problem 2.1. Given a smooth function on a closed manifold, does the Reeb space always have the structure of a graph?

Problem 2.2. Given a graph, is it realized as the Reeb space of a certain smooth function?

In this article, we consider these problems and give some answers.

3. Reeb graph theorem

Let M be a smooth manifold of dimension $n \geq 2$ which is closed (compact and $\partial M = \emptyset$), and $f : M \to \mathbb{R}$ a smooth function. Then, we have the following.

Theorem 3.1. *If f has at most finitely many critical values, then the Reeb space W_f has the structure of a graph. Furthermore, $\bar{f} : W_f \to \mathbb{R}$ is an embedding on each edge.*

Remark 3.2. The above theorem has been known for smooth functions with finitely many critical points. A proof can be found in [26].

Remark 3.3. (1) If f has infinitely many critical values, then the above theorem does not hold in general.

(2) A similar assertion holds also for smooth functions $f : M \to \mathbb{R}$, $n \geq 2$, on compact manifolds with $\partial M \neq \emptyset$, provided that both f and $f|_{\partial M}$ have at most finitely many critical values.

Remark 3.4. It is known that if $f : M \to N$ with $n = \dim M > \dim N = p \geq 1$ is a smooth map between manifolds with M being compact, and if f is triangulable, then its Reeb space W_f has the structure of a p–dimensional finite simplicial complex (or a compact polyhedron) in such a way that $\bar{f}$ is an embedding on each simplex [6]. In particular, if f is C^0–stable, then the Reeb space W_f is a p–dimensional polyhedron. Therefore, if a smooth function $f : M \to \mathbb{R}$ on a compact manifold is triangulable (for example, if f is a Morse function), then it follows that W_f is a graph.

As we will see later, there exist smooth functions with finitely many critical values that are not triangulable.

Example 3.5. Let M be an arbitrary smooth closed manifold of dimension $n \geq 2$. Then, by [27], there always exists a smooth function $f : M \to [0, \infty)$ such that $f^{-1}(0)$ is a Cantor set embedded in M. In particular, $f^{-1}(0)$ has uncountably many connected components. Thus, Theorem 3.1 does not hold for such an f.

In this example, we can show that f has infinitely many critical values.

Let us give another example.

Example 3.6. Given an arbitrary smooth closed manifold M of dimension $n \geq 2$, there always exists a smooth function $f : M \to \mathbb{R}$ with finitely many critical values that is not triangulable. Nevertheless, even in such a situation, W_f is a graph.

In fact, we can construct such a function so that for a critical value r, the closed set $f^{-1}(r)$ cannot be triangulated. See Figure 2.

4. Realization I

Let G be a graph without loops. The following theorem states that G is always realized as the Reeb space of a certain smooth function on a closed manifold with finitely many critical values. In fact, our theorem is stronger: regular fibers can also be pre-assigned.

For $n \geq 2$, consider maps

$$\Phi : \{\text{edges of } G\} \to \{\text{closed connected } (n-1)\text{–dimensional manifolds}\},$$

$$\Gamma : \{\text{vertices of } G\} \to \{\text{compact connected } n\text{–dimensional manifolds}\},$$

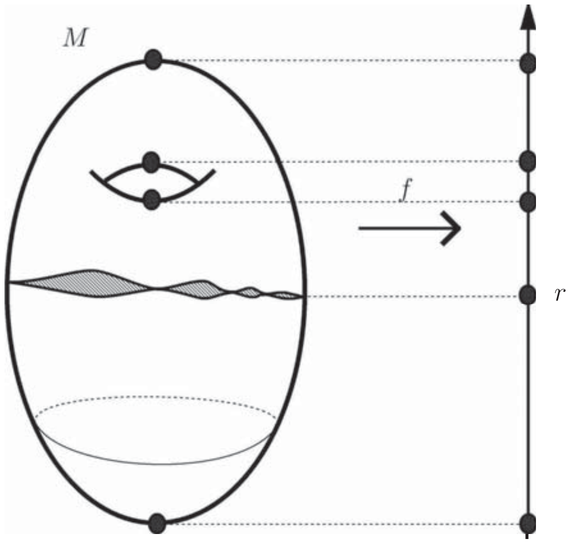

FIGURE 2. Non-triangulable function with finitely many critical values

such that for each vertex v of G, we have

$$\partial(\Gamma(v)) \cong \sqcup_{v \in e} \Phi(e),$$

i.e. the boundary of $\Gamma(v)$ is diffeomorphic to the disjoint union of the finitely many manifolds $\Phi(e)$, where e runs over all edges incident to v.

Then, we say that (G, Φ, Γ) is *realizable* if there exists a smooth function $f : M \to \mathbb{R}$ on a closed n–dimensional manifold M with finitely many critical values such that

(1) G is identified with W_f,
(2) for each edge e of $G = W_f$, we have $q_f^{-1}(x) \cong \Phi(e)$, $\forall x \in \operatorname{Int} e$,
(3) for each vertex v of $G = W_f$, we have $q_f^{-1}(N(v)) \cong \Gamma(v)$, where $N(v)$ is a regular neighborhood of v in G.

See Figure 3.

Then, we have the following realization theorem.

Theorem 4.1. *Every* (G, Φ, Γ) *is realizable.*

Remark 4.2. We have a similar theorem for functions on compact manifolds with boundary as well.

Corollary 4.3. *For all* $n \geq 2$, *every graph without loops is the Reeb space of a smooth function on a closed* n*–dimensional manifold with finitely many critical values.*

Remark 4.4. In the above corollary, we can even construct such a smooth function in such a way that

(1) every regular level set is a finite disjoint union of standard $(n-1)$–spheres, and

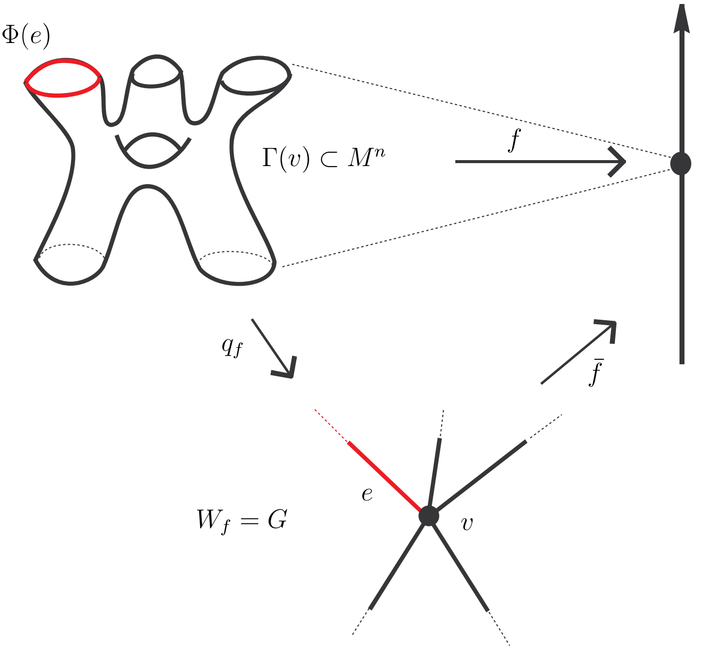

FIGURE 3. Realizing (G, Φ, Γ)

(2) the source manifold is diffeomorphic to S^n or a connected sum of a finite number of copies of $S^1 \times S^{n-1}$.

Remark 4.5. Some results similar to Theorem 4.1 and Remark 4.4 are presented in [13, 14].

5. Realization II

Let $f : M \to \mathbb{R}$ be a smooth function on a closed manifold of dimension $n \geq 2$ with finitely many critical values. Then, it is easy to show that the homomorphism $(q_f)_* : \pi_1(M) \to \pi_1(W_f)$ induced by the quotient map is surjective. See [24, Théorème 6] for a related statement. (In fact, this is true for an arbitrary continuous function f as long as W_f is semilocally simply-connected. See [1].)

We have the following theorem, which corresponds to the converse of this fact.

Theorem 5.1. *Let M be a smooth closed connected manifold of dimension $n \geq 2$, G a connected graph without loops, and $Q : M \to G$ a continuous map such that $Q_* : \pi_1(M) \to \pi_1(G)$ is surjective. Then, there exists a smooth function $f : M \to \mathbb{R}$ with finitely many critical values such that*

(1) *G can be identified with W_f,*
(2) *$q_f : M \to W_f = G$ is homotopic to Q.*

Remark 5.2. A similar result also holds for functions on compact manifolds with non-empty boundary.

Remark 5.3. A stronger result has been obtained by Michalak [22, 23] (see also Gelbukh [4], Marzantowicz and Michalak [17]). For $n \geq 3$, one can realize a given graph by a Morse function.

For a group H, set

$$\mathrm{corank}(H) = \max\{r \mid \text{There exists an epimorphism } H \to F_r\},$$

where F_r is the free group of rank r. This is called the *co-rank* of the group H (for example, see [2, 3]).

Corollary 5.4. *Let M be a smooth closed connected manifold of dimension $n \geq 2$, and G a connected graph without loops. Then, G arises as the Reeb space of a certain smooth function on M with finitely many critical values if and only if $\beta_1(G) \leq \mathrm{corank}(\pi_1(M))$, where β_1 denotes the first betti number.*

Remark 5.5. About realization of Reeb graphs, there have been a lot of studies, e.g. by Sharko [26], Martínez-Alfaro, Meza-Sarmiento and Oliveira [18, 19, 20], Masumoto and Saeki [21], Gelbukh [2, 3, 4, 5], Kaluba, Marzantowicz and Silva [12], Michalak [22, 23], Michalak and Marzantowicz [17], Kitazawa [13, 14, 15], etc. Our theorems generalize some of them.

Acknowledgment

The author would like to thank Dr. Naoki Kitazawa for helpful discussions. This work was supported by JSPS KAKENHI Grant Number JP17H06128. This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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