ON THE MILNOR FIBRATION FOR $f(\mathbf{z})\bar{g}(\mathbf{z})$ II

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1. Locally tame non-degenerate complete intersection pair

1.1. **Introduction.** Let $f(\mathbf{z})$ and $g(\mathbf{z})$ be holomorphic functions vanishing at the origin. For $h(\mathbf{z}) := f(\mathbf{z})g(\mathbf{z})$, there exists a tubular Milnor fibration $h : E(r,\delta)^* \to D_\delta^*$ or a spherical Milnor fibration $h/|h| : S_r \setminus K_r \to S^1$ for small r and $\delta \ll r$ ([15, 11]. Here $E(r,\delta)^* := \{\mathbf{z} \in B_r^{2n} \,|\, 0 \neq |f(\mathbf{z}) \leq \delta\}$ and $K_r := f^{-1}(0) \cap S_r^{2n-1}$. We consider the mixed function $H(\mathbf{z}, \bar{\mathbf{z}}) := f(\mathbf{z})\bar{g}(\mathbf{z})$ and the existence problem of its Milnor fibration. The link of H is the same as the complex link given by $h(\mathbf{z})$ but the fibration structure along the link of $g = 0$ is conversely oriented. It turns out that such a fibration does not exist for an arbitrary pair. This problem has been studied by several authors but there are not yet satisfactory results ([26, 27, 28, 24]). For a non-degenerate mixed function, it is known that the Milnor fibration exists ([19]). However for $n \geq 3$, H can not be non-degenerate as $f = g = 0$ is a non-isolated singular locus for H. In our previous paper [23], we have shown the existence of Milnor fibrations for H under the assumption that f, g are convenient non-degenerate functions, satisfying the multiplicity condition. A convenient non-degenerate function f has an isolated singularity at the origin. In this paper, we consider the same problem without assuming the convenience. That is, we consider the case that $f = 0$ or $g = 0$ may have non-isolated singularity at the origin.

1.2. **Vanishing coordinate subspaces and locally tameness.** Let $f(\mathbf{z})$ be a holomorphic function of n complex variables $z_1, \dots, z_n$ which vanishes at the origin. Consider a coordinate subspace $\mathbb{C}^I := \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, z_j = 0,\, j \notin I\}$ where $I \subset \{1, 2, \dots, n\}$. $\mathbb{C}^I$ is called *a vanishing coordinate subspace* of f if the restriction of f to $\mathbb{C}^I$ is identically zero. The restriction of f is denoted as f^I. We denote the set of vanishing subspaces of f (respectively of g) by $\mathcal{V}_f$ (resp. by $\mathcal{V}_g$). Let $P = (p_1, \dots, p_n)$ be a semi-positive weight vector. We put $I(P) := \{i \,|\, p_i = 0\}$. Take a vanishing coordinate subspace $\mathbb{C}^I$ and take an arbitrary semi-positive weight vector $P = (p_1, \dots, p_n)$ such that $I(P) = I$. Then the face function f_P is a weighted homogeneous function of the variables $(z_j)_{j \notin I}$ with a positive degree $d(P; f)$ with respect to the weight vector P.

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Recall that f is non-degenerate if for any strictly positive weight vector P (i.e., $I(P)=\emptyset$), $f_P:\mathbb{C}^{*n}\to\mathbb{C}$ has no critical points ([16]). We say that the function f (or the hypersurface $V(f):=f^{-1}(0)$) is *locally tame and non-degenerate* if it is non-degenerate and for any vanishing coordinate subspace $\mathbb{C}^I$, there exists a positive number r_I such that for any weight vector P with $I(P)=I$, f_P is a non-degenerate function of $(z_j)_{j\notin I}$ with the other variables $(z_i)_{i\in I}\in\mathbb{C}^{*I}$ being fixed in the ball $\sum_{i\in I}|z_i|^2\le r_I$ ([21, 8]). Put $V(f)^\sharp:=\cup_{\mathbb{C}^I\notin\mathcal{V}_f}V(f^I)\cap\mathbb{C}^{*I}$. Recall that $V(f)^\sharp$ is smooth near the origin (Lemma (2.2), [17]).

For the pair of function $\{f,g\}$, consider the following conditions.

1. (1) The hypersurfaces $V(f)=f^{-1}(0), V(g)=g^{-1}(0)$ are locally tame and non-degenerate.
2. (2) The variety $V(f,g)=\{f=g=0\}$ is a locally tame non-degenerate complete intersection variety. Namely (2-a) for any strictly positive weight vector P, the variety $\{\mathbf{z}\in\mathbb{C}^{*n}\,|\,f_P(\mathbf{z})=g_P(\mathbf{z})=0\}$ is a smooth complete intersection variety. (2-b) For any common vanishing coordinate subspace $\mathbb{C}^I$, there exists a positive number r_I such that for any weight vector P with $I(P)=I$, $\{f_P=g_P=0\}$ is a non-degenerate complete intersection variety in $\mathbb{C}^{*J}$ with $J=I^c$ and $\mathbf{z}_I\in\mathbb{C}^{*I}$ is fixed in the ball $\sum_{i\in I}|z_i|^2\le r_I$.

We say $\{f,g\}$ is *a locally tame non-degenerate pair* if it satisfies only (1) and (2). The pair $\{f,g\}$ is *a disjoint locally tame non-degenerate complete intersection pair* if it satisfies (1), (2-a) and (3) $\{f,g\}$ satisfies the disjointness of vanishing subspaces, i.e. $\mathcal{V}_f\cap\mathcal{V}_g=\emptyset$.

Note that if $\mathbb{C}^I\in\mathcal{V}_f\setminus\mathcal{V}_g$, there exists a positive number r_I such that for any semi-positive weight vector P with $I(P)=I$, $g_P=g^I$ and $f_P=g_P=0$ is a non-degenerate complete intersection variety.

2. Isolatedness of the critical value

2.0.1. *Multiplicity condition.* We slightly generalize the multiplicity condition which is introduced in [23]. We say that $H:=f\bar{g}$ satisfies *the multiplicity condition* if there exists a good resolution $\pi:X\to\mathbb{C}^n$ of the holomorphic function $h:=fg$ such that

- (i) $\pi:X\setminus\pi^{-1}(V(h))\to\mathbb{C}^n\setminus V(h)$ is biholomorphic and the divisor defined by $\pi^*(fg)=0$ has only normal crossing singularities and the respective strict transforms $\tilde{V}(f)$ of $V(f)$ and $\tilde{V}(g)$ of $V(g)$ are smooth.
- (ii) Put $\pi^{-1}(\mathbf{0})=\cup_{j=1}^s D_j$ where $D_1,\dots,D_s$ are smooth compact divisors in X. Denote the respective multiplicities of π^*f and π^*g along D_j by m_j and n_j. Then $m_j\ne n_j$ for $j=1,\dots,s$.

Assume that there exists a regular simplicial cone subdivision Σ^* of the dual Newton diagram $\Gamma^*(fg)$ and let $\hat{\pi}:X\to\mathbb{C}^n$ be the corresponding admissible toric modification. Let $\mathcal{V}^+$ be the set of strictly positive vertices of Σ^*. Then it gives a good resolution of the function fg and the compact exceptional divisors are bijectively correspond to $\{\hat{E}(P) \mid P \in \mathcal{V}^+\}$ ([17]). Recall that the multiplicity of $\hat{\pi}^* f$ and $\hat{\pi}^* g$ along the divisor $\hat{E}(P)$ are given by $d(P, f)$ and $d(P, g)$ respectively. We say that $\hat{\pi}$ satisfies *the toric multiplicity condition for H* if

$$d(P, f) \neq d(P, g), \quad \forall P \in \mathcal{V}^+.$$

For further detail about the toric modification $\hat{\pi} : X \to \mathbb{C}^n$, we refer to [17].

Lemma 1 (Isolatedness of the critical value, Lemma 3 [23])**.** *Assume that $\{f, g\}$ is a locally tame and non-degenerate complete intersection pair. Assume that there exists an admissible toric modification $\hat{\pi} : X \to \mathbb{C}^n$ which satisfies the toric multiplicity condition. Then there exist positive numbers r_1 such that 0 is the unique critical value of H on $B_{r_1}^{2n}$.*

The proof follows by the exact same argument as Lemma 3,[23].

2.0.2. *A sufficient condition for the toric multiplicity condition.* We consider the following truncated cone. Let $h(\mathbf{z}) = \sum_\nu a_\nu \mathbf{z}^\nu$ be a holomorphic function which is not necessarily convenient. Let $\Gamma_+(h)$ be the convex hull of the union $\bigcup_{\nu, a_\nu \neq 0}\{\nu + (\mathbb{R}^+)^n\}$ as usual. The Newton boundary $\Gamma(h)$ is defined by the union of compact faces of $\Gamma_+(h)$. To give a sufficient condition for the multiplicity condition, we further consider following.

Definition 2. *We define the set $\Gamma_{++}(h)$ and $\mathrm{Int}\Gamma_{++}(\mathrm{h})$ as*

$$\Gamma_{++}(h) = \{r\nu \mid r \geq 1,\, \nu \in \Gamma(h)\},\ \mathrm{Int}\Gamma_{++}(\mathrm{h}) = \{\mathrm{r}\nu \mid \mathrm{r} > 1,\, \nu \in \Gamma(\mathrm{h})\}$$

Note that $\Gamma_{++}(h) \subset \Gamma_+(h)$ and the equality holds if and only if h is convenient. The following gives a sufficient condition for the multiplicity condition.

Lemma 3. *Assume $\{f, g\}$ is a locally tame non-degenerate complete intersection pair. Suppose the following condition is satisfied.*
($\sharp$): $\Gamma(f) \subset \mathrm{Int}\,\Gamma_{++}(\mathrm{g})$ or $\Gamma(g) \subset \mathrm{Int}\,\Gamma_{++}(\mathrm{f})$.
Then the multiplicity condition is satisfied with respect to any admissible toric modification.

See Figure 1 which shows the situation $\mathrm{Int}\,\Gamma_{++}(\mathrm{f}) \supset \Gamma(\mathrm{g})$. The condition ($\sharp$) is a generalization of Newton multiplicity condition in [23] for non-convenient f and g. We call ($\sharp$) *the tame Newton multiplicity condition.*

Example 4. *1. Assume that $f(\mathbf{z})$ (respectively g) is a convenient function and assume that $\Gamma(f) \cap \Gamma(g) = \emptyset$ and $\Gamma(g)$ is above $\Gamma(f)$(resp. $\Gamma(f)$ is above $\Gamma(g)$). Then the tame Newton multiplicity condition is satisfied.*

2. Assume $\{f, g\}$ is a locally tame non-degenerate complete intersection pair and let $\hat{\pi} : X \to \mathbb{C}^n$ is an admissible toric modification. Let $\mathcal{V}^+$ be the strictly positive vertices of Σ^*. Consider the mapping $\varphi_m : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\varphi(\mathbf{z}) = (z_1^m, \ldots, z_n^m)$ and put $f_m(\mathbf{z}) := \varphi^* f(\mathbf{z})$ and $g_m(\mathbf{z}) := \varphi^* g(\mathbf{z})$.

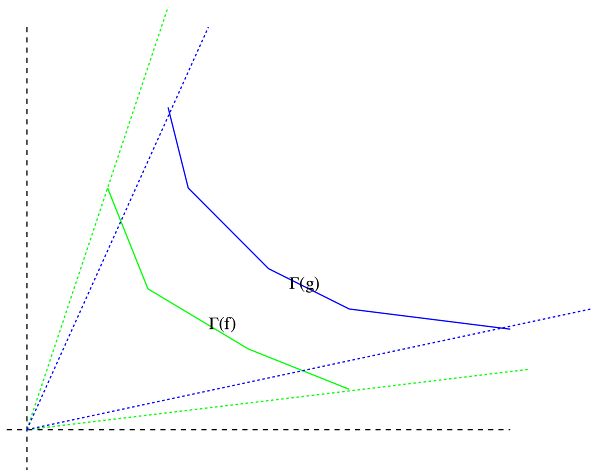

FIGURE 1. $\Gamma(g) \subset \mathrm{Int}\,\Gamma_{++}(\mathrm{f})$

Then there exists a sufficiently large m such that $\hat{\pi} : X \to \mathbb{C}^n$ satisfies the toric multiplicity condition for $f_m\bar{g}$ and $f\bar{g}_m$ respectively. This follows from the canonical equality $d(P, f_m) = m\,d(P, f)$ and $d(P, g_m) = m\,d(P, g)$ and the stability of the dual Newton diagrams $\Gamma^*(f) = \Gamma^*(f_m)$, $\Gamma^*(g) = \Gamma^*(g_m)$.

3. Let $f(\mathbf{z}) = c_1 z_1^a + \cdots + c_n z_n^a$. Let $g(\mathbf{z})$ be any locally tame non-degenerate function. Then $\{f, g\}$ is a locally tame non-degenerate pair for generic coefficients $c_1, \dots, c_n$ and satisfies the Newton multiplicity condition if $a = 1$. If $a > 1$, $\{f, g \prod_{i=1}^n z_i^a\}$ satisfies the Newton multiplicity condition, as $\Gamma_{++}(f) \supset \mathrm{Int}\Gamma_{++}(\mathrm{g})$.

Remark 5. *If f is locally tame and non-degenerate and if $\mathbb{C}^I$ is not a vanishing coordinate subspace for f, f^I is also locally tame and non-degenerate as a function on $\mathbb{C}^I$. See the argument in Proposition (1.5), Chapter III* [17]*. Locally tameness has been defined for mixed functions (Definition 2.7,* [8]*). If a holomorphic function $f(\mathbf{z})$ is locally tame, it is also locally tame as a mixed function.*

3. FIBRATION PROBLEM FOR FUNCTION $f\bar{g}$

We study the existence problem for the Milnor fibration of the mixed function $H(\mathbf{z}, \bar{\mathbf{z}}) := f(\mathbf{z})\bar{g}(\mathbf{z})$ in a more general situation. In this paper, we do not assume the convenience of f and g and therefore $V(f)$ or $V(g)$ may have non-isolated singularities at the origin. There are also interesting works from more general viewpoint in Parameswaran and Tibar [25, 24] and Araujo dos Santos, Ribeiro and Tibar [5] where authors consider the case of critical values being not isolated.

3.1. **Canonical stratification.** We assume that $\{f, g\}$ is a locally tame non-degenerate complete intersection pair. Consider the hypersurface $V(fg) = V(f) \cup V(g)$. Note that the mixed hypersurface $V(f\bar{g})$ is equal to $V(fg)$ as real algebraic varieties. We consider the following canonical stratification $\mathcal{S}$ of $\mathbb{C}^{*n}$ which also give a stratification of $V(fg)$. Put $V^{*I}(f) = V(f^I) \cap \mathbb{C}^{*I}$ if $\mathbb{C}^I$ is not a vanishing coordinate subspace.
Here $\mathbb{C}^{*I} = \{(z_i) \in \mathbb{C}^I \mid \forall z_i \neq 0,\ i \in I\}$. We first define a stratification $\mathcal{S}^I$ of $\mathbb{C}^{*I}$ as follows.

$$\left\{\mathbb{C}^{*I} \setminus (V^{*I}(f) \cup V^{*I}(g)), V^{*I}(f)', V^{*I}(g)',\ V^{*I}(f) \cap V^{*I}(g)\right\}, \text{if } f^I \neq 0,\ g^I \neq 0$$
$$\left\{\mathbb{C}^{*I} \setminus V^{*I}(f), V^{*I}(f)\right\}, \text{if } g^I \equiv 0, f^I \neq 0$$
$$\left\{\mathbb{C}^{*I} \setminus V^{*I}(g), V^{*I}(g)\right\}, \text{if } f^I \equiv 0, g^I \neq 0$$
$$\left\{\mathbb{C}^{*I}\right\}, \quad \text{if } f^I \equiv 0, g^I \equiv 0.$$

and we define $\mathcal{S} = \cup_I \mathcal{S}^I$. If $\{f, g\}$ is a disjoint locally tame non-degenerate pair, the last case does not exist. Here $V^{*I}(f)' = V^{*I}(f) \setminus V^{*I}(g)$ and $V^{*I}(g)' = V^{*I}(g) \setminus V^{*I}(f)$. $V^I(f)$ is empty only if f^I is a monomial.

We call $\mathcal{S}$ *the canonical toric strafitication* of $V(fg) = V(f\bar{g})$. Note that $\mathcal{S}$ is a complex analytic stratification.

3.2. **Transversality and Thom's a_f-regularity.** We use the notation $V(H, \mathbf{z}) := H^{-1}(H(\mathbf{z}))$ hereafter. Another key condition for the existence of the Milnor fibration is the transversality of the nearby fibers $H^{-1}(\eta)$, $\eta \neq 0$ and the sphere S_r^{2n-1}. Assume that 0 is the unique critical value of H in $B_{r_1}^{2n}$.

Transversality of nearby fibers: For any pair $r_2 \leq r_1$, there exists a positive number δ such that for any r, $r_2 \leq r \leq r_1$ and non-zero η with $|\eta| \leq \delta$, $H^{-1}(\eta)$ and S_r^{2n-1} intersect transversely. This condition follows if H satisfies the Thom's a_f-regularity (See for example, Proposition 11, [21]). Recall that H satisfies a_f-condition at the origin if there exists a stratification $\mathcal{S}$ of $H^{-1}(0) \cap B_{r_1}^{2n}$ for some $r_1 > 0$ such that for any sequence $\mathbf{q}_\nu$, $\nu = 1, 2, \ldots$ which converges $\mathbf{q}_0 \in M$, $M \in \mathcal{S}$ and $\mathbf{q}_0 \neq \mathbf{0}$, the limit of the tangent space $T_{\mathbf{q}_\nu} V(H, \mathbf{q}_\nu)$ (if it exists) includes the tangent space of M at $\mathbf{q}_0$.

Theorem 6. *Assume that either (i) $\{f, g\}$ is a locally tame non-degenerate complete intersection pair which satisfies also the tame Newton multiplicity condition ($\sharp$) or (ii) $\{f, g\}$ is a disjoint locally tame non-degenerate complete intersection pair. In the case (ii), we assume also that H has a unique critical value 0 in a ball $B_{r_1}^{2n}$. Then $H = f\bar{g}$ satisfies a_f-regularity.*

Note that in case (i), the tame Newton multiplicity condition guarantees the isolatedness of the critical value of H. For the proof, we consider the canonical toric stratification $\mathcal{S}$ on $V(fg)$. We choose $r_0, r_1 \ge r_0 > 0$ sufficiently small so that for any $r \le r_0$, the canonical toric strata are smooth in B_r^{2n} and any sphere S_ρ^{2n-1} with $0 < \rho \le r_0$ meets transversally with every strata of $\mathcal{S}$ of positive dimension. We use Curve selection lemma (see [15, 10]). Suppose we have a real analytic curve $\mathbf{z}(t)$, $0 \le t \le 1$ such that $\mathbf{z}(0) = \mathbf{a} \in V(H) \cap B_{r_0}^{2n}$, $\mathbf{a} \ne \mathbf{0}$ and $\mathbf{z}(t) \in \mathbb{C}^n \setminus V(H)$ for $t > 0$. Put $K = \{i \mid z_i(t) \not\equiv 0\}$ and write the expansion as

$$\begin{aligned} z_i(t) &= \alpha_i t^{p_i} + (\text{higher terms}),\ \alpha_i \ne 0,\ i \in K \\ &\equiv 0,\ i \notin K \end{aligned}$$

Let $M \in \mathcal{S}^I$ be the stratum which contains $\mathbf{a}$. We have to show that the limit of the tangent space of the fiber $V(H, \mathbf{z}(t))$ at $\mathbf{z}(t)$ for $t \to 0$ contains the tangent space of the stratum M at $\mathbf{a}$. The restriction of f, g and H on $\mathbb{C}^K$ satisfy also the locally tame non-degenerate assumption. As the argument for the proof is exactly the same, we assume for simplicity that $K = \{1, \dots, n\}$ hereafter. That is, we assume that $\mathbf{z}(t) \in \mathbb{C}^{*n}$ for $t \ne 0$ and $\mathbf{z}(0) = \mathbf{a}$. Put $P = (p_1, \dots, p_n)$ and

$$(1)\qquad \begin{cases} I := \{i \mid p_i = 0\}, \\ J = I^c = \{1, \dots, n\} \setminus I, \quad \mathbf{w} := (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^{*n}. \end{cases}$$

Note that $p_i = 0$ if and only if $i \in I$. Thus $\mathbf{a} = \mathbf{w}_I$ and $0 \ne \|\mathbf{a}\| \le r_0$. We will show that

$$\lim_{t \to 0} T_{\mathbf{z}(t)} V(H, \mathbf{z}(t)) \supset T_{\mathbf{a}} M.$$

We use the key property that the tangent space of the level hypersurface $V(H, \mathbf{z}(t))$ at $\mathbf{z}(t)$ contains the intersection of two tangent spaces of the level complex hypersurfaces $V(f, \mathbf{z}(t))$ and $V(g, \mathbf{z}(t))$ by Proposition 14, [23]. Here we are assuming that r_0 is sufficiently small so that 0 is the only critical value for f and g on $B_{r_0}^{2n}$. We divide the situation into three cases.

(a) $\mathbb{C}^I \notin \mathcal{V}_f \cup \mathcal{V}_g$, i.e. $f^I \not\equiv, g^I \not\equiv 0$.
(b) $\mathbb{C}^I \in \mathcal{V}_f$ and $\mathbb{C}^I \notin \mathcal{V}_g$, i.e. $f^I \equiv 0$, $g^I \not\equiv 0$.
$(b)'$ $\mathbb{C}^I \in \mathcal{V}_g$ and $\mathbb{C}^I \notin \mathcal{V}_f$, i.e. $f^I \not\equiv 0$, $g^I \equiv 0$.
(c) $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$, i.e. $f^I \equiv 0$, $g^I \equiv 0$.

As (b) and $(b)'$ is symmetric, it is enough to consider three cases (a), (b) and (c).

We first consider the case (a). The case (a) can be divided into two subcases:

(a-1) $\mathbf{a} \in M = V^{*I}(f) \cap V^{*I}(g)$.
(a-2) $\mathbf{a} \in V^{*I}(f)' = V^{*I}(f) \setminus V^{*I}(g)$, or $\mathbf{a} \in V^{*I}(g)' = V^{*I}(g) \setminus V^{*I}(f)$.

In the case (a-1), $\mathbf{a}$ is a non-singular point of $\mathbf{a} \in V(f, g)$. As the tangent space $T_{\mathbf{z}(t)} V(f, \mathbf{z}(t))$ converges to $T_{\mathbf{a}} V(f)$ which includes $T_{\mathbf{a}} V^{*I}(f)$ and $T_{\mathbf{z}(t)} V(g, \mathbf{z}(t)))$ converges to $T_{\mathbf{a}} V(g)$ which includes $T_{\mathbf{a}} V^{*I}(g)$ and $T_{\mathbf{a}} M =$

$T_{\mathbf{a}}V^{*I}(f) \cap T_{\mathbf{a}}V^{*I}(g)$ by the Newton non-degeneracy assumption, the assertion follows from Proposition 14, [23].

In the case (a-2), $\mathbf{a} \in V^{*I}(f)'$ or $\mathbf{a} \in V^{*I}(g)'$, $\mathbf{a}$ is a non-singular point of $V(H)$ and the assertion is obvious from the continuity of the tangent space.

Consider the case (b). Thus we assume that $\mathbb{C}^I \in \mathcal{V}_f \setminus \mathcal{V}_g$. By the local tameness assumption, the limit of the normalized holomorphic gradient vector $\lim_{t\to 0} \overline{\partial f}(\mathbf{z}(t))/\|\overline{\partial f}(\mathbf{z}(t))\|$ along $\mathbf{z}(t)$ is a vector in $\mathbb{C}^J$. Here $J = \{1,\dots,n\} \setminus I$. (Recall $\partial f(\mathbf{z}) = (\frac{\partial f(\mathbf{z})}{\partial z_1},\dots,\frac{\partial f(\mathbf{z})}{\partial z_n})$.) Thus the limit of the tangent space of $V(f,\mathbf{z}(t))$ contains $\mathbb{C}^I$ by the local tameness assumption. There are two subcases.

(b-1) $\mathbf{a} \in V^{*I}(g)$, or
(b-2) $\mathbf{a} \in \mathbb{C}^{*I} \setminus V^{*I}(g)$.

Note that $M = V^{*I}(g)$ in the case (b-1) and $M = \mathbb{C}^{*I} \setminus V^{*I}(g)$ in the case (b-2) respectively. In the case of (b-1), the limit of the normalized vector of $\overline{\partial f}(\mathbf{z}(t))$ is a vector in $\mathbb{C}^J$ by the local tameness assumption of f. Thus the limit of $T_{\mathbf{z}(t)}V(f,\mathbf{z}(t))$ includes $\mathbb{C}^I$. On the other hand, as $\overline{\partial g^I}(\mathbf{a})$ is non-zero, $T_{\mathbf{a}}V(g)$ is transverse to $\mathbb{C}^I$ at $\mathbf{a}$. Thus for any sufficiently small t, they are transverse and the limit of the intersection of two tangent space of the tangent space of $V(f,\mathbf{z}(t))$ and $V(g,\mathbf{z}(t))$ contains $T_{\mathbf{a}}V^{*I}(g)$.

Now we consider the case (b-2). We claim that the limit of the tangent space $T_{\mathbf{z}(t)}V(H,\mathbf{z}(t))$ includes $\mathbb{C}^I$, the tangent space of the stratum $M = \mathbb{C}^{*I} \setminus V^{*I}(g)$ at $\mathbf{a}$. First we prepare a sublemma.

Sublemma 7. *Let f be a holomorphic function and write $f(z) = k(\mathbf{z},\bar{\mathbf{z}}) + i\ell(\mathbf{z},\bar{\mathbf{z}})$ where $k = \Re f$, $\ell = \Im f$. Then we have $\bar\partial k = \frac{1}{2}\overline{\partial f}$ and $\bar\partial \ell = \frac{i}{2}\overline{\partial f}$. In particular, two gradient vectors $\bar\partial k$ and $\bar\partial \ell$ are linearly dependent over $\mathbb{C}$ but linearly independent over $\mathbb{R}$ at a non-critical point $\mathbf{z}$ of f.*

The assertion follows from the identities:

$$\overline{\partial k} = \bar\partial k,\ \overline{\partial \ell} = \bar\partial \ell,\ \bar\partial f = \bar\partial k + i\bar\partial\ell = 0,\ \partial f = \partial k + i\partial \ell.$$

Put $p_{min} = \min\{p_j \mid j \notin I\}$. First we can write

Lemma 8. *The orders of $\bar\partial\Re f(\mathbf{z}(t))$ and $\bar\partial\Im f(\mathbf{z}(t))$ are equal to the order of $\overline{\partial f}(\mathbf{z}(t))$. Put $s = \operatorname{order}\overline{\partial f}(\mathbf{z}(t))$. Then s and strictly less than $d(P;f) - p_{min}$. We can write further as follows.*

$$\overline{\partial f}(\mathbf{z}(t)) = \mathbf{v}t^s + \textit{(higher terms)},\ \exists \mathbf{v} \in \mathbb{C}^J$$
$$\bar\partial\Re f(\mathbf{z}(t)) = \frac{1}{2}\mathbf{v}t^s + \textit{(higher terms)}$$
$$\bar\partial\Im f(\mathbf{z}(t)) = \frac{i}{2}\mathbf{v}t^s + \textit{(higher terms)}$$

In particular, $\lim_{t\to 0}T_{\mathbf{z}(t)}V(f,\mathbf{z}(t))$ is the complex orthogonal of $\mathbf{v}$.

Now we are ready to analyze the case (b-2). Note that the limit of normalized gradient vector $\overline{\partial f}(\mathbf{z}(t))$ is $\mathbf{v}/\|\mathbf{v}\|$. For a vector $\mathbf{v}$, let $\mathbf{v}^{\perp_{\mathbb{C}}}$ be the subspace of $\mathbb{C}^n$ which are complex orthogonal to $\mathbf{v}$. Namely $\mathbf{v}^{\perp_{\mathbb{C}}} = \{\mathbf{w} \in \mathbb{C}^n \,|\, (\mathbf{w}, \mathbf{v}) = 0\}$. Now we claim

Assertion 9. *Assume* $\mathbf{a} \in \mathbb{C}^{*I} \setminus V^{*I}(g)$. *Then* $\lim_{t\to 0} T_{\mathbf{z}(t)} V(H, \mathbf{z}(t)) = \lim_{t\to 0} T_{\mathbf{z}(t)} V(f, \mathbf{z}(t))$.

Proof. Put $b := \bar{g}(\mathbf{a})$ and write $b = b_1 + ib_2$ with $b_1, b_2 \in \mathbb{R}$. First we use the equalities:

$$\Re(H) = \Re(f)\Re(\bar{g}) - \Im(f)\Im(\bar{g}), \quad \Im(H) = \Re(f)\Im(\bar{g}) + \Im(f)\Re(\bar{g}).$$

Then the gradient vectors are given as

$$\begin{aligned}
\bar{\partial}\Re(H)(\mathbf{z}(t)) &= (\bar{\partial}\Re(f)\Re(\bar{g})(\mathbf{z}(t)) + (\Re(f)\bar{\partial}\Re(\bar{g}))(\mathbf{z}(t)) \\
&\quad - (\bar{\partial}\Im(f)\Im(\bar{g}))(\mathbf{z}(t)) - (\Im(f)\bar{\partial}\Im(\bar{g}))(\mathbf{z}(t)) \\
&\equiv b_1\bar{\partial}\Re(f)(\mathbf{z}(t)) - b_2\bar{\partial}\Im f(\mathbf{z}(t)) \text{ modulo } (\mathrm{t}^{\mathrm{s}+1}) \\
&\equiv \frac{\mathbf{v}\bar{b}}{2} t^s \text{ modulo } (\mathrm{t}^{\mathrm{s}+1}) \\
\bar{\partial}\Im(H)(\mathbf{z}(t)) &= (\bar{\partial}\Re(f)\Im\bar{g})(\mathbf{z}(t)) + (\Re(f)\bar{\partial}\Im(\bar{g}))(\mathbf{z}(t)) \\
&\quad + \bar{\partial}\Im(f)\Re(\bar{g})(\mathbf{z}(t)) + \Im(f)\bar{\partial}\Re(\bar{g})(\mathbf{z}(t)) \\
&\equiv b_2(\bar{\partial}\Re(f)(\mathbf{z}(t)) + b_1\bar{\partial}\Im(f)(\mathbf{z}(t)) \text{ modulo } (\mathrm{t}^{\mathrm{s}+1}) \\
&\equiv \frac{(i\mathbf{v}\bar{b})}{2} t^s \text{ modulo } (\mathrm{t}^{\mathrm{s}+1})
\end{aligned}$$

and therefore the normalized vector of these gradient vectors $\bar{\partial}\Re(H)(\mathbf{z}(t))$ and $\bar{\partial}\Im(H)(\mathbf{z}(t))$ converges to the vectors

$$\frac{\mathbf{v}\bar{b}}{\|\mathbf{v}\bar{b}\|}, \qquad i\frac{\mathbf{v}\bar{b}}{\|\mathbf{v}\bar{b}\|}$$

respectively. This implies the limit of the tangent space $T_{\mathbf{z}(t)} V(H, \mathbf{z}(t))$ is the real orthogonal of the real 2-dimensional subspace span by these two vectors, that is nothing but the complex subspace $\mathbf{v}^{\perp_{\mathbb{C}}}$ which is equal to the limit of $T_{\mathbf{z}(t)} V(f, \mathbf{z}(t))$. The proof of the assertion for (b-2) is now completed. The case $\{f, g\}$ is a disjoint tame non-degenerate complete intersection pair is now proved.

Now we consider the last case (c) $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$. Recall that $\mathbf{w} = (\alpha_1, \dots, \alpha_n)$. We divide the situation into three subcases.

(c-1) $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$.
(c-2) $f_P(\mathbf{w}) = 0$ and $g_P(\mathbf{w}) \neq 0$ or (c-2)' $f_P(\mathbf{w}) \neq 0$ and $g_P(\mathbf{w}) = 0$.
(c-3) $f_P(\mathbf{w}) \neq 0$, $g_P(\mathbf{w}) \neq 0$.

We restate the assertion as the following lemma.

Lemma 10. *Assume that* $\mathbb{C}^I \in \mathcal{V}_f \cap \mathcal{V}_g$. *The the limit of the tangent space* $T_{\mathbf{z}(t)} V(H, \mathbf{z}(t))$ *includes* $\mathbb{C}^I$ *as a subspace.*

Proof. First assume that $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$. Put $\overline{\partial f}(\mathbf{z}(t)) = (u_1(t), \dots, u_n(t))$ and $\overline{\partial g}(\mathbf{z}(t)) = (v_1(t), \dots, v_n(t))$. We can write as

$$u_j(t) = \overline{\frac{\partial f_P}{\partial z_j}}(\mathbf{w})t^{d(P;f)-p_j} + \text{(higher terms)}$$
$$v_j(t) = \overline{\frac{\partial g_P}{\partial z_j}}(\mathbf{w})t^{d(P;g)-p_j} + \text{(higter terms)}.$$

Put o_f and o_g be the orders of $\overline{\partial f}(\mathbf{z}(t))$ and $\overline{\partial g}(\mathbf{z}(t))$ respectively. That is $o_f = \min\{\mathrm{ord}_t u_i(t) \,|\, i = 1, \dots, n\}$ and $o_g = \min\{\mathrm{ord}\, v_i(t) \,|\, i = 1, \dots, n\}$. Then the limit of $\overline{\partial f}(\mathbf{z}(t))$ and $\overline{\partial g}(\mathbf{z}(t))$ up to scalar multiplications are represented respectively by

$$\lim_{t\to 0} \frac{1}{t^{o_f}}\overline{\partial f}(\mathbf{z}(t)), \quad \lim_{t\to 0} \frac{1}{t^{o_g}}\overline{\partial g}(\mathbf{z}(t)). \tag{2}$$

We denote these limit vectors as $\lim^{(n)}_{t\to 0} \overline{\partial f}(\mathbf{z}(t))$ and $\lim^{(n)}_{t\to 0} \overline{\partial g}(\mathbf{z}(t))$. If these two limits are linearly independent over $\mathbb{C}$, the intersection

$$T_{\mathbf{z}(t)}V(f, \mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g, \mathbf{z}(t))$$

converges to the the complex orthogonal subspace to these two limit vectors. That is,

$$< \overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t)) >^{\perp_{\mathbb{C}}} \mapsto < \lim_{t\to 0}{}^{(n)}\overline{\partial f}(\mathbf{z}(t)), \lim_{t\to 0}{}^{(n)}\overline{\partial g}(\mathbf{z}(t)) >^{\perp_{\mathbb{C}}}.$$

The problem happens if these two limits are linearly dependent. We use a similar argument as the one which is used in the proof of Theorem 20, [21] or Theorem 3.14, [8] to solve this problem. For the simplicity of the argument, we assume that $J = \{1, \dots, m\}$ and $I = \{m+1, \dots, n\}$ and we assume that

$$p_1 \ge p_2 \ge \dots \ge p_m > 0,\ p_{m+1} = \dots = p_n = 0.$$

Note that $p_{min} = p_m$ under the above assumption and

$$\mathrm{ord}\, u_j(t) \ge d(P; f) - p_j, \quad \mathrm{ord}\, v_j(t) \ge d(P; g) - p_j,\ j = 1, \dots, m$$

while for $j \ge m+1$,

$$\mathrm{ord}\, u_j t) \ge d(P; f), \quad \mathrm{ord}\, v_j(t) \ge d(P; g),\ j = m+1, \dots, n.$$

Now we consider first (c-1): $f_P(\mathbf{w}) = g_P(\mathbf{w}) = 0$. By the locally tame non-degeneracy assumption, there exists $1 \le a, b \le m$, $a \ne b$ so that we have

$$\det \begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ \overline{\frac{\partial g_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \ne 0. \tag{3}$$

Here we assume that $a \ne b$ but we do not assume that $a < b$. In particular,

$$o_f \le \max\{d(P; f) - p_a, d(P; f) - p_b\} \le d(P; f) - p_m,$$
$$o_g \le \max\{d(P; g) - p_a, d(P; g) - p_b\} \le d(P; g) - p_m.$$

For simplicity, we may assume that $o_f \le o_g$ and consider

$$\ell_0 := \min\{j \,|\, \operatorname{ord} u_j(t) = o_f\}, \quad m_0 := \min\{j \,|\, \operatorname{ord} v_j(t) = o_g\}.$$

We call ℓ_0, m_0 *the leading indices of* $\overline{\partial f}(\mathbf{z}(t)$ *and* $\overline{\partial g}(\mathbf{z}(t))$.

Case 1. Assume that $\ell_0 \neq m_0$. Then the two limit gradient vectors given by (2) are already linearly independent. There are nothing to do further.

Case 2. Assume that $\ell_0 = m_0$. Then we take a monomial function $\rho(t) = ct^{o_g - o_f}$, $c \in \mathbb{C}$ and replace $\partial g(\mathbf{z}(t))$ by

$$\mathbf{v}^{(1)}(t) = \overline{\partial g}(\mathbf{z}(t)) - \rho(t)\overline{\partial f}(\mathbf{z}(t)).$$

We choose a constant c so that $\operatorname{ord} v_{m_0}^{(1)}(t) > o_g$. We put $\operatorname{ord} v_{m_0}^{(1)}(t) = \infty$ if $v_{m_0}^{(1)}(t) \equiv 0$. Here $v_{m_0}^{(1)}(t)$ is the m_0-th component of $\mathbf{v}^{(1)}(t)$. Note that the two dimensional complex subspace $W = \langle \overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t)) \rangle$ generated by $\{\overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t))\}$ is the same with subspace $\langle \overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(1)}(t) \rangle$ generated by $\{\overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(1)}(t)\}$. Thus their complex orthogonal subspaces are also equal. We continue this operation

$$\overline{\partial g} \to \mathbf{v}^{(1)} \to \cdots \to \mathbf{v}^{(k)}$$

until the leading index of $\mathbf{v}^{(k)}$ changes. Note that the k-times operation $\partial g(\mathbf{z}(t)) \to \mathbf{v}^{(k)}(t)$ is given as

$$\mathbf{v}^{(k)}(t) = \overline{\partial g}(\mathbf{z}(t)) - \rho_k(t)\overline{\partial f}(\mathbf{z}(t))$$

where $\rho_k(t)$ is a polynomial of variable t whose lowest degree is $o_g - o_f$. By (3), we may assume that $\partial g_P/\partial z_a(\mathbf{w}) \neq 0$. Note that $\operatorname{ord} v_{m_0}^{(\nu)}(t)$ is strictly increasing as long as $\nu \le k-1$ and $\operatorname{ord} \mathbf{v}^{(\nu)}(t) = \operatorname{ord} v_{m_0}^{(\nu)}(t)$. Let us look the components a, b which is given by

$$v_\tau^{(k)}(t) = \overline{\frac{\partial g}{\partial z_\tau}}(\mathbf{z}(t)) - \rho_k(t)\overline{\frac{\partial f}{\partial z_\tau}}(\mathbf{z}(t)),\ \tau = a, b.$$

Assertion 11. *One of the following inequalities holds.*

$$\operatorname{ord} v_a^{(k)}(t) \le d(P;g) - p_a, \quad \textit{or} \quad \operatorname{ord} v_b^{(k)}(t) \le d(P;g) - p_b.$$

Proof. Assume that $\operatorname{ord} v_a^{(k)} > d(P;g) - p_a$. We will show that $\operatorname{ord} v_b^{(k)}(t) \le d(P;g) - p_b$. As the first term of $\rho_k(t)\overline{\frac{\partial f}{\partial z_a}}(\mathbf{z}(t))$ kill the first term of $\overline{\frac{\partial g}{\partial z_a}}(t)$, the order of $\rho_k(t)\overline{\frac{\partial f}{\partial z_a}}(t)$ is equal to $d(P;g) - p_a$. There are two cases to be considered.

(A) $\frac{\partial f_P}{\partial z_a}(\mathbf{w}) \neq 0$ or (B) $\frac{\partial f_P}{\partial z_a}(\mathbf{w}) = 0$.

Assume the case (A). Then we have $\operatorname{ord} \rho_k(t) = d(P;g) - d(P;f) = o_g - o_f$ and which implies that

$$\operatorname{ord} \rho_k(t)\overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{z}(t)) \ge (d(P;f) - p_b) + (d(P;g) - d(P;f)) = d(P;g) - p_b.$$

Then putting λ be the coefficient of $t^{d(P;g)-d(P;f)}$ in $\rho_k(t)$, we have

$$\begin{aligned}
0 \neq \det &\begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ \overline{\frac{\partial g_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \\
= \det &\begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ \overline{\frac{\partial g_P}{\partial z_a}}(\mathbf{w}) + \lambda\overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda\overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix} \\
= \det &\begin{pmatrix} \overline{\frac{\partial f_P}{\partial z_a}}(\mathbf{w}) & \overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \\ 0 & \overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda\overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \end{pmatrix}
\end{aligned}$$

and thus $\overline{\frac{\partial g_P}{\partial z_b}}(\mathbf{w}) + \lambda\overline{\frac{\partial f_P}{\partial z_b}}(\mathbf{w}) \neq 0$ by (3), $\operatorname{ord} v_b^{(k)} = d(P;g) - p_b$.

Consider the case (B) now. Then $\operatorname{ord}\rho_k(t) < d(P;g) - d(P;f)$. By (3), $\frac{\partial f_P}{\partial z_b}(\mathbf{w}) \neq 0$ and $\operatorname{ord} v^{(k)}(t)_b < d(P;g) - p_b$. Thus in both cases, under the above operation, $\operatorname{ord} v^{(k)}(t) \leq d - p_m$. □

The above argument implies that the number k of operations is bounded by $k \leq d(P;g) - p_m - o_g$. At the last operation, the leading index of $\mathbf{v}^{(k)}(t)$ is different from m_0 and the limit vector of $\mathbf{v}^{(k)}(t)$ and $\overline{\partial f}(\mathbf{z}(t))$ are linearly independent and they are in the subspace $\mathbb{C}^J$. As $T_{\mathbf{z}(t)}V(f,\mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g,\mathbf{z}(t))$ is the complex orthonormal subspace of the two dimensional subspace $\langle\overline{\partial f}(\mathbf{z}(t)), \overline{\partial g}(\mathbf{z}(t))\rangle_{\mathbb{C}}$ and it is equal to the complex orthonormal subspace of $\langle\overline{\partial f}(\mathbf{z}(t)), \mathbf{v}^{(k)}(t)\rangle_{\mathbb{C}}$, the limit of $T_{\mathbf{z}(t)}V(f,\mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g,\mathbf{z}(t))$ includes the vanishing subspace $\mathbb{C}^I$. As $T_{\mathbf{z}(t)}V(H,\mathbf{z}(t))$ includes $T_{\mathbf{z}(t)}V(f,\mathbf{z}(t)) \cap T_{\mathbf{z}(t)}V(g,\mathbf{z}(t))$ as a subspace, $\lim_{t\to 0} T_{\mathbf{z}(t)}V(H,\mathbf{z}(t)) \supset \mathbb{C}^I$. Thus the proof of case (c-1) is done.

Now we consider the case (c-2): $f_P(\mathbf{w}) = 0, g_P(\mathbf{w}) \neq 0$. Consider the hypersurface $V := \{(\mathbf{z}_J \in \mathbb{C}^{*J} \mid f_P(\mathbf{z}) = 0, \mathbf{z}_I = \mathbf{a}\}$. By the locally tameness assumption, V is a non-singular hypersurface in $\mathbb{C}^{*I}$. Let us consider the restriction $g_P : V \to \mathbb{C}$. As g_P is a weighted homogeneous polynomial function of weight $P_J = (p_1, \dots, p_m)$ and V is invariant under the associated $\mathbb{C}^*$ action on $\mathbb{C}^J$, g_P has no non-zero critical value on V. Namely $\bar{\partial} g_P(\mathbf{w})$ is non-zero and linearly independent with $\bar{\partial} f_P(\mathbf{w})$. Thus there is a pair $a \leq b \leq m$ which satisfies (3). Thus the rest of the argument is the exact same as above and $\lim_{s\to 0} T_{\mathbf{z}(s)}V(H,\mathbf{z}(s)) \supset \mathbb{C}^I$. The case (c-2)': $f_P(\mathbf{w}) \neq 0$ and $g_P(\mathbf{w}) = 0$ is treated similarly.

Now we consider the case (c-3): $f_P(\mathbf{w}), g_P(\mathbf{w}) \neq 0$. In this case, we need the assumption that f, g satisfies the tame Newton multiplicity condition. Let $\deg_P f = d_f$ and $\deg_P g = d_g$. Put $d_r := d_f + d_g$ and $d_p = d_f - d_g$. Then the tame Newton multiplicity condition implies $d_p \neq 0$. The mixed function $H_P = f_P\bar{g}_P : \mathbb{C}^J \to \mathbb{C}$ is a strongly mixed weighted homogeneous polynomial which satisfies $H(\rho e^{i\theta} \circ \mathbf{z}) = \rho^{d_r} e^{d_p\theta} H(\mathbf{z})$ and thus 0 is the only critical value. Thus $\bar{\partial}\Re H_P(\mathbf{w}), \bar{\partial}\Im H_P(\mathbf{w})$ are linearly independent over $\mathbb{R}$ and we can proceed the same argument as above replacing $\bar{\partial} f, \bar{\partial} g$

to $\bar\partial\Re H(\mathbf{z}(t)), \bar\partial\Im H(\mathbf{z}(t))$ to conclude the real two dimensional subspace $\langle\bar\partial\Re H(\mathbf{z}(t)), \bar\partial\Im H(\mathbf{z}(t))\rangle_{\mathbb{R}}$ has a limit which is a real 2-dimensional subspace of $\mathbb{C}^J$. Thus $\lim_{t\to 0} T_{\mathbf{z}(t)}V(H,\mathbf{z}(t)) \supset \mathbb{C}^I$. See the proof of Theorem 20, [21] for further detail. □

Now the proof of Theorem 6 is completed.

□

By Proposition 11, [21], we get the transversality assertion:

Corollary 12. *Let $\{f,g\}$ be as in Theorem 6. Then there exists a positive number r_0 such that for any r_1, $0 < r_1 \le r_0$, there exists a positive number $\delta(r_1)$ so that for any $\eta \ne 0$ with $|\eta| \le \delta(r_1)$ and and any ρ, $r_1 \le \rho \le r_0$, the nearby fiber $H^{-1}(\eta)$ is non-singular in $B^{2n}_{r_0}$ and intersects transversely with the sphere S^{2n-1}_ρ.*

3.3. **Existence of a tubular Milnor fibration.** By Lemma 1, Theorem6 and Corollary 12, we apply Ehresmann's fibration theorem ([35]) to obtain:

Theorem 13. *Assume that $\{f,g\}$ satisfies the same assumption as in Theorem 6. Then there exists a positive number ε and a sufficiently small $\delta \ll \varepsilon$ such that*

$$H = f\bar g : E(\varepsilon;\delta)^* \to D^*_\delta$$

is a locally trivial fibration where $E(\varepsilon,\delta)^* := \{(\mathbf{z}) \,|\, 0 \ne |H(\mathbf{z})| \le \delta,\ \|\mathbf{z}\| \le \varepsilon\}$ and $D^*_\delta := \{\zeta \in \mathbb{C} \,|\, 0 \ne |\zeta| \le \delta\}$,

By Corollary 12, the fibration does not depend on the choice of ε and δ. For a disjoint locally tame non-degenerate pair $\{f,g\}$, applying the argument of 2 of Example 4, we have:

Corollary 14. *Asssume that $\{f,g\}$ is a disjoint locally tame non-degenerate complete intersection pair. Fix an admissible toric modification $\hat\pi : X \to \mathbb{C}^n$. Take $m > 0$ large so that $\{f_m, g\}$ and $\{f, g_m\}$ satisfy the toric multiplicity condition with $\hat\pi$. Then changing the coefficients of f_m or g_m slightly if necessary, $\{f_m, g\}$ and $\{f, g_m\}$ are locally tame non-degenerate complete intersections respectively for which $\hat\pi : X \to \mathbb{C}^n$ satisfies the toric multiplicity condition. Thus $H_m := f_m\bar g$ and $K := f\bar g_m$ have tubular Milnor fibrations.*

For the definition f_m, see Example 4.

4. Spherical Milnor fibration

In this section, we study the existence of the spherical Milnor fibration. For a fixed small $r > 0$, we consider the mapping $\varphi : S^{2n-1}_r \setminus K \to S^1$ where $K = V(H) \cap S^{2n-1}_r$ and $\varphi(\mathbf{z}) = H(\mathbf{z})/|H(\mathbf{z})|$.

Lemma 15 (Lemma 10,[23])**.** *We assume $\{f,g\}$ is a weak locally tame non-degenerate complete intersection pair satisfying the assumption in Theorem 6. Then there exists a positive number r_3 so that $\varphi : S^{2n-1}_r \setminus K \to S^1$ has no critical points for any r, $0 < r \le r_3$.*

In the proof of Lemma 17 below, we will simultaneously reprove Lemma 15. Using Lemma15 and the transversality property of the fibers $H^{-1}(\eta)$, $0 \neq |\eta| \le \delta$ and the sphere S_r^{2n-1} (Corollary 12), we obtain the following.

Theorem 16. *Assume $\{f, g\}$ is a weak locally tame non-degenerate complete intersection pair as in Theorem 6. For a sufficiently small r, $\varphi : S_r^{2n-1} \setminus K \to S^1$ is a locally trivial fibration.*

Proof. Consider the neighborhood of K defined by $N(K) := \{\mathbf{z} \in S_r^{2n-1} \setminus K \,|\, |H(\mathbf{z})| \le \delta\}$. Corollary 12 says that three vectors $\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ are linearly independent over $\mathbb{R}$ on $N(K)$. For the definition of $\mathbf{v}_1, \mathbf{v}_2$, see the next section. Construct a vector filed $\mathcal{V}$ on $S_r^{2n-1} \setminus K$ such that $\Re(\mathcal{V}(\mathbf{z}), \mathbf{v}_2(\mathbf{z})) = 1$ and furthermore if $\mathbf{z} \in N(K)$, it also satisfies $\Re(\mathcal{V}(\mathbf{z}), \mathbf{v}_1(\mathbf{z})) = 0$. Then along the integral curves of $\mathcal{V}$, the argument of $H(\mathbf{z})$ is monotonic increase and the absolute value of H is constant when it enters in the neighborhood $N(K)$. Thus the integral curves exists for any time interval. For the local triviality, we use the integration of $\mathcal{V}$. □

4.1. **Equivalence of tubular and spherical Milnor fibrations.** In this section, we consider the equivalence problem of two Milnor fibrations. Let us recall two vector fields on the complement of $V(H)$ which are defined as follows ([19]).

$$\begin{aligned}
\mathbf{v}_1 &= \overline{\partial \log H} + \bar{\partial} \log H = \frac{\overline{\partial f}}{\bar{f}} + \frac{\overline{\partial g}}{\bar{g}} \\
\mathbf{v}_2 &= i(\overline{\partial \log H} - \bar{\partial} \log H) = i\left(\frac{\overline{\partial f}}{\bar{f}} - \frac{\overline{\partial g}}{\bar{g}}\right).
\end{aligned}$$

$\mathbf{v}_1, \mathbf{v}_2$ are real orthogonal. Let $\mathbf{z}(t)$ be a real analytic curve in $\mathbb{C}^n \setminus V(H)$. Then we have

$$\begin{aligned}
&\frac{d}{dt} \log H(\mathbf{z}(t)) \\
&= \frac{1}{f(\mathbf{z}(t))} \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(\mathbf{z}(t)) \frac{dz_i(t)}{dt} + \frac{1}{\bar{g}(\mathbf{z}(t))} \sum_{i=1}^{n} \overline{\frac{\partial g}{\partial z_i}(\mathbf{z}(t))} \overline{\frac{dz_i(t)}{dt}} \\
&= \frac{1}{2}\left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t)) - i\mathbf{v}_2(\mathbf{z}(t))\right) + \frac{1}{2}\overline{\left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t)) + i\mathbf{v}_2(\mathbf{z}(t))\right)} \\
&= \Re\left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_1(\mathbf{z}(t))\right) + i\Re\left(\frac{d\mathbf{z}(t)}{dt}, \mathbf{v}_2(\mathbf{z}(t))\right).
\end{aligned}$$

Thus we $\mathbf{v}_1(\mathbf{z})$ and $\mathbf{v}_2(\mathbf{z})$ are gradient vectors of $\Re \log H(\mathbf{z}) = \log |H(\mathbf{z})|$ and $\Im \log H(\mathbf{z}) = i \arg H(\mathbf{z})$. They are defined on $\mathbb{C}^n \setminus V(H)$. A key lemma is the following.

Lemma 17. *Assume that $\{f, g\}$ is as in Theorem 6. There exists a positive number r_0 such that for any $\mathbf{z} \in B_{r_0}^{2n} \setminus V(H)$, either three vectors*

$\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ *are linearly independent over* $\mathbb{R}$ *or they are linearly dependent and the relation takes the following form:*

$$\mathbf{z} = \lambda \mathbf{v}_1(\mathbf{z}) + \mu \mathbf{v}_2(\mathbf{z}),\ \lambda, \mu \in \mathbb{R},$$

where λ *is positive.*

Proof. Assume that there exists a real analytic curve $\mathbf{z}(t)$ in $\mathbb{C}^n \setminus V(H)$ and real valued rational functions $\lambda(t), \mu(t)$ such that

$$\mathbf{z}(t) = \lambda(t)\mathbf{v}_1(\mathbf{z}(t)) + \mu(t)\mathbf{v}_2(\mathbf{z}(t)) \tag{4}$$

and $\mathbf{z}(0) = \mathbf{0}$. If $\mu(t) \equiv 0$, the assertion follows from Corollary 3.4, [15]. Thus we may assume that $\mu(t) \not\equiv 0$. Let $I = \{j \mid z_j(t) \not\equiv 0\}$. As f^I, g^I satisfies the same assumption, we assume for simplicity that $I = \{1, \dots, n\}$. Thus $\mathbf{z}(t) \in \mathbb{C}^{*n}$ for $t \neq 0$. Consider their Taylor or Laurent expansions

$$\begin{aligned} f(\mathbf{z}(t)) &= \gamma t^{m_f} + (\text{higher terms}),\ \gamma \in \mathbb{C}^* \\ g(\mathbf{z}(t)) &= \beta t^{m_g} + (\text{higher terms}),\ \beta \in \mathbb{C}^* \\ z_i(t) &= a_i t^{p_i} + (\text{higher terms}),\ a_i \in \mathbb{C}^* \\ \lambda(t) &= \lambda_0 t^{\nu_1} + (\text{higher terms}),\ \ \lambda_0 \in \mathbb{R} \\ \mu(t) &= \mu_0 t^{\nu_2} + (\text{higher terms}),\ \mu_0 \in \mathbb{R}^*. \end{aligned}$$

In the proof, we reprove Lemma 15. Thus $\lambda_0 = 0$ only if $\lambda(t) \equiv 0$ and in that case, we understand $\nu_1 = +\infty$. If this is the case, $\mathbf{z}(t)$ is a critical point of $\varphi : S_\tau^{2n-1} \setminus K_\tau$ where $\tau = \|\mathbf{z}(t)\|$ and K_τ is the link of $H^{-1}(0)$ in this sphere. Put $\ell := \min\{d(f; P) - m_f,\ d(P; g) - m_g\}$ and let us define

$$\varepsilon_f = \begin{cases} 1, \text{ if } d(P; f) - m_f = \ell \\ 0, \text{ if } d(P; f) - m_f > \ell \end{cases}$$

$$\varepsilon_g = \begin{cases} 1, \text{ if } d(P; g) - m_g = \ell \\ 0, \text{ if } d(P; g) - m_g > \ell \end{cases}$$

As $\operatorname{ord} f(\mathbf{z}(t)) \geq d(P; f)$, $\operatorname{ord} g(\mathbf{z}(t)) \geq d(P; g)$, we have that $\ell \leq 0$. Put $\mathbf{v}_k(\mathbf{z}(t)) = (v_k^1(t), \dots, v_k^n(t))$ for $k = 1, 2$. Observe that

$$v_1^j(t) = \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f + \frac{\overline{g_{P,i}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\ell - p_j} + \dots \tag{5}$$

$$v_2^j(t) = i \left(\frac{\overline{f_{P,j}(\mathbf{a})}}{\bar{\gamma}} \varepsilon_f - \frac{\overline{g_{P,j}(\mathbf{a})}}{\bar{\beta}} \varepsilon_g \right) t^{\ell - p_j} + \dots \tag{6}$$

Put

$$\begin{aligned} &P(\mathbf{a}) := (p_1 a_1, \dots, p_n a_n),\ p_{min} = \min\{p_j \mid j \in I\}, \\ &J = \{j \mid p_j = p_{min}\},\ \nu_0 = \min\{\nu_1, \nu_2\} \end{aligned}$$

and put $\delta_i = 1$ or 0 according to $\nu_i = \nu_0$ or $\nu_i > \nu_0$ respectively for $i = 1, 2$. By (4), we get

$$\begin{aligned} a_j t^{p_j} &+ \dots \\ &= \lambda_0 \left(\overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f + \overline{\frac{g_{P,i}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \right) t^{\nu_1+\ell-p_j} + \dots \\ &+ \mu_0 i \left(\overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f - \overline{\frac{g_{P,j}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \right) t^{\nu_2+\ell-p_j} + \dots \\ &= e_j t^{\nu_0+\ell-p_j} + \dots \end{aligned}$$

where

$$e_j = \left\{ \lambda_0 \delta_1 \left(\overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f + \overline{\frac{g_{P,i}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \right) + \mu_0 \delta_2 i \left(\overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f - \overline{\frac{g_{P,j}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \right) \right\}.$$

If $\nu_0 + \ell - 2p_{min} > 0$, we get a contradiction $a_j = 0, j \in J$. Thus $\nu_0 + \ell - 2p_{min} \le 0$. Consider the vectors

$$\begin{aligned} \mathbf{v}_1^{(0)} &= (w_1^1, \dots, w_1^n),\ w_1^j = \overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f + \overline{\frac{g_{P,j}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \\ \mathbf{v}_2^{(0)} &= (w_2^1, \dots, w_2^n),\ w_2^j = i \left(\overline{\frac{f_{P,j}(\mathbf{a})}{\bar{\gamma}}} \varepsilon_f - \overline{\frac{g_{P,j}(\mathbf{a})}{\bar{\beta}}} \varepsilon_g \right) \end{aligned}$$

Assume that $\nu_0 + \ell - 2p_{min} < 0$. By (7), we get

$$\lambda_0 \delta_1 w_1^j + \mu_0 \delta_2 w_2^j = 0,\ j = 1, \dots, n. \tag{7}$$

If $\ell < 0$, $\varepsilon_f f_P(\mathbf{a}) = 0$ and $\varepsilon_g g_P(\mathbf{a}) = 0$. The above equality gives a contradiction to the non-degeneracy condition either for $V(f)$ if $\varepsilon_f = 1, \varepsilon_g = 0$, or for $V(g)$ if $\varepsilon_f = 0, \varepsilon_g = 1$ or for the intersection variety $V(f, g)$ if $\varepsilon_f = \varepsilon_g = 1$.

Assume $\ell = 0$. Then $\nu_0 < 2p_{min}$, $\gamma = f_P(\mathbf{a})$ and $\beta = g_P(\mathbf{a})$. We consider the equality

$$\sum_{j \in J} \frac{dz_j(t)}{dt} z_j(t) = \sum_{j \in J} \frac{dz_j(t)}{dt} \left(\overline{\lambda(t)} \overline{v_1^j}(t) + \overline{\mu(t)} \overline{v_2^j}(t) \right). \tag{8}$$

The left hand side has order $2p_{min} - 1$ as

$$\sum_{j \in J} \frac{z_j(t)}{dt} z_j(t) = \sum_{j \in J}^{n} p_j |a_j|^2 t^{2p_{min}-1} + \dots.$$

Using (7) and Euler equality, we see that the leading term of the right hand is $t^{\nu_0 - 1}$ which has the coefficient

$$\lambda_0 \delta_1 (d(P; f) + d(P; g)) + i \mu_0 \delta_2 (d(P; f) - d(P; g)) \neq 0.$$

The coefficient is non-zero. (If $\delta_1 = 0$, we use the Newton multiplicity condition to see the imaginary part is non-zero.) Thus the order is strictly smaller than $2p_{min} - 1$, which is a contradiction. Thus the case $\nu_0 + \ell - 2p_{min} < 0$ does not occur. Thus the following equality holds:

$$\nu_0 + \ell - 2p_{min} = 0.$$

(7) implies the following equality.

$$\lambda_0\delta_1 w_1^j + \mu_0\delta_2 w_2^j \quad = \quad \begin{cases} a_j & j \in J \\ 0, & j \notin J. \end{cases}$$

We consider the equality (8) again.

The left side of (7) has order $2p_{min}-1$ with the coefficient $\sum_{j\in J} p_{min}|a_j|^2 > 0$. The right side has order $2p_{min-1}$ and the coefficient is given through Euler equality as

$$\lambda_0\delta_1\left(\frac{d(P;f)f_P(\mathbf{a})}{\gamma}\varepsilon_f + \frac{d(P;g)g_P(\mathbf{a})}{\beta}\varepsilon_g\right) \\ + \mu_0\delta_2 i\left(\frac{d(P;f)f_P(\mathbf{a})}{\gamma}\varepsilon_f - \frac{d(P;g)g_P(\mathbf{a})}{\beta}\varepsilon_g\right)$$

If $\ell < 0$, $f_P(\mathbf{a})\varepsilon_f = g_P(\mathbf{a})\varepsilon_g = 0$ and the above coefficient is zero. Thus we get a contradiction. Thus the only possible case is $\ell = 0$ and therefore

$$f_P(\mathbf{a}),\, g_P(\mathbf{a}) \neq 0,\, \nu_0 = 2p_{min}.$$

We observe also $\delta_1 \neq 0$, as otherwise the coefficient is purely imaginary. Thus we should have

$$\ell = 0,\, \nu_1 \leq \nu_2,\, \gamma = f_P(\mathbf{a}),\, \beta = g_P(\mathbf{a}).$$

The leading coefficients of (8) gives the equality:

$$\sum_{j\in J} p_{min}|a_j|^2 \quad = \quad \lambda_0\left(d(P;f) + d(P;g)\right) \; + \; i\delta_2\mu_0\left(d(P;f) - d(P;g)\right).$$

Thus taking the real part of this equality, we conclude that $\lambda_0 > 0$. This also proves $\lambda(t) \equiv 0$ does not occur as $\lambda_0 \neq 0$. This gives another proof of Corollary 12. $\square$

Now we are ready to prove the equivalence theorem.

Theorem 18. *Assume that $\{f, g\}$ is a locally tame non-degenerate complete intersection pair as in Theorem 6. Consider the tubular and spherical Milnor fibrations*

$$H : \partial E(r,\delta)^* \to S^1_\delta$$
$$\varphi : S_r^{2n-1} \setminus K \to S^1.$$

These two fibrations are equivalent. Here we use the notations

$$\partial E(r,\delta)^* := \{\mathbf{z} \in B_r^{2n} \,|\, |H(\mathbf{z})| = \delta\},\; K = S_r^{2n-1} \cap V(H).$$

Proof. Let δ be sufficiently small and put $\partial N(K) := \{\mathbf{z} \in S_r^{2n-1} \,|\, |H(\mathbf{z})| < \delta\}$. By the transversality, $N(K)$ is contractible to K and $N(K) \setminus K$ is diffeomorphic to $\partial N(K) \times (0,1)$ and $\varphi : S_r^{2n-1} \setminus N(K) \to S^1$ is equivalent to the spherical fibration $\varphi : S_r^{2n-1} \setminus K \to S^1$. Note that vectors $\mathbf{v}_1, \mathbf{v}_2$ are real orthogonal. Take a locally finite open covering $\mathcal{U} = \{U_\alpha,\, \alpha \in A\} \cup \{V_\beta,\, \beta \in B\}$ of $B_r^{2n} \cap \{\mathbf{z} \,|\, |H(\mathbf{z})| \geq \delta\}$ as follows. Each U_α, V_β are open disk with center p_α, p_β. Secondly in each U_α, $\{\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})\}$ are linearly independent over $\mathbb{R}$, while in V_β, p_β can be written as

$$p_\beta = \lambda \mathbf{v}_1(p_\beta) + \mu \mathbf{v}_2(p_\beta), \quad \lambda > 0$$

and we take the radius of V_β is small enough so that $\Re(\mathbf{z}, \mathbf{v}_1(\mathbf{z})) > 0$ for any $\mathbf{z} \in V_\beta$. (There might exist a point $\mathbf{z} \in V_\beta$ where $\mathbf{z}, \mathbf{v}_1(\mathbf{z}), \mathbf{v}_2(\mathbf{z})$ are linearly independent.) Construct a vector field $\mathbf{w}_\alpha$ on U_α so that

$$\Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{v}_2(\mathbf{z})) = 0, \Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{v}_1(\mathbf{z})) = 1,\ \Re(\mathbf{w}_\alpha(\mathbf{z}), \mathbf{z}) = 1,\ \forall \mathbf{z} \in U_\alpha.$$

Such a vector field is called a Milnor vector field in [5, 6]. Along any integration curve $\mathbf{z}(t)$ starting a point $p \in \partial E(r,\delta)^*$, $\arg H(\mathbf{z}(t))$ is constant and $|H(\mathbf{z}(t))|$, $\|\mathbf{z}(t)\|$ are strictly increasing. This curve arrives at $\mathbf{z}(s(p)) \in S_r^{2n-1} \setminus N(K)$ for a finite time $s(p) > 0$. Using this integration and the correspondence $p \mapsto \mathbf{z}(s(p))$, we can construct a diffeomorphism $\psi : \partial E(r,\delta)^* \to S_r^{2n-1} - N(K)$ and ψ gives the commutative diagram:

$$\begin{array}{ccc} \partial E(r,\delta)^* & \xrightarrow{\psi} & S_r^{2n-1} - N(K) \\ \Big\downarrow H & & \Big\downarrow \varphi \\ S_\delta^1 & \xrightarrow{\iota} & S^1 \end{array}$$

where $\iota(\eta) = \eta/\delta$. Thus ψ gives an isomorphism of the two fibrations. $\square$

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