LOCC estimator for pure state model in quantum parameter estimation with nuisance parameters

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1 Introduction

Recently, quantum information technology, such as quantum communication and quantum computation, has been researched rapidly. In the quantum information technology, we use quantum devices to obtain information from quantum particles. This process is referred to as a quantum measurement, or a measurement simply. However, we cannot obtain the complete information from the quantum particles because of the law of quantum mechanics. For example, the Heisenberg uncertainty relation states that we cannot obtain the complete information of the position and momentum simultaneously. Therefore, in quantum information science, designing optimal quantum measurements is a really fundamental problem.

Quantum parameter estimation, one of the well established fields in quantum information science, is a study about estimating the properties of a quantum particle as precise as possible. More formally speaking, in quantum parameter estimation, we consider a parametric quantum state family $\mathcal{M}$ and we try to estimate the parameter θ of a given state $\rho_\theta \in \mathcal{M}$. This subject is useful when you want to design optimal quantum devices for quantum computation and quantum communication. In quantum parameter estimation, our goal is to construct accurate and appropriate estimators for a given parametric state model. More general discussion about quantum statistical inference is found in [1, 2].

For the design of quantum devices, eliminating the effect of noises is a crucial problem. You have to obtain the information of quantum particles as much as possible, while eliminating the effect of noise. In quantum parameter estimation, many problems of noises are formulated as nuisance parameter problems. For a given model $\mathcal{M}$ and a given state $\rho_\theta \in \mathcal{M}$, the nuisance parameter problem aims at estimating some elements $(\theta_1, \dots, \theta_{d_I})$ $(d_I < d)$ of the entire parameters $\theta = (\theta_1, \dots, \theta_d)$. For eliminating the effect of noise, We usually regard the noises as the rest of parameters $(\theta_{d_I+1}\ldots,\theta_d)$ and just focus on the estimation of some elements $(\theta_1,\ldots,\theta_{d_I})$ of the parameters. More details about the nuisance parameter problem are found in [3, 4], for example.

When the quantum particles are spatially different, quantum operations over the entire particles are usually difficult to perform. One of the reasons is simply that we have to prepare bigger devices. For example, if one particle is in America and the other particle is in Japan, quantum devices over the two different countries are difficult to implement in practice. To overcome this difficulty, we consider LOCC(Local Operations and Classical Communication) operations, a restricted class of quantum operations. In any LOCC operation, operations over two or more particles are prohibited. Only local operations and classical communications to other particles are allowed in the LOCC operations. Since we do not need to use the bigger operations over the entire particles, the LOCC operation is much easier to realize physically.

We show the existence of an optimal estimator for the estimation with nuisance parameters even if the operations of the estimator is restricted to LOCC operations. Also, we give the explicit construction of the optimal LOCC estimator.

The contents of this paper is as follows. In Section 2, we introduce quantum state and quantum measurement, which is needed for later sections. In Section 3, we explain the basics of quantum parameter estimation. In Section 4, we show our main result in Proposition 1 and Proposition 2. Section 5 and Section 6 devotes for an application and an example of our result.

2 Basics of quantum information theory

Quantum theory has many curious properties such as non-locality. Most of these properties can be described by the definitions of quantum state and measurement. In this section, we briefly summarize the definitions of quantum state and measurement. To describe any of the definitions, a finite-dimensional Hilbert space is used and is called a quantum system.

First, we define the quantum state as follows.

Quantum state

Definition 1. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. A linear operator ρ on $\mathcal{H}$ is a quantum state if*

1. $\mathrm{Tr}\rho = 1$,
2. $\rho \geq 0$

holds.
A set of all quantum states on Hilbert space $\mathcal{H}$ is denoted by $\mathcal{S}(\mathcal{H})$.
A quantum state ρ is pure if there exists a unit vector $v \in \mathcal{H}$ such that

$$\rho = |v\rangle\langle v| \tag{1}$$

holds (The Bra-ket notation is used.). Otherwise, the state ρ is called a mixed state.

Next, we introduce the quantum measurement. Mathematically, quantum measurement is described as POVM (Positive Operator Valued Measure) which is defined as follows.

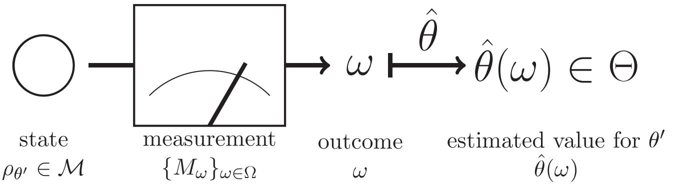

Figure 1: Estimation Process

POVM(Positive Operator Valued Measure)

Definition 2. *An indexed set $\{M_\omega\}_{\omega\in\Omega}$ of positive linear operators M_ω on Hilbert space $\mathcal{H}$ is a POVM if*

$$\sum_{\omega\in\Omega} M_\omega = I \tag{2}$$

holds where I denotes the identity operator on Hilbert space $\mathcal{H}$.
When a measurement $\{M_\omega\}_{\omega\in\Omega}$ is performed on a state ρ, the outcome is ω with probability

$$p^{\mathbf{M}}(\omega|\rho) := \mathrm{Tr}\rho M_\omega. \tag{3}$$

Next, we define Adaptive measurement which is a LOCC operation. Suppose a quantum system $\mathcal{H}$ is a composite system $\mathcal{H} = \bigotimes_{i\le n}\mathcal{H}_i$ of quantum systems $\mathcal{H}_1,\ldots,\mathcal{H}_n$. A measurement $\{M_\omega\}_{\omega\in\Omega}$ is said to be adaptive if there exist measurements $\{M_{\omega_i}^{(\omega_1,\cdots,\omega_{i-1})}\}_{\omega_i}$ on $\mathcal{H}_i$ such that

$$M_{\omega=(\omega_1,\ldots,\omega_n)} = M_{\omega_1}\otimes M_{\omega_2}^{(\omega_1)}\otimes\cdots\otimes M_{\omega_n}^{(\omega_1,\ldots,\omega_{n-1})} \tag{4}$$

holds.

For more properties of quantum information theory, see the textbooks[5, 6] for example.

3 Quantum parameter estimation

For a given parametric state model

$$\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta\in\Theta\}, \quad \Theta\subset\mathbb{R}^d \tag{5}$$

and a given state $\rho_{\theta'}\in\mathcal{M}$, quantum parameter estimation is a subject to estimate the true parameter θ' of the state $\rho_{\theta'}$.
A general procedure of the estimation process is as follows, as illustrated in Fig 1. First, we perform a measurement $\{M_\omega\}_{\omega\in\Omega}$ to the given state $\rho_{\theta'}$ and get an outcome ω with probability $\mathrm{Tr}M_\omega\rho_{\theta'}$. Based on this outcome, we estimate the true parameter using the map $\hat{\theta}:\Omega\to\Theta$. Since the measurement $\mathbf{M}=\{M_\omega\}_{\omega\in\Omega}$ and the map $\hat{\theta}$ determine the whole process of estimation, our goal is to select the measurement and the map appropriately so that we can estimate the true parameter as precise as possible. The pair $(\mathbf{M},\hat{\theta})$ is called an estimator.

In this section, we summarize the quantum parameter estimation for one-parameter case and for nuisance parameters.

3.1 Quantum estimation for one-parameter model

Here, we consider a parametric state model

$$\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta \in \Theta\} \tag{6}$$

whose open set Θ is a subset of one-dimensional real space $\mathbb{R}$.
In quantum one-parameter estimation, the locally unbiased condition is defined as follows.

locally unbiased estimator

Definition 3. *For a given parametric model*

$$\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta \in \Theta \subset \mathbb{R}\}, \tag{7}$$

an estimator $(\mathbf{M} = \{M_\omega\}_{\omega\in\Omega}, \hat{\theta})$ *is locally unbiased at* $\theta_0 \in \Theta$ *if*

$$\mathbf{E}_{\theta_0}(\mathbf{M}, \hat{\theta}) = \theta_0, \tag{8}$$

$$\frac{d}{d\theta}\Big|_{\theta=\theta_0} \mathbf{E}_\theta(\mathbf{M}, \hat{\theta}) = 1 \tag{9}$$

where

$$\mathbf{E}_\theta(\mathbf{M}, \hat{\theta}) := \sum_{\omega\in\Omega} \hat{\theta}(\omega)\mathrm{Tr}(M_\omega \rho_\theta). \tag{10}$$

An estimator is said to be an unbiased estimator if the estimator is locally unbiased at all points in Θ.

One of the most important problem in quantum parameter estimation is to find an estimator which is most precise among all locally unbiased estimators. The MSE (Minimum Squared Error) is usually used to characterize the accuracy of estimators, and is defined as follows.

Minimum Squared Error

Definition 4. *For a given parametric model* $\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta \in \Theta \subset \mathbb{R}\}$ *and an estimator* $(\mathbf{M} = \{M_\omega\}_{\omega\in\Omega}, \hat{\theta})$*, the MSE at* $\theta = \theta_0$ *is defined as*

$$\mathbf{V}_{\theta_0}(\mathbf{M}, \hat{\theta}) := \sum_{\omega\in\Omega} \left(\hat{\theta}(\omega) - \theta_0\right)^2 \mathrm{Tr}(M_\omega \rho_{\theta_0}). \tag{11}$$

If the MSE of an estimator at θ is smaller, the estimator is regarded as more precise at the point θ. By definition, the MSE is non-negative. Therefore, an estimator is the most precise estimator at θ if its MSE at θ is zero. However, according to the following proposition which is firstly proven in [7], there exists a lower bound of MSEs if we assume that estimators are locally unbiased.

SLD Cramer-Rao Bound

Theorem 1. *For a given parametric model $\mathcal{M} = \{\rho_\theta \mid \theta \in \Theta \subset \mathbb{R}^d\}$ and a locally unbiased estimator $(\mathbf{M}, \hat{\theta})$ at θ_0, the following inequality holds:*

$$V_{\theta_0}(\mathbf{M}, \hat{\theta}) \geq (J^S_{\theta_0})^{-1}. \tag{12}$$

Note that J^S_θ is called SLD fisher information which is defined as

$$(J^S_\theta) := \frac{1}{2}\mathrm{Tr}\rho_\theta(L^S_\theta)^2 \tag{13}$$

where the SLD operator L^S_θ is defined implicitly to satisfy

$$\frac{d\rho_\theta}{d\theta} = \frac{1}{2}(\rho_\theta L^S_\theta + L^S_\theta \rho_\theta). \tag{14}$$

This bound (12) can be attained by the estimator constructed as the spectral decomposition of $(J^S_\theta)^{-1}L^S_\theta$.

According to Theorem 1, the spectral decomposition of $(J^S_\theta)^{-1}L^S_\theta$ is an optimal estimator. However, this estimator is not usually a LOCC operation. Therefore, the realization of this estimator is difficult when the quantum system is multi-partite. As a current result, The recent paper [9] states the existence of the optimal LOCC estimator if the parametric model is composed of pure states.

3.2 Quantum parameter estimation with nuisance parameters

Next, we consider the quantum parameter estimation with nuisance parameters. For a parametric model with nuisance parameters

$$\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta = (\theta_I, \theta_N) \in \Theta, \theta_I \in \mathbb{R}^{d_I}\} \quad (\Theta \subset \mathbb{R}^d, \quad d_I < d), \tag{15}$$

consider the case when we want to estimate some elements $\theta_I \in \mathbb{R}^{d_I}$ of $\theta = (\theta_I, \theta_N)$, not the entire elements $(\theta_1, \ldots, \theta_d)$. This is called nuisance parameter problem. In nuisance parameter problem, we say the elements θ_I, which we want to estimate, as parameters of interest and say other parameters θ_N as nuisance parameters. In the nuisance parameter problem, the locally unbiasedness condition for estimators is defined as follows.

locally unbiased estimator (nuisance)

Definition 5. *An estimator* $(\mathbf{M} = \{M_\omega\}_{\omega\in\Omega}, \hat{\theta}_I : \Omega \to \mathbb{R}^{d_I})$ *is locally unbiased at* $\theta = (\theta_I, \theta_N)$ *for* θ_I *if*

$$\forall i \in \{1,\dots,d_I\}, \forall j \in \{1,\dots,d\}, \tag{16}$$

$$\mathbf{E}^i_\theta(\mathbf{M},\hat{\theta}) = \theta_i \text{ and } \frac{\partial}{\partial\theta_j}E^i_\theta(\mathbf{M},\hat{\theta}) = \delta_{ij} \tag{17}$$

holds. Note that $\mathbf{E}^i_\theta(\mathbf{M},\hat{\theta})$ *is defined as*

$$\mathbf{E}^i_\theta(\mathbf{M},\hat{\theta}) := \sum_{\omega\in\Omega}\hat{\theta}^i(\omega)\mathrm{Tr}M_\omega\rho_\theta \tag{18}$$

where $\hat{\theta}^i$ *denotes the i-th element of the map* $\hat{\theta} : \Omega \to \mathbb{R}^d$.

As mentioned in Theorem 1, the SLD Cramer-Rao bound characterize the precision bound of locally unbiased estimators in quantum one-parameter estimation. In quantum parameter estimation with nuisance parameters, there also exists a Cramer-Rao type bound. Here, we introduce the bound for the case when $d_I = 1$.

MSE Bound (nuisance parameter)

Theorem 2. *For a given parametric model*

$$\mathcal{M} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta = (\theta_I,\theta_N) \in \Theta, \theta_I \in \mathbb{R}\} \quad (\Theta \subset \mathbb{R}^d), \tag{19}$$

we define the MSE at θ *for* θ_I *as*

$$\mathbf{V}_{\theta;I}(\mathbf{M},\hat{\theta}_I) := \sum_{\omega\in\Omega}\left(\hat{\theta}_I(\omega) - \theta_I\right)^2 \mathrm{Tr}(M_\omega\rho_\theta). \tag{20}$$

In this case,

$$\mathbf{V}_{\theta;I}(\mathbf{M},\hat{\theta}_I) \geq [(\mathbf{J}^S_\theta)^{-1}]_{11} \tag{21}$$

holds for any locally unbiased estimator $(\mathbf{M},\theta_I)$ *at* θ *for* θ_I. *Note that* $\mathbf{J}^S_\theta$ *is the SLD matrix which is defined as*

$$[\mathbf{J}^S]_{ij} := \frac{1}{2}\mathrm{Tr}\rho_\theta(L^S_{\theta,i}L^S_{\theta,j} + L^S_{\theta,j}L^S_{\theta,i}) \quad (1 \leq i,j \leq d) \tag{22}$$

where $L^S_{\theta,i}$ $(1 \leq i \leq d)$ *are defined implicitly to satisfy*

$$\frac{\partial\rho_\theta}{\partial\theta^i} = \frac{1}{2}(\rho_\theta L^S_{\theta,i} + L^S_{\theta,i}\rho_\theta). \tag{23}$$

This bound (21) is attained by the estimator constructed as the spectral decomposition of $\sum_{i\leq d}[(\mathbf{J}^S_\theta)^{-1}]_{1j}L^S_{\theta;j}$.

For the proof of Theorem 2, see [4, Theorem 5.3].

In quantum parameter estimation with nuisance parameters, the most important problem is to construct the locally unbiased estimator satisfying the equality in (21). If you need more about nuisance parameter problem, see [3, 4] for example. Also, the constructions of asymptotically optimal estimators are found in [3, 8], which are based on the method called a two-step method.

4 Construction of LOCC estimator for pure state model

In this section, we consider the following model

$$\mathcal{M}_{n,p} := \{|u_\theta\rangle\langle u_\theta| \in \mathcal{S}(\mathcal{H}) \mid \theta = (\theta_I, \theta_N) \in \Theta, \theta_I \in \mathbb{R}\}, \quad \Theta \subset \mathbb{R}^d. \tag{24}$$

Assume that the Hilbert space $\mathcal{H} = \bigotimes_{i \le N} \mathcal{H}_i$ of the model $\mathcal{M}_{n,p}$ consists of the sub-spaces $\mathcal{H}_i$. In this case, the optimal estimator mentioned in Theorem 2 should be difficult to realize in many cases. To overcome this difficulty, we consider adaptive measurements to estimate the parameter of interest θ_I. The following proposition states the existence of an optimal estimator even if we assume that the measurement is adaptive.

existence of LOCC measurement

Proposition 1. *There exists a locally unbiased estimator* $(\mathbf{M}, \hat{\theta}_I)$ *at* θ *for* $\theta_I \in \mathbb{R}$ *satisfying*

$$\mathbf{V}_{\theta_I}(\mathbf{M}, \hat{\theta}_I) = [(\mathbf{J}_\theta^S)^{-1}]_{11} \tag{25}$$

and the measurement $\mathbf{M}$ *is adaptive.*

Proposition 1 is proven by constructing an appropriate estimator. To construct the desired estimator, we use the following proposition. This proposition is inspired by the method in [9, 10].

construction of estimator for pure state model (nuisance)

Proposition 2. *Consider the model $\mathcal{M}_{n,p}$ is defined as (24). Suppose an orthonormal basis $\{v_k\}_{k\leq \dim\mathcal{H}}$ of $\mathcal{H}$ satisfies*

$$\forall k,\ \langle u_\theta|v_k\rangle = 0 \Rightarrow (\ \langle \tilde{u}_\theta|v_k\rangle = 0) \tag{26}$$

and

$$\forall k,\ \langle u_\theta|v_k\rangle \neq 0 \Rightarrow (\frac{\langle \tilde{u}_\theta|v_k\rangle}{\langle u_\theta|v_k\rangle} \in \mathbb{R}), \tag{27}$$

where $|\tilde{u}_\theta\rangle$ is defined as

$$|\tilde{u}_\theta\rangle := |u_\theta^1\rangle + \sum_{1\leq k\leq d-1} [-\mathbf{J}^S_{\theta;IN}(\mathbf{J}^S_{\theta;NN})^{-1}]_k |u_\theta^{k+1}\rangle, \tag{28}$$

$$|u_\theta^k\rangle := 2\,(I - |u_\theta\rangle\langle u_\theta|)\,|\frac{\partial u_\theta}{\partial \theta^k}\rangle. \tag{29}$$

Note that $\mathbf{J}^S_{\theta;IN} \in \mathbb{R}^{d-1}, \mathbf{J}^S_{\theta;NN} \in \mathbb{R}^{(d-1)\times(d-1)}$ are the block sub-matrices of $\mathbf{J}^S_\theta$ which is defined as

$$\mathbf{J}^S_\theta = \begin{pmatrix} \mathbf{J}^S_{\theta;II} & \mathbf{J}^S_{\theta;IN} \\ \mathbf{J}^S_{\theta;NI} & \mathbf{J}^S_{\theta;NN} \end{pmatrix}. \tag{30}$$

Then, defining the estimator $\hat{\theta}_I : \Omega \to \mathbb{R}$ as

$$\hat{\theta}_I(k) := \begin{cases} [(\mathbf{J}^S_\theta)^{-1}]_{11} \frac{\langle \tilde{u}_\theta|v_k\rangle}{\langle u_\theta|v_k\rangle} + \theta_I & \text{if } \langle u_\theta|v_k\rangle \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

and the estimator

$$\Big(\{|v_k\rangle\langle v_k|\}_k, \hat{\theta}_I\Big) \tag{32}$$

is a locally unbiased estimator at θ for θ_I and attains the Cramer-Rao type bound.

You can directly prove that the estimator given in Proposition 2 satisfies the locally unbiasedness condition and attains the Cramer-Rao bound.
Therefore, the measurement of the estimator can be chosen as an adaptive measurement if there exists an orthonormal basis $\{v_{\mathbf{j}} = v_{j_1} \otimes v_{j_2}^{(j_1)} \otimes \cdots \otimes v_{j_n}^{(j_1,\ldots,j_{n-1})}\}_{\mathbf{j}=(j_1,\ldots,j_n)}$ of $\mathcal{H} = \bigotimes_{i\leq N} \mathcal{H}_i$ satisfying

$$\forall \mathbf{j},\ \langle u_\theta|v_{\mathbf{j}}\rangle = 0 \Rightarrow (\ \langle \tilde{u}_\theta|v_{\mathbf{j}}\rangle = 0) \tag{33}$$

and

$$\forall \mathbf{j},\ \langle u_\theta|v_{\mathbf{j}}\rangle \neq 0 \Rightarrow (\frac{\langle \tilde{u}_\theta|v_{\mathbf{j}}\rangle}{\langle u_\theta|v_{\mathbf{j}}\rangle} \in \mathbb{R}) \tag{34}$$

where each $\{v_{j_i}^{(j_1,\ldots,j_{i-1})}\}_{j_i}$ forms an orthonormal basis of $\mathcal{H}_i$. Fortunately, the existence of such orthonormal basis $\{v_{\mathbf{j}}\}$ is ensured by the following proposition.

construction of adaptive basis

Proposition 3. *Any traceless operator A on $\mathcal{H} = \bigotimes_{i\le n} \mathcal{H}_i$ has an orthonormal basis $\{v_{\mathbf{j}} = v_{j_1} \otimes v_{j_2}^{(j_1)} \otimes \cdots \otimes v_{j_n}^{(j_1,\ldots,j_{n-1})}\}_{\mathbf{j}=(j_1,\ldots,j_n)}$ on $\mathcal{H}$ satisfying*

$$\forall \mathbf{j}, \ \langle v_{\mathbf{j}}|A|v_{\mathbf{j}}\rangle = 0 \tag{35}$$

where each $\{v_{j_i}^{(j_1,\ldots,j_{i-1})}\}_{j_i}$ forms an orthonormal basis of $\mathcal{H}_i$.

For the proof of Proposition 3, see the paper [11]. Using Proposition 3 to the traceless matrix $|\phi\rangle\langle\phi + \frac{\pi}{2}|$ where $|\phi\rangle$ is defined as

$$|\phi\rangle = \sin\phi|u_\theta\rangle + \cos\phi|\tilde{u}_\theta\rangle, \ \phi \in \mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}, \tag{36}$$

we obtain the CONS satisfying the conditions given in Proposition 2. Using this adaptive measurement $\{|v_{\mathbf{j}}\rangle\langle v_{\mathbf{j}}|\}$ and Proposition 2, we obtain the optimal estimator whose measurement is adaptive.

5 Estimation of $f(\theta)$

One of the most important application of the nuisance parameter problem is the estimation of a smooth, real valued function $f : \mathcal{M} \to \mathbb{R}$. For example, the Von Neumann entropy is the real valued function. Since the model is parameterized, each state in $\mathcal{M}$ has a corresponding parameter $\theta \in \Theta$. Therefore, the function f can be understood as a map from Θ to $\mathbb{R}$. In this section, we investigate the estimation of the smooth function $f : \Theta \to \mathbb{R}$.

For an estimator $(\mathbf{M} = \{M_\omega\}_{\omega\in\Omega}, \hat{f} : \Omega \to \mathbb{R})$ of the function f, we impose the locally unbiasedness condition at θ_0 which is defined as follows.

Definition 6. *An estimator $(\mathbf{M} = \{M_\omega\}_{\omega\in\Omega}, \hat{f} : \Omega \to \mathbb{R})$ of the function $f : \Theta \to \mathbb{R}$ is locally unbiased at θ_0 if*

$$\mathbf{E}_{\theta_0}(\mathbf{M}, \hat{f}) = f(\theta_0) \tag{37}$$

and

$$\forall i \le d, \ \frac{\partial}{\partial\theta_i}\mathbf{E}_\theta(\mathbf{M}, \hat{f}) \mid_{\theta=\theta_0} = \frac{\partial f}{\partial\theta_i}(\theta_0) \tag{38}$$

holds.

For the estimators $(\mathbf{M}, \hat{f})$ of the function f, The variance $\mathbf{V}_\theta(\mathbf{M}, \hat{f})$ at θ is defined as

$$\mathbf{V}_\theta(\mathbf{M}, \hat{f}) := \sum_{\omega\in\Omega} \Big(f(\theta) - \hat{f}(\omega)\Big)^2 \operatorname{Tr}(\rho_\theta M_\omega). \tag{39}$$

Our purpose is to find an estimator $(\mathbf{M}, \hat{f})$ satisfying the following two conditions:

- the locally unbiasedness condition at θ.

- attaining the minimum variance among all locally unbiased estimators.

The following proposition states Cramer-Rao type inequality for estimating a real-valued function f.

bound for estimator of function

Proposition 4. *For any locally unbiased estimator* $(\mathbf{M}, \hat{f})$ *at* θ_0 *of a real-valued function* f*, its MSE* $\mathbf{V}_{\theta_0}(\mathbf{M}, \hat{f})$ *satisfies*

$$\mathbf{V}_{\theta_0}(\mathbf{M}, \hat{f}) \geq (\frac{\partial f}{\partial \theta_1}(\theta_0), \ldots, \frac{\partial f}{\partial \theta_d}(\theta_0))(\mathbf{J}^S_{\theta_0})^{-1} \begin{pmatrix} \frac{\partial f}{\partial \theta_1}(\theta_0) \\ \vdots \\ \frac{\partial f}{\partial \theta_d}(\theta_0) \end{pmatrix}. \tag{40}$$

A construction of a locally unbiased estimator attaining the bound (40) is explained in Section 4.

Proof. Consider the following transformation of the parameter $\theta = (\theta_1, \ldots, \theta_d) \mapsto \xi = (\xi_1, \ldots, \xi_d)$:

$$\xi_1 = f(\theta_0) + \sum_{j \leq d} \frac{\partial f}{\partial \theta_j}(\theta_0)(\theta_j - \theta_{0;j}), \tag{41}$$

and define other $\xi_i (i \neq 1)$ appropriately so that this transformation becomes invertible. For example, for $j \in \{1, \ldots, d\}$ satisfying $\frac{\partial f}{\partial \theta_j}(\theta_0) \neq 0$, define the transformation as

$$\begin{pmatrix} \frac{\partial f}{\partial \theta_1}(\theta_0) & \cdots & \frac{\partial f}{\partial \theta_j}(\theta_0) & \cdots & \frac{\partial f}{\partial \theta_d}(\theta_0) & r_0 \\ & I_{j-1} & & & & 0 \\ 1 & & 0 & & & \vdots \\ & & & I_{d-j-1} & & 0 \\ 0 & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \\ 1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \\ 1 \end{pmatrix} \tag{42}$$

where r_0 is defined as

$$r_0 = f(\theta_0) - \sum \frac{\partial f}{\partial \theta_i}(\theta_0)\theta_{0;i}. \tag{43}$$

Using this transformation, our problem is reduced to the estimation of ξ_1 because ξ_1 is the Taylor approximation of $f(\theta)$ by the definition (41). This estimation is completely same as the estimation with nuisance parameters. Applying Proposition 2, we obtain the desired inequality (40). Therefore, by the same method in Section 4, we obtain the optimal estimator whose measurement is adaptive. □

6 Example

Here, we give an example of an estimation for a real-valued function whose basic aspects are described in Section 5. Assume that two-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are given. Also, we assume that $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$ form CONSs of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Define the pure state model $\mathcal{M}$ as

$$\mathcal{M} := \{|u_\theta\rangle\langle u_\theta| \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid \theta \in \Theta\} \tag{44}$$

where

$$\Theta := \left\{\theta = (\theta_C, \phi_C, \theta_A, \phi_A, \theta_B, \phi_B) \in \mathbb{R}^6 \mid 0 \leq \theta_S \leq \frac{\pi}{2}, 0 \leq \phi_{S'} \leq 2\pi, S, S' \in \{A, B, C\}\right\} \tag{45}$$

and

$$|u_\theta\rangle := \cos\theta_C e^{-i\phi_C}|u_A\otimes u_B\rangle + \sin\theta_C e^{i\phi_C}|u_A^\perp \otimes u_B^\perp\rangle, \tag{46}$$
$$|u_A\rangle := \cos\theta_A e^{-i\phi_A}|0_A\rangle + \sin\theta_A e^{i\phi_A}|1_A\rangle, \tag{47}$$
$$|u_B\rangle := \cos\theta_B e^{-i\phi_B}|0_B\rangle + \sin\theta_B e^{i\phi_B}|1_B\rangle. \tag{48}$$

As an example of a real-valued function, we choose the entropy of entanglement which is characterized as $H(\mathrm{Tr}_{\mathcal{H}_B}\rho)$, the von-Neumann entropy of the reduced density operator $\mathrm{Tr}_{\mathcal{H}_B}\rho$ for a state $\rho \in \mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$. For a state $|u_\theta\rangle\langle u_\theta| \in \mathcal{M}$, this value $H(\mathrm{Tr}_{\mathcal{H}_B}|u_\theta\rangle\langle u_\theta|)$ is calculated as

$$H(\mathrm{Tr}_{\mathcal{H}_B}(|u_\theta\rangle\langle u_\theta|)) = -\cos^2\theta_C\log\cos^2\theta_C - \sin^2\theta_C\log\sin^2\theta_C. \tag{49}$$

Therefore, defining the function $f:\Theta\to\mathbb{R}$ as

$$f(\theta) = H(\mathrm{Tr}_{\mathcal{H}_B}|u_\theta\rangle\langle u_\theta|), \tag{50}$$

the partial derivatives of this function f are calculated as

$$\frac{\partial f}{\partial\theta_C}(\theta) = 4\sin\theta_C\cos\theta_C\log\frac{\cos\theta_C}{\sin\theta_C}, \quad \frac{\partial f}{\partial\theta_i}(\theta) = 0 \ (1 < i \le 6). \tag{51}$$

Since $(1,j)$-elements $(1\le j\le 6)$ of the SLD fisher information matrix at $\theta\in\Theta$ are calculated as

$$(\mathbf{J}_\theta^S)_{11} = 4, \quad (\mathbf{J}_\theta^S)_{1j} = 0 \ (1 < j \le 6), \tag{52}$$

the bound (4) is calculated as

$$\mathbf{V}_\theta(\mathbf{M},\hat{f}) \ge 4\sin\theta_C\cos\theta_C\log\frac{\cos\theta_C}{\sin\theta_C}\cdot\frac{1}{4}\cdot 4\sin\theta_C\cos\theta_C\log\frac{\cos\theta_C}{\sin\theta_C} = 4\sin^2\theta_C\cos^2\theta_C\log^2\frac{\cos\theta_C}{\sin\theta_C}. \tag{53}$$

The construction of a locally unbiased estimator attaining this bound (53) is explained in Section 5.

Next, let us compare this result to the case when operations for only $\mathcal{H}_A$ is allowed. In this case, we consider the model $\tilde{\mathcal{M}}$ which is defined as

$$\tilde{\mathcal{M}} = \{\mathrm{Tr}_{\mathcal{H}_B}|u_\theta\rangle\langle u_\theta| \in \mathcal{S}(\mathcal{H}_A)\}. \tag{54}$$

Since

$$\mathrm{Tr}_{\mathcal{H}_B}|u_\theta\rangle\langle u_\theta| = \cos^2\theta_C|u_A\rangle\langle u_A| + \sin^2\theta_C|u_A^\perp\rangle\langle u_A^\perp| \tag{55}$$

holds, this model $\tilde{\mathcal{M}}$ is three-dimensional and its parameters are θ_C, θ_A and ϕ_A. Since the $(1,j)$-elements $(1\le j\le 3)$ of the SLD fisher information matrix $\tilde{\mathbf{J}}_\theta^S$ for the model $\tilde{\mathcal{M}}$ are calculated as

$$(\tilde{\mathbf{J}}_\theta^S)_{11} = 4,\ (\tilde{\mathbf{J}}_\theta^S)_{12} = 0,\ (\tilde{\mathbf{J}}_\theta^S)_{13} = 0, \tag{56}$$

the bound is calculated as

$$\left(\frac{\partial f}{\partial\theta_C}(\theta), \frac{\partial f}{\partial\theta_A}(\theta), \frac{\partial f}{\partial\phi_A}(\theta)\right)(\mathbf{J}_\theta^S)^{-1}\begin{pmatrix}\frac{\partial f}{\partial\theta_C}(\theta)\\ \frac{\partial f}{\partial\theta_A}(\theta)\\ \frac{\partial f}{\partial\phi_A}(\theta)\end{pmatrix} = 4\cos^2\theta_C\sin^2\theta_C\log^2\frac{\cos\theta_C}{\sin\theta_C}. \tag{57}$$

This bound is the same as the bound (53). This means that, in this case, the precision bound is attained even if operations are restricted to only on $\mathcal{H}_A$. According to Theorem 1, the spectral decomposition of $[(\frac{\partial f}{\partial \theta_C}(\theta))^2(\mathbf{J}_\theta^S)_{11}]^{-1}L_{\theta;1}^S$ gives an optimal estimator. Since $L_{\theta;1}^S$ is decomposed to

$$L_{\theta;1}^S = -\tan\theta|u_A\rangle\langle u_A| + (\tan\theta)^{-1}|u_A^\perp\rangle\langle u_A^\perp|, \tag{58}$$

an optimal measurement on $\mathcal{H}_A$ is given as $\{|u_A\rangle\langle u_A|, |u_A^\perp\rangle\langle u_A^\perp|\}$.

7 Conclusion

In quantum technology which has been developed recently, designing optimal and accurate quantum devices is a really difficult and important problem. One of the reasons is because we cannot get the complete information from quantum particles. It usually varies. Therefore, the estimating process for the desired information is a fundamental problem. In this paper, we show the existence and the construction of an optimal LOCC estimator for nuisance parameter problem. This estimator is significantly important when dealing with the effect of noise and the physical realization. Since the quantum technology has been paid much attention to recently, quantum statistical inference will become more fundamental and essential in the future.

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