Mixed type curves in Minkowski 3-space

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Abstract

In this paper, we study mixed type curves in Minkowski 3-space. Mixed type curves are regular curves, and there are both non-lightlike points and lightlike points in a mixed type curve. For non-lightlike curves and null curves in Minkowski 3-space, we can study them by a Frenet frame or a Cartan frame respectively. But for mixed type curves, the two frames will not work. And as far as we know, no one has yet given a frame to study them in Minkowski 3-space. So we give the lightcone frame in order to provide a tool for studying this type curves in mathematical and physical research. As an application of the lightcone frame, we define an evolute of a mixed type curve. And we also give some examples to show the evolutes.

1 Preliminaries

Let $\mathbb{R}^3$=$\{(x_1,\ x_2,\ x_3)|x_1,\ x_2,\ x_3 \in \mathbb{R}\}$ be a real vector space. The Minkowski 3-space $\mathbb{R}^3_1$ is $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle .,\ .\rangle = -dx_1^2 + dx_2^2 + dx_3^2.$$

A non-zero vector $\mathbf{v} \in \mathbb{R}^3_1$ is said to be spacelike, timelike or lightlike if $\langle \mathbf{v},\ \mathbf{v}\rangle > 0,\ \langle \mathbf{v},\ \mathbf{v}\rangle < 0$ or $\langle \mathbf{v},\ \mathbf{v}\rangle = 0$, respectively. We usually consider the zero vector as a spacelike vector.

A curve $\gamma = \gamma(t)$ in $\mathbb{R}^3_1$ is said to be spacelike, timelike or null if its tangent vector field $\gamma'(t)$ is spacelike, timelike or lightlike, respectively, for all t.

But a regular curve in $\mathbb{R}^3_1$ may not be of one of the above three types. If there are both non-lightlike points and lightlike points in a regular curve in $\mathbb{R}^3_1$, we call it the mixed type curve.

Let γ be a spacelike or timelike curve in $\mathbb{R}^3_1$ parametrized by arc-length, we suppose that

$$\langle \gamma'',\ \gamma''\rangle \neq 0.$$

Then there is a Frenet frame $\{\gamma; \mathbf{T} = \gamma',\ \mathbf{N} = \frac{\gamma''}{\|\langle \gamma'',\ \gamma''\rangle\|^{\frac{1}{2}}},\ \mathbf{B} = \mathbf{T} \wedge \mathbf{N}\}$ satisfying the following Frenet equations:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\delta_1\delta_2\kappa & 0 & \delta_1\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where

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$$\delta_1 = \langle \mathbf{T},\ \mathbf{T}\rangle,\ \delta_2 = \langle \mathbf{N},\ \mathbf{N}\rangle.$$

The vector fields $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are called the tangent, principal normal and binormal of γ, respectively. The functions κ and τ are called the curvature and torsion of γ, respectively (see [5]).

As we all know, an evolute of a regular space curve γ in $\mathbb{R}^3$ (see [2]) is defined by

$$Ev(\gamma)(t) = \gamma(t) + \tfrac{1}{k}\mathbf{N}(t) - \tfrac{\dot{\kappa}}{\kappa^2\tau}\mathbf{B}(t).$$

By using the method, we can define an evolute of a non-lightlike curve γ in $\mathbb{R}^3_1$ by

$$Ev(\gamma)(t) = \gamma(t) + \delta_1\delta_2\tfrac{1}{k}\mathbf{N}(t) + \delta_1\delta_2\tfrac{\dot{\kappa}}{\kappa^2\tau}\mathbf{B}(t).$$

But for a mixed type curve, the frame will not work. We want to define an evolute of a mixed type curve, so we need a new frame. In the following work, we consider mixed type curves with isolated lightlike points and we suppose $\dot{\gamma} \wedge \ddot{\gamma} \neq 0$.

2 Lightcone frame

In this section,we will introduce the lightcone frame in Minkowski 3-space.

We denote

$$L^+_{\theta(t)} = (1,\ \ \cos\theta(t),\ \ \sin\theta(t)),$$

$$L^-_{\theta(t)} = (1,\ \ -\cos\theta(t),\ \ -\sin\theta(t))$$

and

$$M_{\theta(t)} = L^+_{\theta(t)} \wedge L^-_{\theta(t)} = (0,\ \ \sin\theta(t),\ \ -\cos\theta(t)),$$

where $\theta(t)$ is a smooth function. We call $\{L^+_{\theta(t)},\ L^-_{\theta(t)},\ M_{\theta(t)}\}$ a lightcone frame in $\mathbb{R}^3_1$. And we give a figure to show that (see Figure 1).

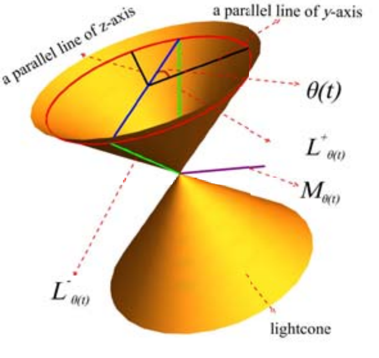

Figure 1: the lightcone frame

Let γ be a regular curve (or a mixed type curve) in $\mathbb{R}^3_1$. There exists a smooth function $(\alpha,\ \beta,\ \theta) : I \longrightarrow \mathbb{R}^3 \setminus \{(0,\ 0,\ \theta)\}$ such that

$$\dot{\gamma}(t) = \alpha(t)L^+_{\theta(t)} + \beta(t)L^-_{\theta(t)}$$

for all $t \in I$. We say that a regular curve γ with the lightcone semi-polar coordinates $(\alpha,\ \beta,\ \theta)$ if the above condition holds.

Since

$$\langle \dot{\gamma}(t),\ \dot{\gamma}(t) \rangle = -4\alpha(t)\beta(t),$$

$\gamma(t_0)$ is a

$$\begin{cases} \textit{spacelike point}: & \alpha(t_0)\beta(t_0) < 0, \\ \textit{timelike point}: & \alpha(t_0)\beta(t_0) > 0, \\ \textit{lightlike point}: & \alpha(t_0)\beta(t_0) = 0. \end{cases}$$

We show that in the following figure (see Figure 2).

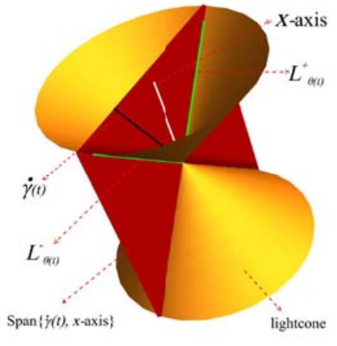

Figure 2: the lightcone semi-polar coordinates of $\dot{\gamma}(t)$

For convenience, let

$$\begin{aligned} \varepsilon_1(t) &= \langle \dot{\gamma}(t) \wedge \ddot{\gamma}(t),\ \dot{\gamma}(t) \wedge \ddot{\gamma}(t) \rangle \\ &= 4(\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t))^2 + 4\dot{\theta}^2(t)\alpha(t)\beta(t)(\beta(t) - \alpha(t))^2, \end{aligned}$$

$$\begin{aligned} \varepsilon_2(t) &= det(\dot{\gamma}(t),\ \ddot{\gamma}(t),\ \dddot{\gamma}(t)) \\ &= -2\dot{\theta}(t)(\beta(t) - \alpha(t))(\alpha(t)\ddot{\beta}(t) - \beta(t)\ddot{\alpha}(t)) \\ &\quad + 2(\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t))(2\dot{\theta}(t)(\dot{\beta}(t) - \dot{\alpha}(t)) + \ddot{\theta}(t)(\beta(t) - \alpha(t))) \\ &\quad + \dot{\theta}^3(t)(\beta(t) - \alpha(t))^2(\beta^2(t) - \alpha^2(t)). \end{aligned}$$

Theorem 2.1. *(The Existence Theorem) Let* $(\alpha,\ \beta,\ \theta) : I \longrightarrow \mathbb{R}^3 \setminus \{(0,\ 0,\ \theta)\}$ *be a smooth function. There exists a regular curve* $\gamma : I \to \mathbb{R}^3_1$ *with the lightcone semi-polar coordinates* $(\alpha,\ \beta,\ \theta)$.

Remark 2.2. *If* γ *and* $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3_1$ *are regular curves with the same lightcone semi-polar coordinates* $(\alpha,\ \beta,\ \theta)$*, then there exists a constant vector* $\mathbf{c} \in \mathbb{R}^3_1$ *such that* $\widetilde{\gamma}(t) = \gamma(t) + \mathbf{c}$.

Before we give the uniqueness theorem, we need to make some preparations.

Definition 2.3. *Let* γ *and* $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3_1$ *be regular curves. We say that* γ *and* $\widetilde{\gamma}$ *are congruent through a Lorentz motion if there exist a matrix* $\mathbf{A}$ *and a constant* $\mathbf{c} \in \mathbb{R}^3_1$ *such that* $\widetilde{\gamma}(t) = \mathbf{A}(\gamma(t)) + \mathbf{c}$ *for all* $t \in I$*, where* $\mathbf{A}$ *satisfies*

$$\mathbf{A}^{\mathbf{T}}\mathbf{G}\mathbf{A} = \mathbf{G},\ det(\mathbf{A}) = 1,\ \mathbf{G} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any vector $\mathbf{v} \in \mathbb{R}^3_1$ and $\mathbf{w} \in \mathbb{R}^3_1$, we can calculate

$$\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{A}(\mathbf{v}),\ \mathbf{A}(\mathbf{w}) \rangle,$$

$$\mathbf{v} \wedge \mathbf{w} = \mathbf{A}(\mathbf{v}) \wedge \mathbf{A}(\mathbf{w}).$$

So we have

$$\alpha(t)\beta(t) = \widetilde{\alpha}(t)\widetilde{\beta}(t), \quad \varepsilon_1(t) = \widetilde{\varepsilon_1}(t),\ \varepsilon_2 = \widetilde{\varepsilon_2}(t).$$

Proposition 2.4. *If* $\gamma : I \longrightarrow \mathbb{R}^3_1$ *is a non-lightlike curve, then*

$$\kappa(t) = \tfrac{(-\delta_1\delta_2\varepsilon_1(t))^{\frac{1}{2}}}{8(-\delta_1\alpha(t)\beta(t))^{\frac{3}{2}}},\ \tau(t) = \delta_1\tfrac{\varepsilon_2(t)}{\varepsilon_1(t)}.$$

So we have

$$\kappa(t) = \widetilde{\kappa}(t),\ \tau(t) = \widetilde{\tau}(t).$$

The fundamental theorem of non-lightlike curves has been given in [1, 4]. Using them, we get the uniqueness theorem.

Theorem 2.5. *(The Uniqueness Theorem) Let* γ *and* $\widetilde{\gamma} : I \longrightarrow \mathbb{R}^3_1$ *be regular curves with the lightcone semi-polar coordinates* $(\alpha,\ \beta,\ \theta)$ *and* $(\widetilde{\alpha},\ \widetilde{\beta},\ \widetilde{\theta})$*. Suppose the lightlike points are isolated. If*

$$\alpha(t)\beta(t) = \widetilde{\alpha}(t)\widetilde{\beta}(t),\ \varepsilon_1(t) = \widetilde{\varepsilon_1}(t),\ \varepsilon_2 = \widetilde{\varepsilon_2}(t)$$

for all $t \in I$*, then* γ *and* $\widetilde{\gamma}$ *are congruent through a Lorentz motion.*

3 Evolutes of mixed type curves

In this section we give the definition of mixed type curves. In the following work, we suppose $\varepsilon_2(t) \neq 0$. Firstly, we define an evolute of a mixed type curve with $\varepsilon_1(t) \neq 0$.

Definition 3.1. *Let* $\gamma : I \longrightarrow \mathbb{R}^3_1$ *be a regular curve (*$\varepsilon_1(t) \neq 0$*) with the lightcone semi-polar coordinates* $(\alpha,\ \beta,\ \theta)$*, then we define an evolute* $Ev(\gamma) : I \to \mathbb{R}^3_1$ *of* γ *by*

$$\begin{aligned} &Ev(\gamma)(t) = \gamma(t) \\ &+ 4(\frac{2\alpha\beta}{\varepsilon_1}(\alpha\dot{\beta} - \beta\dot{\alpha}) + \dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta - \alpha)(\frac{\alpha\beta\dot{\varepsilon_1}}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}))(t)(\alpha(t)L^+_{\theta(t)} - \beta(t)L^-_{\theta(t)}) \\ &+ 8(\frac{2\alpha\beta}{\varepsilon_1}\dot{\theta}\alpha\beta(\alpha - \beta) + |\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta} - \beta\dot{\alpha})(\frac{\alpha\beta\dot{\varepsilon_1}}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}))(t)M_{\theta(t)}. \end{aligned}$$

Proposition 3.2. *If $\gamma : I \longrightarrow \mathbb{R}^3_1$ is a non-lightlike curve ($\varepsilon_1(t) \neq 0$) with the lightcone semi-polar coordinates ($\alpha,\ \beta,\ \theta$), then*

$$\begin{aligned}&Ev(\gamma)(t) = \gamma(t)\\&+4(\frac{2\alpha\beta}{\varepsilon_1}(\alpha\dot{\beta}-\beta\dot{\alpha})+\dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta-\alpha)(\frac{\alpha\beta\dot{\varepsilon_1}}{\varepsilon_1\varepsilon_2}-3\frac{\alpha\dot{\beta}+\beta\dot{\alpha}}{\varepsilon_2}))(t)(\alpha(t)L^+_{\theta(t)}-\beta(t)L^-_{\theta(t)})\\&+8(\frac{2\alpha\beta}{\varepsilon_1}\dot{\theta}\alpha\beta(\alpha-\beta)+|\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta}-\beta\dot{\alpha})(\frac{\alpha\beta\dot{\varepsilon_1}}{\varepsilon_1\varepsilon_2}-3\frac{\alpha\dot{\beta}+\beta\dot{\alpha}}{\varepsilon_2}))(t)M_{\theta(t)}\\&=\gamma(t)+\delta_1\delta_2\frac{1}{k}\mathbf{N}(t)+\delta_1\delta_2\frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{B}(t).\end{aligned}$$

Remark 3.3. *If $\gamma(t_0)$ is a lightlike point of $\gamma(t)$ ($\varepsilon_1(t) \neq 0$), we have*

$$\alpha(t_0) = 0,\ \ \beta(t_0) \neq 0$$

or

$$\alpha(t_0) \neq 0,\ \ \beta(t_0) = 0.$$

So

$$Ev(\gamma)(t_0) = \gamma(t_0).$$

In appropriate conditions, we also define an evolute of a mixed type curve with $\varepsilon_1(t_0) = 0$.

Definition 3.4. *The evolute $Ev(\gamma) : I \to \mathbb{R}^3_1$ of γ is given by*

$$\begin{aligned}&Ev(\gamma)(t) = \gamma(t)\\&+2(2\lambda(\alpha\dot{\beta}-\beta\dot{\alpha})+\dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta-\alpha)(\lambda\frac{\dot{\varepsilon_1}}{\varepsilon_2}-6\frac{\alpha\dot{\beta}+\beta\dot{\alpha}}{\varepsilon_2}))(t)(\alpha(t)L^+_{\theta(t)}-\beta(t)L^-_{\theta(t)})\\&+4(2\lambda\dot{\theta}\alpha\beta(\alpha-\beta)+|\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta}-\beta\dot{\alpha})(\lambda\frac{\dot{\varepsilon_1}}{\varepsilon_2}-6\frac{\alpha\dot{\beta}+\beta\dot{\alpha}}{\varepsilon_2}))(t)M_{\theta(t)},\end{aligned}$$

if there exists a unique smooth function $\lambda : I \to \mathbb{R}$ such that

$$2\alpha(t)\beta(t) = \lambda(t)\varepsilon_1(t).$$

4 Examples

In this section we give some examples.

Example 4.1. Let $\gamma : \mathbb{R} \to \mathbb{R}^3_1$ be a regular curve defined by

$$\gamma(t) = (\tfrac{2}{3}t^3 + t,\ \ \sin t,\ \ -\cos t).$$

We can calculate

$$2\alpha(t)\beta(t) = 2t^2(1+t^2),\ \ \varepsilon_1(t) = 4t^2(t^2+5),$$

so

$$\lambda(t) = \frac{2\alpha(t)\beta(t)}{\varepsilon_1(t)} = \frac{1+t^2}{2(t^2+5)}.$$

The expression of $Ev(\gamma)(t)$ (the evolute of $\gamma(t)$) is too long and complicated, so we do not write it here and we show it in the following figures (see Figure 3 and Figure 4).

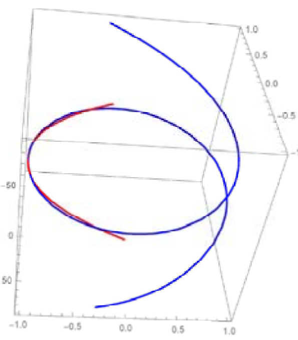

Figure 3: $\gamma(t)$(blue) and $Ev(\gamma)(t)$(red)

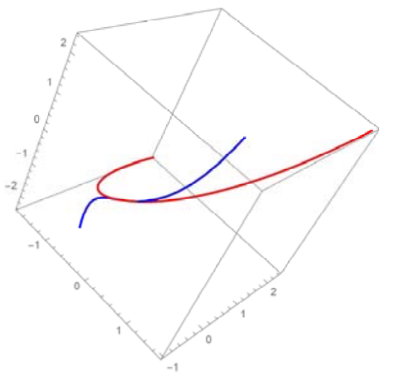

Figure 4: $\gamma(t)$(blue) and $Ev(\gamma)(t)$(red) around the lightlike point

Example 4.2. Let $\gamma : [0,\ 4\pi) \to \mathbb{R}^3_1$ be a regular curve defined by

$$\gamma(t) = (\tfrac{1}{2}\sin 2t,\ \ -2(-\cos\tfrac{1}{2}t - \tfrac{1}{3}\cos\tfrac{3}{2}t),\ \ -2(\sin\tfrac{1}{2}t - \tfrac{1}{3}\sin\tfrac{3}{2}t)).$$

We can calculate

$$2\alpha(t)\beta(t) = 2\cos^2 t(4\cos^2 t - 3),\ \varepsilon_1(t) = 4\cos^2 t(12\sin^4 t + 17\sin^2 t + 4),$$

so

$$\lambda(t) = \frac{2\alpha(t)\beta(t)}{\varepsilon_1(t)} = \frac{4\cos^2 t - 3}{2(12\sin^4 t + 17\sin^2 t + 4)}.$$

The expression of $Ev(\gamma)(t)$ (the evolute of $\gamma(t)$) is too long and complicated, so we do not write it here and we show it in the following figures (see Figure 5 and Figure 6).

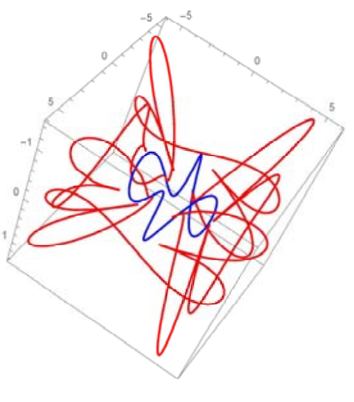

Figure 5: $\gamma(t)$(blue) and $Ev(\gamma)(t)$(red)

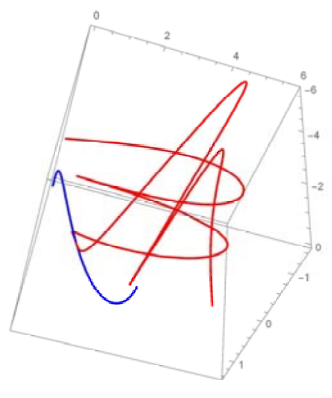

Figure 6: $\gamma(t)$(blue) and $Ev(\gamma)(t)$(red) around the lightlike point

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