On the duality property of Blaschke products and its application

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Abstract

We study geometric properties of finite Blaschke products. For a Blaschke product B of degree d, the interior curve and the exterior curve are defined. In this paper, we explain the existence of duality-like geometrical property lies between the interior curve and the exterior curve. Using this property, we construct some examples of Blaschke products whose interior curves consist of two ellipses.

1 Geometry of Blaschke products

1.1 Blaschke Products

A *Blaschke product* of degree d is a rational function defined by

$$B(z) = e^{i\theta}\prod_{k=1}^{d}\frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

Note that we only need to consider a canonical Blaschke product for the following discussions. Moreover, we remark that there are d distinct preimages $z_1, \cdots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative B' has no zeros on $\partial\mathbb{D}$ (for instance, see [Mas13]).

1.2 The interior curves and exterior curves

For a Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let ℓ_λ be the set of lines joining each distinct two preimages in $B^{-1}(\lambda)$. Then, the envelope of the family of lines $\{\ell_\lambda\}_\lambda$ called the *interior curve* associated with B.

While, for a canonical Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages of λ. Then, the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle called the *exterior curve* associated with B.

For a canonical Blaschke product of degree d, the exterior curve is an algebraic curve of degree at most $d-1$ ([Fuj17]).

Example 1
For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}$ *of degree* 3, *the interior curve is the ellipse* (*see* [*DGM02*])

$$|z-a| + |z-b| = |1-\overline{a}b|, \tag{1}$$

and the exterior curve is the non-degenerate conic (*see* [*Fuj17*])

$$\overline{a}\overline{b}z^2 + (-|ab|^2 + |a+b|^2 - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a}+\overline{b})z - 2(a+b)\overline{z} + 4 = 0. \tag{2}$$

Example 2
For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}\dfrac{z-c}{1-\overline{c}z}$ *of degree 4, the interior curve is defined by the equation S of degree 6. We describe the defining equation S in Appendix A below. The exterior curve is written as follows (see [Fuj17])*

$$\begin{aligned}&\overline{\sigma_3}z^3 + (\sigma_1\overline{\sigma_2} - \sigma_2\overline{\sigma_3} - \overline{\sigma_1})z^2\overline{z} - (\sigma_1 - \sigma_2\overline{\sigma_1} + \sigma_3\overline{\sigma_2})z\overline{z}^2 + \sigma_3\overline{z}^3\\&- 2\overline{\sigma_2}z^2 - (2\sigma_1\overline{\sigma_1} - 2\sigma_3\overline{\sigma_3} - 4)z\overline{z} - 2\sigma_2\overline{z}^2 + 4\overline{\sigma_1}z + 4\sigma_1\overline{z} - 8 = 0,\end{aligned}$$

where σ_k are the elementary symmetric polynomials on three variables $a,\ b,\ c$ of degree k ($k = 1, 2, 3$), i.e. $\sigma_1 = a + b + c,\ \sigma_2 = ab + bc + ca,$ and $\sigma_3 = abc$.

1.3 Duality-like property

There exists a duality-like relationship between the interior and exterior curves.

Theorem 1 ([Fuj18])
Let B be a canonical Blaschke product of degree d, and E_B^* the dual curve of the homogenized exterior curve E_B. Then the interior curve is given by

$$I_B:\ u_B^*(-z) = 0,$$

where $u_B^*(z) = 0$ is a defining equation of the affine part of E_B^*.

Equivalently, the converse also holds.

Corollary 2
Let B be a canonical Blaschke product of degree d, and I_B^* be the dual curve of the homogenized interior curve I_B. Then the exterior curve is given by

$$E_B:\ v_B^*(-z) = 0,$$

where $v_B^*(z) = 0$ is a defining equation of the affine part of I_B^*.

In general, the defining equation of the interior curve is hard to calculate, even using an algebraic computation system. On the other hand, the defining equation of the exterior curve is relatively simple, as seen in Example 2 above. Theorem 1 allows us to get the defining equation of the interior curve via the exterior curve. In the next section, we will show some examples as an application of this theorem.

2 Examples

Here, we construct Blaschke products of degree 5 whose interior curve consists of two ellipses. The defining equation of the exterior curve is an algebraic curve of degree four. We need to find an example of Blaschke product whose exterior curve can be resolved into two conics because the dual curve of a conic is a conic.

Example 3
Let

$$B_A(z) = z\frac{z^2-a}{1-az^2}\frac{z^2-b}{1-bz^2} \qquad (0 < a, b < 1),$$

where a, b satisfy $a^3b^3 - 2a^2b^2 - (b^2 + a^2) + 3ab = 0$. Then the exterior curve is given by

$$\begin{aligned}E_{B_A}:\ &\Big(a(b+1)^2x^2 + a(b-1)^2y^2 - 4b\Big)\\&\times\Big((a^2b^3 - ab^2 + 2b^2 + 3b - a)x^2 + (a^2b^3 - ab^2 - 2b^2 + 3b - a)y^2 - 4b\Big) = 0,\end{aligned}$$

and the interior curve consists of two ellipses

$$I_{B_A}: \Big(\frac{4b}{a(b+1)^2}x^2 + \frac{4b}{a(b-1)^2}y^2 - 1\Big)\Big(\frac{4a}{b(a+1)^2}x^2 + \frac{4a}{b(a-1)^2}y^2 - 1\Big) = 0,$$

where we set $z = x + iy$. *Their foci are* $\pm\sqrt{a}$ *(the first factor) and* $\pm\sqrt{b}$ *(the second factor).*

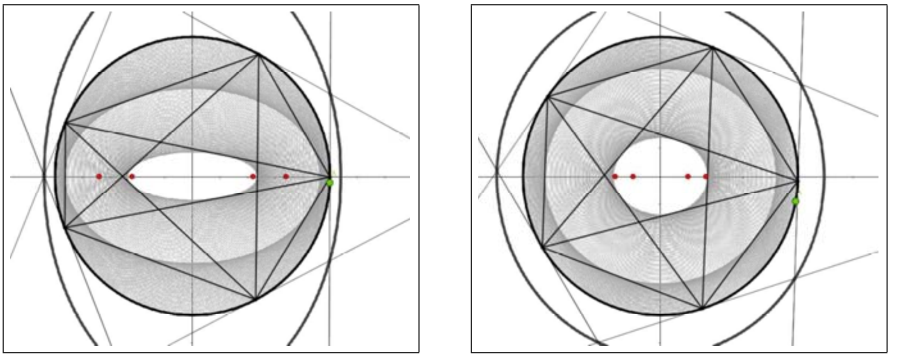

Figure 1: The interior curve I_{B_A} for $a = 0.4801\cdots$, $b = 0.2$ (left) and $a = 0.04$, $b = 0.1043\cdots$ (right). The interior curve consists of two ellipses, one is inscribed in a family of pentagons and the other is inscribed in a family of pentagrams.

The interior curve of a finite Blaschke product is closely related to the numerical range of a matrix (see [GMR18], [DGSV18], for example). In fact, the interior curve in the above example corresponds to the elliptic domain that appears in the following result of Chien and Nakazato ([CN17, Theorem 3.2]).

Theorem: Let A *be a* 4×4 *unitary bordering matrix with real eigenvalues* $\pm\sqrt{a}$, $\pm\sqrt{b}$ *for some* $0 < a \neq b < 1$. *Then* $F_A(x, y, z) := \det\big(x \cdot Re(A) + y \cdot Im(A) + z \cdot I_4\big)$ *is a product of two quadratic forms if and only if* $a^3b^3 - 2a^2b^2 - a^2 + 3ab - b^2 = 0$, *where* $Re(A) = \frac{A+A^*}{2}$, $Im(A) = \frac{A-A^*}{2i}$. *In this case, the higher rank numerical range* $\Lambda_k(A)$ *is an elliptical disc.*

Here we construct some more examples of Blaschke products of degree 5 whose interior curve consists of two ellipses.

Example 4
Let

$$B_B(z) = z\Big(\frac{z-a}{1-az}\Big)^2\Big(\frac{z-b}{1-bz}\Big)^2 \qquad (0 < a, b < 1),$$

where a, b *satisfy* $a^2b^3 - 2a^2b^2 - (a^2 + b^2) + 3ab = 0$. *Then the exterior curve is given by*

$$E_{B_B}: \Big((a(b^2-1)^2 - 4b^3)x^2 + 8b^2x + a(b^2-1)^2y^2 - 4b\Big) \\ \times \Big(((a^2-1)b - 4a^3)x^2 + 8a^2x + (a^2-1)by^2 - 4a\Big) = 0,$$

and the interior curve consists of two circles (see Figures 2, 3, and 4).

$$I_{B_B}: \Big(4a(x-a)^2 + 4ay^2 - (a^2-1)^2b\Big)\Big(4b(x-b)^2 + 4by^2 - a(b^2-1)^2\Big) = 0,$$

where we set $z = x + iy$. *Their centers coincide with non-zero zeros of* B_B, *i.e.* a *(the first factor) and* b *(the second factor).*

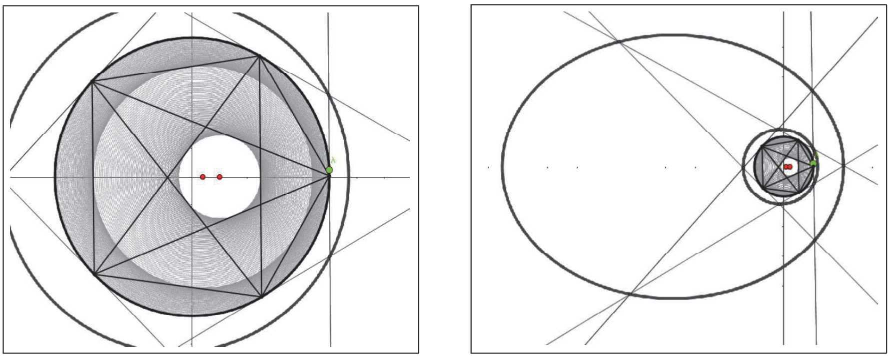

Figure 2: The interior and exterior curves of B_B for $a = 0.07746\cdots, b = \frac{1}{5}$. In this case, the exterior and interior curves consist of two ellipses and two circles, respectively.

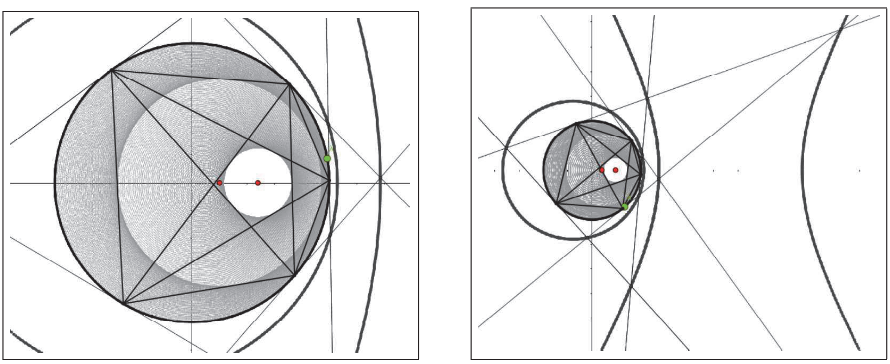

Figure 3: The interior and exterior curves of B_B for $a = 0.48012\cdots, b = \frac{1}{5}$. In this case, the exterior curve consists of an ellipse and a hyperbolic curve.

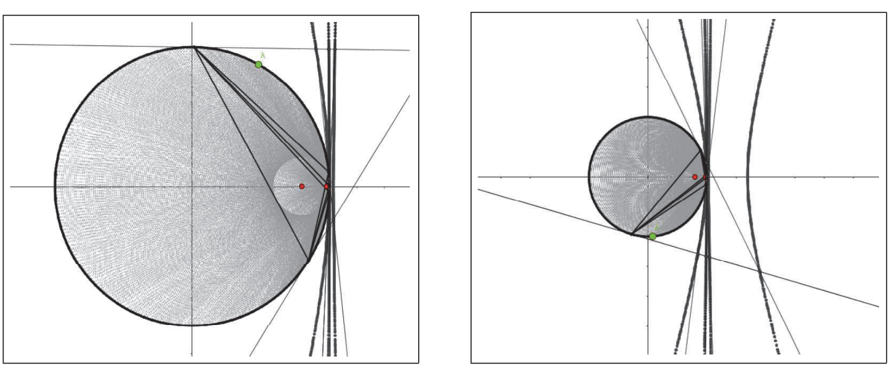

Figure 4: The interior and exterior curves of B_B for $a = 0.98697541\cdots,\ b = \frac{4}{5}$. In this case, the exterior curve consists of two hyperbolic curves.

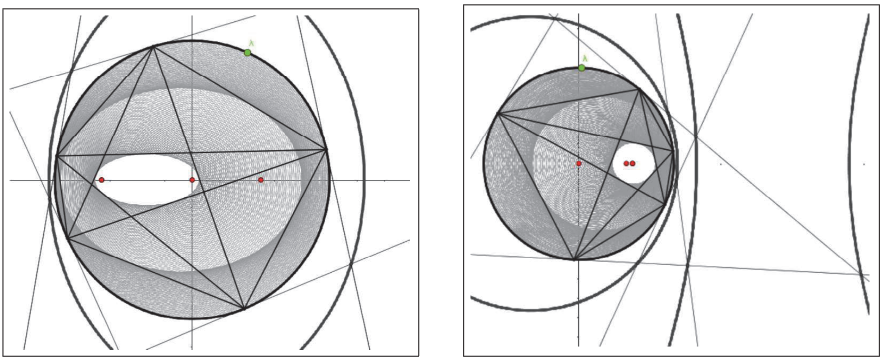

Figure 5: The interior and exterior curves of B_C. The left figure indicates the case of $P=0$ (I_P for $a=0.5,\ b=-0.660442\cdots$). The right figure indicates the case of $Q=0$ (I_Q for $a=0.5,\ b=0.5653036\cdots$).

Example 5
Let

$$B_C(z)=z^2\frac{z-a}{1-az}\Big(\frac{z-b}{1-bz}\Big)^2 \qquad (-1<a,b<1),$$

where a,b satisfy

$$P(a,b)=a^2+(b^3-b)a-b^2=0 \quad \text{or} \quad Q(a,b)=(b^3-3b)a^3-2(b^2-2)a^2+(3b^3-b)a-2b^2=0.$$

Then, the exterior curve for $P=0$ and $Q=0$ are given by

$$\begin{aligned}E_{P(a,b)}:&\Big((a^3+3ba^2-a+b)x^2-4a(a+b)x+(a^2-1)(a-b)y^2+4a\Big)\\ &\times\Big((b^2-1)(a+b)x^2-4b^2x+(b^2-1)(a+b)y^2+4b\Big)=0\end{aligned}$$

and

$$\begin{aligned}E_{Q(a,b)}:&\Big(2\big((b^2+1)a+b^3-b\big)x^2-8abx-2(b^2-1)(a-b)y^2+4a\Big)\\ &\times\Big(\big(ba^2+(2b^2-2)a-b\big)x^2-4abx+\big(ba^2+(2b^2-2)a-b\big)y^2+4b\Big)=0\end{aligned}$$

respectively. Here, we assume that a,b satisfy $E_{P(a,b)}\cap\partial\mathbb{D}=\emptyset$ or $E_{Q(a,b)}\cap\partial\mathbb{D}=\emptyset$. The interior curve for $P=0$ is given by

$$\begin{aligned}I_{P(a,b)}:&\Big(a(a^2-1)\big(2x-(a+b)\big)^2+4a(ab-1)y^2+(a^2-1)(a-b)(ab-1)\Big)\\ &\times\Big(b(b^2-1)(a+b)(2x-b)^2+4b\big((b^2-1)a-b\big)y^2+(b^2-1)(a+b)\big((b^2-1)a-b\big)\Big)=0,\end{aligned}$$

if $E_{P(a,b)}\cap\partial\mathbb{D}=\emptyset$. Similarly, the interior curve for $Q=0$ is given by

$$\begin{aligned}I_{Q(a,b)}:&\Big(b\big(ba^2+2(b^2-1)a-b\big)(2x-a)^2+4b\big(2(b^2-1)a-b\big)y^2\\ &+\big(2(b^2-1)a-b\big)\big(ba^2+2(b^2-1)a-b\big)\Big)\\ &\times\Big(2a(x-b)^2+2ay^2-(b^2-1)(a-b)\Big)=0,\end{aligned}$$

if $E_{Q(a,b)}\cap\partial\mathbb{D}=\emptyset$ (see Figure 5).

The zeros of each examples B_A, B_B, and B_c are on a line passing through the origin (the zero points are placed on the real axis by suitable rotation). The following gives an example of Blaschke product whose zeros are not collinear.

Example 6
Let

$$B_D(z) = z\frac{(z-a)(z-\overline{a})(z-b)(z-\overline{b})}{(1-\overline{a}z)(1-az)(1-\overline{b}z)(1-bz)},$$

where $a \approx -0.44096 + 0.37267i,\ b \approx -0.27103 + 0.65310i$. *Then, the exterior curve consists of two conics*

$$E_D\ :\ \Big(\frac{1}{3}z^2 + \frac{1}{3}\overline{z}^2 + vz + v\overline{z} + 4\Big)\Big(\frac{1}{2}z^2 - \frac{1}{4}(3v\tilde{v}+2)z\overline{z} + \frac{1}{2}\overline{z}^2 + \tilde{v}z + \tilde{v}\overline{z} + 4\Big) = 0,$$

where $v = \frac{\sqrt{28}}{3}$ *and* $\tilde{v}$ *is the unique positive root of* $\tilde{v}^2 + 2v\tilde{v} - 5 = 0$. *Therefore, the interior curve is also written as two conics as follows.*

$$\begin{aligned} I_D\ :\ &\Big((9v^2-48)z^2 - 18v^2 z\overline{z} + (9v^2-48)\overline{z}^2 - 24vz - 24v\overline{z} - 16\Big) \\ &\times\Big(4(\tilde{v}^2-8)z^2 - 8(6\tilde{v}v + \tilde{v}^2 + 4)z\overline{z} + 4(\tilde{v}^2-8)\overline{z}^2 - 12(2\tilde{v} + v\tilde{v}^2)(z+\overline{z}) \\ &\qquad + 9\tilde{v}^2v^2 + 12\tilde{v}v - 12\Big) = 0. \end{aligned}$$

Since these two conics are included in the unit disk, they are necessarily two ellipses.

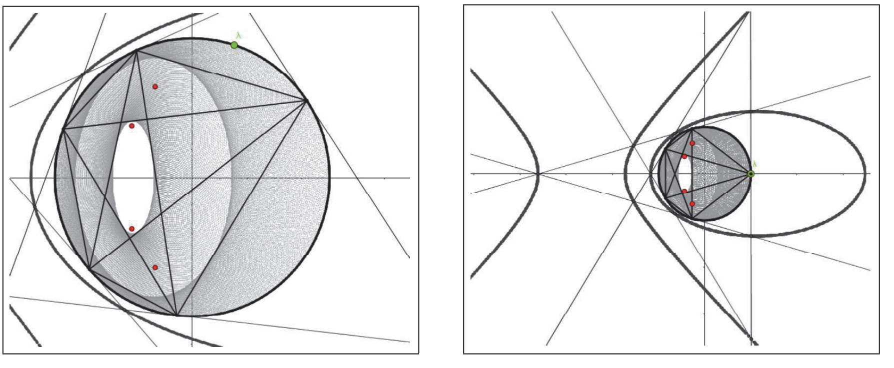

Figure 6: The interior and exterior curves for B_D. In this case, the exterior curve E_D consists of an ellipse and a hyperbola. So, the interior curve I_D consists of two ellipses.

Acknowledgements

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A The interior curve for a Blaschke product of degree 4

For a canonical Blaschke product

$$B(z) = z\frac{z-a}{1-\overline{a}z}\frac{z-b}{1-\overline{b}z}\frac{z-c}{1-\overline{c}z}$$

of degree 4, let σ_k be the elementary symmetric polynomials on three variables $a,\ b,\ c$ of degree k ($k = 1, 2, 3$), i.e.

$$\sigma_1 = a+b+c,\ \sigma_2 = ab+bc+ca,\ \text{and } \sigma_3 = abc.$$

Then, the defining equation of the interior curve is given as follows.

$$\begin{aligned}
&S:\ (-4\overline{\sigma_3\sigma_1}^3 + \overline{\sigma_2}^2\overline{\sigma_1}^2 + 18\overline{\sigma_3\sigma_2\sigma_1} - 4\overline{\sigma_2}^3 - 27\overline{\sigma_3}^2)z^6 + (((2\overline{\sigma_2\sigma_1}^3 - 6\overline{\sigma_3\sigma_1}^2 - 8\overline{\sigma_2}^2\overline{\sigma_1} + 36\overline{\sigma_3\sigma_2})\sigma_1 + \\
&(-4\overline{\sigma_3\sigma_1}^3 + 18\overline{\sigma_3\sigma_2\sigma_1} - 54\overline{\sigma_3}^2)\sigma_2 + (2\overline{\sigma_3\sigma_2\sigma_1}^2 + 18\overline{\sigma_3}^2\overline{\sigma_1} - 12\overline{\sigma_3\sigma_2}^2)\sigma_3 - 4\overline{\sigma_1}^4 + 22\overline{\sigma_2\sigma_1}^2 - 18\overline{\sigma_3\sigma_1} - \\
&24\overline{\sigma_2}^2)\overline{z} + (-12\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_2}^2\overline{\sigma_1}^2 + 54\overline{\sigma_3\sigma_2\sigma_1} - 16\overline{\sigma_2}^3 - 54\overline{\sigma_3}^2)\sigma_1 + (-2\overline{\sigma_3\sigma_2\sigma_1}^2 - 18\overline{\sigma_3}^2\overline{\sigma_1} + 12\overline{\sigma_3\sigma_2}^2)\sigma_2 + \\
&(12\overline{\sigma_3}^2\overline{\sigma_1}^2 - 2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 18\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_2\sigma_1}^3 + 6\overline{\sigma_3\sigma_1}^2 + 8\overline{\sigma_2}^2\overline{\sigma_1} - 36\overline{\sigma_3\sigma_2})z^5 + (((\overline{\sigma_1}^4 - 2\overline{\sigma_2\sigma_1}^2 + 6\overline{\sigma_3\sigma_1} - \\
&8\overline{\sigma_2}^2)\sigma_1^2 + ((-6\overline{\sigma_3\sigma_1}^2 + 36\overline{\sigma_3\sigma_2})\sigma_2 + (-2\overline{\sigma_3\sigma_1}^3 + 2\overline{\sigma_3\sigma_2\sigma_1} - 18\overline{\sigma_3}^2)\sigma_3 - 6\overline{\sigma_1}^3 + 22\overline{\sigma_2\sigma_1} + 18\overline{\sigma_3})\sigma_1 - 27\overline{\sigma_3}^2\sigma_2^2 + \\
&(18\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3 - 4\overline{\sigma_1}^4 + 20\overline{\sigma_2\sigma_1}^2 - 54\overline{\sigma_3\sigma_1} - 12\overline{\sigma_2}^2)\sigma_2 + (\overline{\sigma_3}^2\overline{\sigma_1}^2 - 12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (4\overline{\sigma_2\sigma_1}^3 + 22\overline{\sigma_3\sigma_1}^2 - 18\overline{\sigma_2}^2\overline{\sigma_1} + \\
&6\overline{\sigma_3\sigma_2})\sigma_3 + 13\overline{\sigma_1}^2 - 48\overline{\sigma_2})\overline{z}^2 + ((6\overline{\sigma_2\sigma_1}^3 - 12\overline{\sigma_3\sigma_1}^2 - 24\overline{\sigma_2}^2\overline{\sigma_1} + 72\overline{\sigma_3\sigma_2})\sigma_1^2 + ((-10\overline{\sigma_3\sigma_1}^3 + 52\overline{\sigma_3\sigma_2\sigma_1} - \\
&90\overline{\sigma_3}^2)\sigma_2 + (-2\overline{\sigma_3\sigma_2\sigma_1}^2 + 30\overline{\sigma_3}^2\overline{\sigma_1} - 16\overline{\sigma_3\sigma_2}^2)\sigma_3 - 10\overline{\sigma_1}^4 + 52\overline{\sigma_2\sigma_1}^2 + 6\overline{\sigma_3\sigma_1} - 56\overline{\sigma_2}^2)\sigma_1 - 18\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^2 + \\
&((10\overline{\sigma_3}^2\overline{\sigma_1}^2 + 6\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2\sigma_1}^3 - 28\overline{\sigma_3\sigma_1}^2 + 20\overline{\sigma_2}^2\overline{\sigma_1} - 24\overline{\sigma_3\sigma_2})\sigma_2 + (-4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} - 18\overline{\sigma_3}^3)\sigma_3^2 + (10\overline{\sigma_3\sigma_1}^3 + \\
&4\overline{\sigma_2}^2\overline{\sigma_1}^2 + 26\overline{\sigma_3\sigma_2\sigma_1} - 24\overline{\sigma_2}^3 - 72\overline{\sigma_3}^2)\sigma_3 + 2\overline{\sigma_1}^3 - 4\overline{\sigma_2\sigma_1} - 72\overline{\sigma_3})\overline{z} + (-12\overline{\sigma_3\sigma_1}^3 + 6\overline{\sigma_2}^2\overline{\sigma_1}^2 + 54\overline{\sigma_3\sigma_2\sigma_1} - \\
&24\overline{\sigma_2}^3 - 27\overline{\sigma_3}^2)\sigma_1^2 + ((-6\overline{\sigma_3\sigma_2\sigma_1}^2 - 36\overline{\sigma_3}^2\overline{\sigma_1} + 36\overline{\sigma_3\sigma_2}^2)\sigma_2 + (24\overline{\sigma_3}^2\overline{\sigma_1}^2 - 6\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 36\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 6\overline{\sigma_2\sigma_1}^3 + \\
&24\overline{\sigma_3\sigma_1}^2 + 22\overline{\sigma_2}^2\overline{\sigma_1} - 90\overline{\sigma_3\sigma_2})\sigma_1 + (\overline{\sigma_3}^2\overline{\sigma_1}^2 - 12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_2^2 + ((4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 18\overline{\sigma_3}^3)\sigma_3 - 10\overline{\sigma_3\sigma_1}^3 + 2\overline{\sigma_2}^2\overline{\sigma_1}^2 + \\
&20\overline{\sigma_3\sigma_2\sigma_1} - 4\overline{\sigma_2}^3 - 18\overline{\sigma_3}^2)\sigma_2 + (-12\overline{\sigma_3}^3\overline{\sigma_1} + \overline{\sigma_3}^2\overline{\sigma_2}^2)\sigma_3^2 + (16\overline{\sigma_3\sigma_2\sigma_1}^2 + (-2\overline{\sigma_2}^3 - 30\overline{\sigma_3}^2)\overline{\sigma_1} - 14\overline{\sigma_3\sigma_2}^2)\sigma_3 + \\
&\overline{\sigma_1}^4 - 4\overline{\sigma_2\sigma_1}^2 - 12\overline{\sigma_3\sigma_1} + 4\overline{\sigma_2}^2)z^4 + (((2\overline{\sigma_1}^3 - 8\overline{\sigma_2\sigma_1} + 4\overline{\sigma_3})\sigma_1^3 + (6\overline{\sigma_3\sigma_1}\sigma_2 + (-8\overline{\sigma_3\sigma_1}^2 + 20\overline{\sigma_3\sigma_2})\sigma_3 - \\
&4\overline{\sigma_1}^2 + 4\overline{\sigma_2})\sigma_1^2 + ((-18\overline{\sigma_3}^2\sigma_3 - 8\overline{\sigma_1}^3 + 38\overline{\sigma_2\sigma_1})\sigma_2 + 10\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3^2 + (6\overline{\sigma_2\sigma_1}^2 + 4\overline{\sigma_3\sigma_1} - 36\overline{\sigma_2}^2)\sigma_3 + 22\overline{\sigma_1})\sigma_1 - \\
&36\overline{\sigma_3\sigma_1}\sigma_2^2 + ((20\overline{\sigma_3\sigma_1}^2 + 30\overline{\sigma_3\sigma_2})\sigma_3 + 4\overline{\sigma_1}^2 - 48\overline{\sigma_2})\sigma_2 - 4\overline{\sigma_3}^3\sigma_3^3 + (-18\overline{\sigma_3\sigma_2\sigma_1} - 24\overline{\sigma_3}^2)\sigma_3^2 + (4\overline{\sigma_1}^3 + 6\overline{\sigma_3})\sigma_3 - \\
&32)\overline{z}^3 + ((2\overline{\sigma_1}^4 - 4\overline{\sigma_2\sigma_1}^2 + 6\overline{\sigma_3\sigma_1} - 16\overline{\sigma_2}^2)\sigma_1^3 + ((-4\overline{\sigma_3\sigma_1}^2 + 52\overline{\sigma_3\sigma_2})\sigma_2 + (-6\overline{\sigma_3\sigma_1}^3 + 6\overline{\sigma_3\sigma_2\sigma_1} - 6\overline{\sigma_3}^2)\sigma_3 - \\
&8\overline{\sigma_1}^3 + 14\overline{\sigma_2\sigma_1} + 60\overline{\sigma_3})\sigma_1^2 + (-36\overline{\sigma_3}^2\sigma_2^2 + (10\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3 - 8\overline{\sigma_1}^4 + 40\overline{\sigma_2\sigma_1}^2 - 46\overline{\sigma_3\sigma_1} + 4\overline{\sigma_2}^2)\sigma_2 + (6\overline{\sigma_3}^2\overline{\sigma_1}^2 - \\
&2\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (8\overline{\sigma_2\sigma_1}^3 + 26\overline{\sigma_3\sigma_1}^2 - 50\overline{\sigma_2}^2\overline{\sigma_1} + 34\overline{\sigma_3\sigma_2})\sigma_3 + 46\overline{\sigma_1}^2 - 68\overline{\sigma_2})\sigma_1 + (-32\overline{\sigma_3\sigma_1}^2 - 12\overline{\sigma_3\sigma_2})\sigma_2^2 + \\
&(-6\overline{\sigma_3}^3\sigma_3^2 + (8\overline{\sigma_3\sigma_1}^3 + 72\overline{\sigma_3\sigma_2\sigma_1} - 78\overline{\sigma_3}^2)\sigma_3 - 16\overline{\sigma_1}^3 + 24\overline{\sigma_2\sigma_1} - 132\overline{\sigma_3})\sigma_2 - 2\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^3 + (-8\overline{\sigma_3\sigma_2\sigma_1}^2 - \\
&18\overline{\sigma_3}^2\overline{\sigma_1} - 30\overline{\sigma_3\sigma_2}^2)\sigma_3^2 + (44\overline{\sigma_2\sigma_1}^2 + 6\overline{\sigma_3\sigma_1} - 96\overline{\sigma_2}^2)\sigma_3 - 40\overline{\sigma_1})\overline{z}^2 + ((6\overline{\sigma_2\sigma_1}^3 - 6\overline{\sigma_3\sigma_1}^2 - 24\overline{\sigma_2}^2\overline{\sigma_1} + 36\overline{\sigma_3\sigma_2})\sigma_1^3 + \\
&((-8\overline{\sigma_3\sigma_1}^3 + 50\overline{\sigma_3\sigma_2\sigma_1} - 36\overline{\sigma_3}^2)\sigma_2 + (-10\overline{\sigma_3\sigma_2\sigma_1}^2 + 12\overline{\sigma_3}^2\overline{\sigma_1} + 4\overline{\sigma_3\sigma_2}^2)\sigma_3 - 8\overline{\sigma_1}^4 + 34\overline{\sigma_2\sigma_1}^2 + 54\overline{\sigma_3\sigma_1} - \\
&44\overline{\sigma_2}^2)\sigma_1^2 + (-26\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^2 + ((16\overline{\sigma_3}^2\overline{\sigma_1}^2 - 10\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2\sigma_1}^3 - 48\overline{\sigma_3\sigma_1}^2 + 34\overline{\sigma_2}^2\overline{\sigma_1} + 28\overline{\sigma_3\sigma_2})\sigma_2 + (2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} - \\
&6\overline{\sigma_3}^3)\sigma_3^2 + (16\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_2}^2\overline{\sigma_1}^2 + 52\overline{\sigma_3\sigma_2\sigma_1} - 52\overline{\sigma_2}^3 - 78\overline{\sigma_3}^2)\sigma_3 + 14\overline{\sigma_1}^3 - 6\overline{\sigma_2\sigma_1} - 132\overline{\sigma_3})\sigma_1 + (6\overline{\sigma_3}^3\sigma_3 - \\
&8\overline{\sigma_3\sigma_1}^3 - 4\overline{\sigma_3\sigma_2\sigma_1} - 60\overline{\sigma_3}^2)\sigma_2^2 + (-8\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^2 + (20\overline{\sigma_3\sigma_2\sigma_1}^2 + 4\overline{\sigma_3}^2\overline{\sigma_1} + 22\overline{\sigma_3\sigma_2}^2)\sigma_3 - 8\overline{\sigma_1}^4 + 20\overline{\sigma_2\sigma_1}^2 - 32\overline{\sigma_3\sigma_1} - \\
&16\overline{\sigma_2}^2)\sigma_2 + 2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^3 + (-8\overline{\sigma_3}^2\overline{\sigma_1}^2 - 12\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 42\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (8\overline{\sigma_2\sigma_1}^3 + 10\overline{\sigma_3\sigma_1}^2 - 138\overline{\sigma_3\sigma_2})\sigma_3 - 20\overline{\sigma_1}^2 + \\
&16\overline{\sigma_2})\overline{z} + (-4\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_2}^2\overline{\sigma_1}^2 + 18\overline{\sigma_3\sigma_2\sigma_1} - 16\overline{\sigma_2}^3)\sigma_1^3 + ((-6\overline{\sigma_3\sigma_2\sigma_1}^2 - 18\overline{\sigma_3}^2\overline{\sigma_1} + 36\overline{\sigma_3\sigma_2}^2)\sigma_2 + (12\overline{\sigma_3}^2\overline{\sigma_1}^2 - \\
&6\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 18\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 6\overline{\sigma_2\sigma_1}^3 + 30\overline{\sigma_3\sigma_1}^2 + 18\overline{\sigma_2}^2\overline{\sigma_1} - 72\overline{\sigma_3\sigma_2})\sigma_1^2 + ((2\overline{\sigma_3}^2\overline{\sigma_1}^2 - 24\overline{\sigma_3}^2\overline{\sigma_2})\sigma_2^2 + ((8\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + \\
&18\overline{\sigma_3}^3)\sigma_3 - 20\overline{\sigma_3\sigma_1}^3 + 6\overline{\sigma_2}^2\overline{\sigma_1}^2 + 44\overline{\sigma_3\sigma_2\sigma_1} - 12\overline{\sigma_2}^3)\sigma_2 + (-12\overline{\sigma_3}^3\overline{\sigma_1} + 2\overline{\sigma_3}^2\overline{\sigma_2}^2)\sigma_3^2 + (32\overline{\sigma_3\sigma_2\sigma_1}^2 + (-6\overline{\sigma_2}^3 - \\
&54\overline{\sigma_3}^2)\overline{\sigma_1} - 26\overline{\sigma_3\sigma_2}^2)\sigma_3 + 2\overline{\sigma_1}^4 - 4\overline{\sigma_2\sigma_1}^2 - 42\overline{\sigma_3\sigma_1} + 12\overline{\sigma_2}^2)\sigma_1 + 4\overline{\sigma_3}^3\sigma_2^3 + (-2\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3 - 4\overline{\sigma_3\sigma_2\sigma_1}^2 - 28\overline{\sigma_3}^2\overline{\sigma_1} + \\
&12\overline{\sigma_3\sigma_2}^2)\sigma_2^2 + (-2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^2 + (20\overline{\sigma_3}^2\overline{\sigma_1}^2 + 2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 8\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2\sigma_1}^3 + 16\overline{\sigma_3\sigma_1}^2 + 8\overline{\sigma_2}^2\overline{\sigma_1} - 44\overline{\sigma_3\sigma_2})\sigma_2 + \\
&4\overline{\sigma_3}^4\sigma_3^3 + (-26\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 2\overline{\sigma_3\sigma_2}^3 + 24\overline{\sigma_3}^3)\sigma_3^2 + (-2\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_2}^2\overline{\sigma_1}^2 - 20\overline{\sigma_3\sigma_2\sigma_1} - 6\overline{\sigma_3}^2)\sigma_3 - 8\overline{\sigma_2\sigma_1} + 32\overline{\sigma_3})z^3 + \\
&(((\overline{\sigma_1}^2 - 4\overline{\sigma_2})\sigma_1^4 + (4\overline{\sigma_3}\sigma_2 - 2\overline{\sigma_3\sigma_1}\sigma_3 - 6\overline{\sigma_1})\sigma_1^3 + ((-2\overline{\sigma_1}^2 + 20\overline{\sigma_2})\sigma_2 + \overline{\sigma_3}^2\sigma_3^2 + (-6\overline{\sigma_2\sigma_1} + 22\overline{\sigma_3})\sigma_3 + 13)\sigma_1^2 + \\
&(-18\overline{\sigma_3}\sigma_2^2 + (2\overline{\sigma_3\sigma_1}\sigma_3 + 22\overline{\sigma_1})\sigma_2 + 18\overline{\sigma_3\sigma_2}\sigma_3^2 + (6\overline{\sigma_1}^2 - 54\overline{\sigma_2})\sigma_3)\sigma_1 + (-8\overline{\sigma_1}^2 - 12\overline{\sigma_2})\sigma_2^2 + (-12\overline{\sigma_3}^2\sigma_3^2 + \\
&(36\overline{\sigma_2\sigma_1} + 6\overline{\sigma_3})\sigma_3 - 48)\sigma_2 + (-18\overline{\sigma_3\sigma_1} - 27\overline{\sigma_2}^2)\sigma_3^2 + 18\overline{\sigma_1}\sigma_3)\overline{z}^4 + ((2\overline{\sigma_1}^3 - 8\overline{\sigma_2\sigma_1})\sigma_1^4 + (8\overline{\sigma_3\sigma_1}\sigma_2 + (-6\overline{\sigma_3\sigma_1}^2 + \\
&8\overline{\sigma_3\sigma_2})\sigma_3 - 8\overline{\sigma_1}^2 - 16\overline{\sigma_2})\sigma_1^3 + ((-8\overline{\sigma_3}^2\sigma_3 - 4\overline{\sigma_1}^3 + 40\overline{\sigma_2\sigma_1} + 44\overline{\sigma_3})\sigma_2 + 6\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3^2 + (-4\overline{\sigma_2\sigma_1}^2 + 26\overline{\sigma_3\sigma_1} - \\
&32\overline{\sigma_2}^2)\sigma_3 + 46\overline{\sigma_1})\sigma_1^2 + (-50\overline{\sigma_3\sigma_1}\sigma_2^2 + ((6\overline{\sigma_3\sigma_1}^2 + 72\overline{\sigma_3\sigma_2})\sigma_3 + 14\overline{\sigma_1}^2 + 24\overline{\sigma_2})\sigma_2 - 2\overline{\sigma_3}^3\sigma_3^3 + (10\overline{\sigma_3\sigma_2\sigma_1} - \\
&18\overline{\sigma_3}^2)\sigma_3^2 + (6\overline{\sigma_1}^3 - 46\overline{\sigma_2\sigma_1} + 6\overline{\sigma_3})\sigma_3 - 40)\sigma_1 + (-30\overline{\sigma_3}^2\sigma_3 - 16\overline{\sigma_1}^3 + 4\overline{\sigma_2\sigma_1} - 96\overline{\sigma_3})\sigma_2^2 + (-2\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3^2 + (52\overline{\sigma_2\sigma_1}^2 + \\
&34\overline{\sigma_3\sigma_1} - 12\overline{\sigma_2}^2)\sigma_3 - 68\overline{\sigma_1})\sigma_2 - 6\overline{\sigma_3}^2\overline{\sigma_2}\sigma_3^3 + (-6\overline{\sigma_3\sigma_1}^2 - 36\overline{\sigma_2}^2\overline{\sigma_1} - 78\overline{\sigma_3\sigma_2})\sigma_3^2 + (60\overline{\sigma_1}^2 - 132\overline{\sigma_2})\sigma_3)\overline{z}^3
\end{aligned}$$

$$+((\overline{\sigma_1}^4-2\overline{\sigma_2\sigma_1}^2-8\overline{\sigma_2}^2)\sigma_1^4+((2\overline{\sigma_3\sigma_1}^2+16\overline{\sigma_3\sigma_2})\sigma_2+(-4\overline{\sigma_3\sigma_1}^3+4\overline{\sigma_3\sigma_2\sigma_1})\sigma_3-4\overline{\sigma_1}^3-20\overline{\sigma_2\sigma_1}+30\overline{\sigma_3})\sigma_1^3+$$
$$(-8\overline{\sigma_3}^2\sigma_2^2+(-4\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3-2\overline{\sigma_1}^4+24\overline{\sigma_2\sigma_1}^2+32\overline{\sigma_3\sigma_1}+8\overline{\sigma_2}^2)\sigma_2+(6\overline{\sigma_3}^2\overline{\sigma_1}^2-2\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(2\overline{\sigma_2\sigma_1}^3+16\overline{\sigma_3\sigma_1}^2-$$
$$44\overline{\sigma_2}^2\overline{\sigma_1}+28\overline{\sigma_3\sigma_2})\sigma_3+54\overline{\sigma_1}^2-26\overline{\sigma_2})\sigma_1^2+((-44\overline{\sigma_3\sigma_1}^2+22\overline{\sigma_3\sigma_2})\sigma_2^2+(2\overline{\sigma_3}^3\sigma_3^2+(4\overline{\sigma_3\sigma_1}^3+78\overline{\sigma_3\sigma_2\sigma_1}-$$
$$40\overline{\sigma_3}^2)\sigma_3-20\overline{\sigma_1}^3+76\overline{\sigma_2\sigma_1}-70\overline{\sigma_3})\sigma_2-4\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^3+(-4\overline{\sigma_3\sigma_2\sigma_1}^2-20\overline{\sigma_3}^2\overline{\sigma_1}+6\overline{\sigma_3\sigma_2}^2)\sigma_3^2+(32\overline{\sigma_2\sigma_1}^2+$$
$$34\overline{\sigma_3\sigma_1}-96\overline{\sigma_2}^2)\sigma_3-64\overline{\sigma_1})\sigma_1-30\overline{\sigma_3}^2\sigma_2^3+(6\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3-8\overline{\sigma_1}^4+8\overline{\sigma_2\sigma_1}^2-96\overline{\sigma_3\sigma_1}-8\overline{\sigma_2}^2)\sigma_2^2+((-2\overline{\sigma_3}^2\overline{\sigma_1}^2-$$
$$26\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(16\overline{\sigma_2\sigma_1}^3+28\overline{\sigma_3\sigma_1}^2+22\overline{\sigma_2}^2\overline{\sigma_1}-20\overline{\sigma_3\sigma_2})\sigma_3-26\overline{\sigma_1}^2-8\overline{\sigma_2})\sigma_2+\overline{\sigma_3}^4\sigma_3^4+(2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+8\overline{\sigma_3}^3)\sigma_3^3+$$
$$(-8\overline{\sigma_2}^2\overline{\sigma_1}^2-40\overline{\sigma_3\sigma_2\sigma_1}-30\overline{\sigma_2}^3-84\overline{\sigma_3}^2)\sigma_3^2+(30\overline{\sigma_1}^3-70\overline{\sigma_2\sigma_1}-184\overline{\sigma_3})\sigma_3+16)\overline{z}^2+((2\overline{\sigma_2\sigma_1}^3-8\overline{\sigma_2}^2\overline{\sigma_1})\sigma_1^4+$$
$$((-2\overline{\sigma_3\sigma_1}^3+16\overline{\sigma_3\sigma_2\sigma_1})\sigma_2+(-6\overline{\sigma_3\sigma_2\sigma_1}^2+8\overline{\sigma_3\sigma_2}^2)\sigma_3-2\overline{\sigma_1}^4+30\overline{\sigma_3\sigma_1}-16\overline{\sigma_2}^2)\sigma_1^3+(-8\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^2+((6\overline{\sigma_3}^2\overline{\sigma_1}^2-$$
$$16\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3+4\overline{\sigma_2\sigma_1}^3-16\overline{\sigma_3\sigma_1}^2+8\overline{\sigma_2}^2\overline{\sigma_1}+60\overline{\sigma_3\sigma_2})\sigma_2+6\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}\sigma_3^2+(6\overline{\sigma_3\sigma_1}^3-4\overline{\sigma_2}^2\overline{\sigma_1}^2+34\overline{\sigma_3\sigma_2\sigma_1}-$$
$$32\overline{\sigma_2}^3-30\overline{\sigma_3}^2)\sigma_3+16\overline{\sigma_1}^3+14\overline{\sigma_2\sigma_1}-96\overline{\sigma_3})\sigma_1^2+((8\overline{\sigma_3}^3\sigma_3-12\overline{\sigma_3\sigma_1}^3+8\overline{\sigma_3\sigma_2\sigma_1}-44\overline{\sigma_3}^2)\sigma_2^2+(-6\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^2+$$
$$(16\overline{\sigma_3\sigma_2\sigma_1}^2-2\overline{\sigma_3}^2\overline{\sigma_1}+40\overline{\sigma_3\sigma_2}^2)\sigma_3-12\overline{\sigma_1}^4+32\overline{\sigma_2\sigma_1}^2+14\overline{\sigma_3\sigma_1}-16\overline{\sigma_2}^2)\sigma_2-2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^3+(-6\overline{\sigma_3}^2\overline{\sigma_1}^2-$$
$$4\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}-34\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(12\overline{\sigma_2\sigma_1}^3+26\overline{\sigma_3\sigma_1}^2-10\overline{\sigma_2}^2\overline{\sigma_1}-142\overline{\sigma_3\sigma_2})\sigma_3-38\overline{\sigma_1}^2+16\overline{\sigma_2})\sigma_1-16\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^3+$$
$$((12\overline{\sigma_3}^2\overline{\sigma_1}^2-8\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3-8\overline{\sigma_2\sigma_1}^3-32\overline{\sigma_3\sigma_1}^2+20\overline{\sigma_2}^2\overline{\sigma_1}-40\overline{\sigma_3\sigma_2})\sigma_2^2+(2\overline{\sigma_3}^4\sigma_3^3+(-20\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+18\overline{\sigma_3}^3)\sigma_3^2+$$
$$(12\overline{\sigma_3\sigma_1}^3+16\overline{\sigma_2}^2\overline{\sigma_1}^2+20\overline{\sigma_3\sigma_2\sigma_1}-20\overline{\sigma_2}^3-6\overline{\sigma_3}^2)\sigma_3+8\overline{\sigma_1}^3-36\overline{\sigma_2\sigma_1}+40\overline{\sigma_3})\sigma_2+(2\overline{\sigma_3}^3\overline{\sigma_1}+8\overline{\sigma_3}^2\overline{\sigma_2}^2)\sigma_3^3+$$
$$(-12\overline{\sigma_3\sigma_2\sigma_1}^2+(-8\overline{\sigma_2}^3-42\overline{\sigma_3}^2)\overline{\sigma_1}-10\overline{\sigma_3\sigma_2}^2)\sigma_3^2+(-138\overline{\sigma_3\sigma_1}+20\overline{\sigma_2}^2)\sigma_3+16\overline{\sigma_1})\overline{z}+(\overline{\sigma_2}^2\overline{\sigma_1}^2-4\overline{\sigma_2}^3)\sigma_1^4+$$
$$((-2\overline{\sigma_3\sigma_2\sigma_1}^2+12\overline{\sigma_3\sigma_2}^2)\sigma_2-2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}\sigma_3-2\overline{\sigma_2\sigma_1}^3+12\overline{\sigma_3\sigma_1}^2+2\overline{\sigma_2}^2\overline{\sigma_1}-18\overline{\sigma_3\sigma_2})\sigma_1^3+((\overline{\sigma_3}^2\overline{\sigma_1}^2-12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_2^2+$$
$$(4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}\sigma_3-10\overline{\sigma_3\sigma_1}^3+6\overline{\sigma_2}^2\overline{\sigma_1}^2+28\overline{\sigma_3\sigma_2\sigma_1}-12\overline{\sigma_2}^3+18\overline{\sigma_3}^2)\sigma_2+\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^2+(16\overline{\sigma_3\sigma_2\sigma_1}^2+(-6\overline{\sigma_2}^3-$$
$$24\overline{\sigma_3}^2)\overline{\sigma_1}-10\overline{\sigma_3\sigma_2}^2)\sigma_3+\overline{\sigma_1}^4+4\overline{\sigma_2\sigma_1}^2-42\overline{\sigma_3\sigma_1}+13\overline{\sigma_2}^2)\sigma_1^2+(4\overline{\sigma_3}^3\sigma_2^3+(-2\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3-8\overline{\sigma_3\sigma_2\sigma_1}^2-30\overline{\sigma_3}^2\overline{\sigma_1}+$$
$$24\overline{\sigma_3\sigma_2}^2)\sigma_2^2+(-2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^2+(20\overline{\sigma_3}^2\overline{\sigma_1}^2+4\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}-12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3-8\overline{\sigma_2\sigma_1}^3+36\overline{\sigma_3\sigma_1}^2+14\overline{\sigma_2}^2\overline{\sigma_1}-70\overline{\sigma_3\sigma_2})\sigma_2+$$
$$(-26\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+4\overline{\sigma_3\sigma_2}^3+12\overline{\sigma_3}^3)\sigma_3^2+(-2\overline{\sigma_3\sigma_1}^3+8\overline{\sigma_2}^2\overline{\sigma_1}^2-30\overline{\sigma_3\sigma_2\sigma_1}-2\overline{\sigma_2}^3-6\overline{\sigma_3}^2)\sigma_3-2\overline{\sigma_1}^3-16\overline{\sigma_2\sigma_1}+$$
$$48\overline{\sigma_3})\sigma_1+(2\overline{\sigma_3}^2\overline{\sigma_1}^2-12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_2^3+(\overline{\sigma_3}^4\sigma_3^2+(2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+22\overline{\sigma_3}^3)\sigma_3-8\overline{\sigma_3\sigma_1}^3+\overline{\sigma_2}^2\overline{\sigma_1}^2+2\overline{\sigma_3\sigma_2\sigma_1}+13\overline{\sigma_3}^2)\sigma_2^2+$$
$$((-10\overline{\sigma_3}^3\overline{\sigma_1}-4\overline{\sigma_3}^2\overline{\sigma_2}^2)\sigma_3^2+(24\overline{\sigma_3\sigma_2\sigma_1}^2+(-2\overline{\sigma_2}^3-40\overline{\sigma_3}^2)\overline{\sigma_1}-14\overline{\sigma_3\sigma_2}^2)\sigma_3+2\overline{\sigma_1}^4-4\overline{\sigma_2\sigma_1}^2-4\overline{\sigma_3\sigma_1})\sigma_2+$$
$$12\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^3+(\overline{\sigma_3}^2\overline{\sigma_1}^2-16\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}+\overline{\sigma_2}^4+30\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(-2\overline{\sigma_2\sigma_1}^3-14\overline{\sigma_3\sigma_1}^2+4\overline{\sigma_2}^2\overline{\sigma_1}+12\overline{\sigma_3\sigma_2})\sigma_3+4\overline{\sigma_1}^2)z^2+$$
$$((-4\sigma_1^4+(2\overline{\sigma_1}\sigma_2-4\overline{\sigma_2}\sigma_3)\sigma_1^3+((2\overline{\sigma_3}\sigma_3+22)\sigma_2-6\overline{\sigma_1}\sigma_3)\sigma_1^2+(-8\overline{\sigma_1}\sigma_2^2+18\overline{\sigma_2}\sigma_3\sigma_2+18\overline{\sigma_3}\sigma_3^2-18\sigma_3)\sigma_1+$$
$$(-12\overline{\sigma_3}\sigma_3-24)\sigma_2^2+36\overline{\sigma_1}\sigma_3\sigma_2-54\overline{\sigma_2}\sigma_3^2)\overline{z}^5+(-10\overline{\sigma_1}\sigma_1^4+((6\overline{\sigma_1}^2-4\overline{\sigma_2})\sigma_2+(-10\overline{\sigma_2\sigma_1}+10\overline{\sigma_3})\sigma_3+2)\sigma_1^3+$$
$$(4\overline{\sigma_3}\sigma_2^2+(-2\overline{\sigma_3\sigma_1}\sigma_3+52\overline{\sigma_1})\sigma_2+10\overline{\sigma_3\sigma_2}\sigma_3^2+(-12\overline{\sigma_1}^2-28\overline{\sigma_2})\sigma_3)\sigma_1^2+((-24\overline{\sigma_1}^2+20\overline{\sigma_2})\sigma_2^2+(-4\overline{\sigma_3}^2\sigma_3^2+$$
$$(52\overline{\sigma_2\sigma_1}+26\overline{\sigma_3})\sigma_3-4)\sigma_2+(30\overline{\sigma_3\sigma_1}-18\overline{\sigma_2}^2)\sigma_3^2+6\overline{\sigma_1}\sigma_3)\sigma_1-24\overline{\sigma_3}\sigma_2^3+(-16\overline{\sigma_3\sigma_1}\sigma_3-56\overline{\sigma_1})\sigma_2^2+(6\overline{\sigma_3\sigma_2}\sigma_3^2+$$
$$(72\overline{\sigma_1}^2-24\overline{\sigma_2})\sigma_3)\sigma_2-18\overline{\sigma_3}^2\sigma_3^3+(-90\overline{\sigma_2\sigma_1}-72\overline{\sigma_3})\sigma_3^2-72\sigma_3)\overline{z}^4+((-8\overline{\sigma_1}^2-8\overline{\sigma_2})\sigma_1^4+((6\overline{\sigma_1}^3-4\overline{\sigma_2\sigma_1}+$$
$$8\overline{\sigma_3})\sigma_2+(-8\overline{\sigma_2\sigma_1}^2+16\overline{\sigma_3\sigma_1}-8\overline{\sigma_2}^2)\sigma_3+14\overline{\sigma_1})\sigma_1^3+(4\overline{\sigma_3\sigma_1}\sigma_2^2+((-10\overline{\sigma_3\sigma_1}^2+20\overline{\sigma_3\sigma_2})\sigma_3+34\overline{\sigma_1}^2+20\overline{\sigma_2})\sigma_2+$$
$$(16\overline{\sigma_3\sigma_2\sigma_1}-8\overline{\sigma_3}^2)\sigma_3^2+(-6\overline{\sigma_1}^3-48\overline{\sigma_2\sigma_1}+10\overline{\sigma_3})\sigma_3-20)\sigma_1^2+((-12\overline{\sigma_3}^2\sigma_3-24\overline{\sigma_1}^3+34\overline{\sigma_2\sigma_1})\sigma_2^2+(2\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3^2+$$
$$(50\overline{\sigma_2\sigma_1}^2+52\overline{\sigma_3\sigma_1}-4\overline{\sigma_2}^2)\sigma_3-6\overline{\sigma_1})\sigma_2-8\overline{\sigma_3}^2\overline{\sigma_2}\sigma_3^3+(12\overline{\sigma_3\sigma_1}^2-26\overline{\sigma_2}^2\overline{\sigma_1}+4\overline{\sigma_3\sigma_2})\sigma_3^2+(54\overline{\sigma_1}^2-32\overline{\sigma_2})\sigma_3)\sigma_1-$$
$$52\overline{\sigma_3\sigma_1}\sigma_2^3+((4\overline{\sigma_3\sigma_1}^2+22\overline{\sigma_3\sigma_2})\sigma_3-44\overline{\sigma_1}^2-16\overline{\sigma_2})\sigma_2^2+(2\overline{\sigma_3}^3\sigma_3^3+(-10\overline{\sigma_3\sigma_2\sigma_1}-42\overline{\sigma_3}^2)\sigma_3^2+(36\overline{\sigma_1}^3+$$
$$28\overline{\sigma_2\sigma_1}-138\overline{\sigma_3})\sigma_3+16)\sigma_2+(-6\overline{\sigma_3}^2\overline{\sigma_1}+6\overline{\sigma_3\sigma_2}^2)\sigma_3^3+(-36\overline{\sigma_2\sigma_1}^2-78\overline{\sigma_3\sigma_1}-60\overline{\sigma_2}^2)\sigma_3^2-132\overline{\sigma_1}\sigma_3)\overline{z}^3+$$
$$((-2\overline{\sigma_1}^3-12\overline{\sigma_2\sigma_1})\sigma_1^4+((2\overline{\sigma_1}^4+4\overline{\sigma_2\sigma_1}^2+12\overline{\sigma_3\sigma_1}-8\overline{\sigma_2}^2)\sigma_2+(-2\overline{\sigma_2\sigma_1}^3+6\overline{\sigma_3\sigma_1}^2-12\overline{\sigma_2}^2\overline{\sigma_1}+12\overline{\sigma_3\sigma_2})\sigma_3+$$
$$16\overline{\sigma_1}^2+8\overline{\sigma_2})\sigma_1^3+((-4\overline{\sigma_3\sigma_1}^2+16\overline{\sigma_3\sigma_2})\sigma_2^2+((-6\overline{\sigma_3\sigma_1}^3+16\overline{\sigma_3\sigma_2\sigma_1}-12\overline{\sigma_3}^2)\sigma_3+32\overline{\sigma_2\sigma_1})\sigma_2+(6\overline{\sigma_3\sigma_2\sigma_1}^2-$$
$$6\overline{\sigma_3}^2\overline{\sigma_1}+12\overline{\sigma_3\sigma_2}^2)\sigma_3^2+(-16\overline{\sigma_2\sigma_1}^2+26\overline{\sigma_3\sigma_1}-32\overline{\sigma_2}^2)\sigma_3-38\overline{\sigma_1})\sigma_1^2+(-8\overline{\sigma_3}^2\sigma_2^3+(-4\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3-8\overline{\sigma_1}^4+$$
$$8\overline{\sigma_2\sigma_1}^2-10\overline{\sigma_3\sigma_1}+20\overline{\sigma_2}^2)\sigma_2^2+((6\overline{\sigma_3}^2\overline{\sigma_1}^2-20\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(16\overline{\sigma_2\sigma_1}^3+34\overline{\sigma_3\sigma_1}^2+8\overline{\sigma_2}^2\overline{\sigma_1}+20\overline{\sigma_3\sigma_2})\sigma_3+14\overline{\sigma_1}^2-$$
$$36\overline{\sigma_2})\sigma_2+(-6\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+2\overline{\sigma_3}^3)\sigma_3^3+(-8\overline{\sigma_2}^2\overline{\sigma_1}^2-2\overline{\sigma_3\sigma_2\sigma_1}-16\overline{\sigma_2}^3-42\overline{\sigma_3}^2)\sigma_3^2+(30\overline{\sigma_1}^3+14\overline{\sigma_2\sigma_1}-138\overline{\sigma_3})\sigma_3+$$
$$16)\sigma_1+(-32\overline{\sigma_3\sigma_1}^2-20\overline{\sigma_3\sigma_2})\sigma_2^3+(8\overline{\sigma_3}^3\sigma_3^2+(8\overline{\sigma_3\sigma_1}^3+40\overline{\sigma_3\sigma_2\sigma_1}-10\overline{\sigma_3}^2)\sigma_3-16\overline{\sigma_1}^3-16\overline{\sigma_2\sigma_1}+20\overline{\sigma_3})\sigma_2^2+$$
$$(-2\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^3+(-16\overline{\sigma_3\sigma_2\sigma_1}^2-34\overline{\sigma_3}^2\overline{\sigma_1}-8\overline{\sigma_3\sigma_2}^2)\sigma_3^2+(60\overline{\sigma_2\sigma_1}^2-142\overline{\sigma_3\sigma_1}-40\overline{\sigma_2}^2)\sigma_3+16\overline{\sigma_1})\sigma_2+2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^4+$$
$$(8\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}+18\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^3+(-30\overline{\sigma_3\sigma_1}^2-44\overline{\sigma_2}^2\overline{\sigma_1}-6\overline{\sigma_3\sigma_2})\sigma_3^2+(-96\overline{\sigma_1}^2+40\overline{\sigma_2})\sigma_3)\overline{z}^2+((-4\overline{\sigma_2\sigma_1}^2-4\overline{\sigma_2}^2)\sigma_1^4+$$
$$((4\overline{\sigma_2\sigma_1}^3+4\overline{\sigma_3\sigma_1}^2-6\overline{\sigma_2}^2\overline{\sigma_1}+8\overline{\sigma_3\sigma_2})\sigma_2+(-4\overline{\sigma_2}^2\overline{\sigma_1}^2+8\overline{\sigma_3\sigma_2\sigma_1}-4\overline{\sigma_2}^3)\sigma_3+4\overline{\sigma_1}^3+18\overline{\sigma_2\sigma_1}-24\overline{\sigma_3})\sigma_1^3+$$
$$((-4\overline{\sigma_3\sigma_1}^3+12\overline{\sigma_3\sigma_2\sigma_1}-4\overline{\sigma_3}^2)\sigma_2^2+((-4\overline{\sigma_3\sigma_2\sigma_1}^2-8\overline{\sigma_3}^2\overline{\sigma_1}+18\overline{\sigma_3\sigma_2}^2)\sigma_3-4\overline{\sigma_1}^4+8\overline{\sigma_2\sigma_1}^2+26\overline{\sigma_3\sigma_1}-2\overline{\sigma_2}^2)\sigma_2+$$
$$(8\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}-4\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(4\overline{\sigma_2\sigma_1}^3+10\overline{\sigma_3\sigma_1}^2-18\overline{\sigma_2}^2\overline{\sigma_1}-18\overline{\sigma_3\sigma_2})\sigma_3-24\overline{\sigma_1}^2+4\overline{\sigma_2})\sigma_1^2+(-6\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^3+$$
$$((8\overline{\sigma_3}^2\overline{\sigma_1}^2-24\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3-6\overline{\sigma_2\sigma_1}^3-18\overline{\sigma_3\sigma_1}^2+22\overline{\sigma_2}^2\overline{\sigma_1})\sigma_2^2+((-4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+4\overline{\sigma_3}^3)\sigma_3^2+(8\overline{\sigma_3\sigma_1}^3+12\overline{\sigma_2}^2\overline{\sigma_1}^2+$$
$$52\overline{\sigma_3\sigma_2\sigma_1}-22\overline{\sigma_2}^3-26\overline{\sigma_3}^2)\sigma_3+18\overline{\sigma_1}^3-42\overline{\sigma_2\sigma_1}+4\overline{\sigma_3})\sigma_2-4\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^3+(-8\overline{\sigma_3\sigma_2\sigma_1}^2+(-6\overline{\sigma_2}^3-32\overline{\sigma_3}^2)\overline{\sigma_1}-$$
$$14\overline{\sigma_3\sigma_2}^2)\sigma_3^2+(26\overline{\sigma_2\sigma_1}^2-128\overline{\sigma_3\sigma_1}+18\overline{\sigma_2}^2)\sigma_3+16\overline{\sigma_1})\sigma_1+(10\overline{\sigma_3}^3\sigma_3-4\overline{\sigma_3\sigma_1}^3-22\overline{\sigma_3\sigma_2\sigma_1}+2\overline{\sigma_3}^2)\sigma_2^3+$$
$$(-4\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3^2+(18\overline{\sigma_3\sigma_2\sigma_1}^2-14\overline{\sigma_3}^2\overline{\sigma_1}+22\overline{\sigma_3\sigma_2}^2)\sigma_3-4\overline{\sigma_1}^4-2\overline{\sigma_2\sigma_1}^2+18\overline{\sigma_3\sigma_1})\sigma_2^2+(4\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^3+(-4\overline{\sigma_3}^2\overline{\sigma_1}^2-$$
$$24\overline{\sigma_3\sigma_2}^2\overline{\sigma_1}+4\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2+(8\overline{\sigma_2\sigma_1}^3-18\overline{\sigma_3\sigma_1}^2-98\overline{\sigma_3\sigma_2})\sigma_3+4\overline{\sigma_1}^2)\sigma_2+(4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1}+10\overline{\sigma_3\sigma_2}^3+18\overline{\sigma_3}^3)\sigma_3^3+$$
$$(-4\overline{\sigma_2}^2\overline{\sigma_1}^2-26\overline{\sigma_3\sigma_2\sigma_1}+2\overline{\sigma_2}^3+72\overline{\sigma_3}^2)\sigma_3^2+(-24\overline{\sigma_1}^3+4\overline{\sigma_2\sigma_1}+72\overline{\sigma_3})\sigma_3)\overline{z}$$

$$
-2\overline{\sigma_2}^2\overline{\sigma_1}\sigma_1^4 + ((2\overline{\sigma_2}^2\overline{\sigma_1}^2 + 4\overline{\sigma_3\sigma_2\sigma_1} - 4\overline{\sigma_2}^3)\sigma_2 + (-2\overline{\sigma_2}^3\overline{\sigma_1} + 2\overline{\sigma_3\sigma_2}^2)\sigma_3 + 4\overline{\sigma_2\sigma_1}^2 - 12\overline{\sigma_3\sigma_1} + 6\overline{\sigma_2}^2)\sigma_1^3 +
((-4\overline{\sigma_3\sigma_2\sigma_1}^2 - 2\overline{\sigma_3}^2\overline{\sigma_1} + 12\overline{\sigma_3\sigma_2}^2)\sigma_2^2 + ((2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 4\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2\sigma_1}^3 + 20\overline{\sigma_3\sigma_1}^2 + 4\overline{\sigma_2}^2\overline{\sigma_1} - 28\overline{\sigma_3\sigma_2})\sigma_2 +
2\overline{\sigma_3\sigma_2}^3\sigma_3^2 + (4\overline{\sigma_2}^2\overline{\sigma_1}^2 - 10\overline{\sigma_3\sigma_2\sigma_1} - 4\overline{\sigma_2}^3 + 12\overline{\sigma_3}^2)\sigma_3 - 2\overline{\sigma_1}^3 - 10\overline{\sigma_2\sigma_1} + 24\overline{\sigma_3})\sigma_1^2 + ((2\overline{\sigma_3}^2\overline{\sigma_1}^2 - 12\overline{\sigma_3}^2\overline{\sigma_2})\sigma_2^3 +
((2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 2\overline{\sigma_3}^3)\sigma_3 - 8\overline{\sigma_3\sigma_1}^3 + 2\overline{\sigma_2}^2\overline{\sigma_1}^2 + 6\overline{\sigma_3\sigma_2\sigma_1} + 22\overline{\sigma_3}^2)\sigma_2^2 + (-4\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^2 + (24\overline{\sigma_3\sigma_2\sigma_1}^2 + (-4\overline{\sigma_2}^3 -
38\overline{\sigma_3}^2)\overline{\sigma_1} - 6\overline{\sigma_3\sigma_2}^2)\sigma_3 + 2\overline{\sigma_1}^4 - 22\overline{\sigma_3\sigma_1})\sigma_2 + (-16\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} + 2\overline{\sigma_2}^4 + 6\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (-2\overline{\sigma_2\sigma_1}^3 - 16\overline{\sigma_3\sigma_1}^2 +
12\overline{\sigma_2}^2\overline{\sigma_1} - 6\overline{\sigma_3\sigma_2})\sigma_3 + 4\overline{\sigma_1}^2)\sigma_1 + 4\overline{\sigma_3}^3\sigma_2^4 + (-2\overline{\sigma_3}^3\overline{\sigma_1}\sigma_3 - 2\overline{\sigma_3\sigma_2\sigma_1}^2 - 10\overline{\sigma_3}^2\overline{\sigma_1})\sigma_2^3 + (2\overline{\sigma_3}^3\overline{\sigma_2}\sigma_3^2 + (8\overline{\sigma_3}^2\overline{\sigma_1}^2 +
4\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} + 10\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_2\sigma_1}^3 + 10\overline{\sigma_3\sigma_1}^2)\sigma_2^2 + ((-20\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} - 2\overline{\sigma_3\sigma_2}^3 + 18\overline{\sigma_3}^3)\sigma_3^2 + (-2\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_2}^2\overline{\sigma_1}^2 -
28\overline{\sigma_3\sigma_2\sigma_1} - 18\overline{\sigma_3}^2)\sigma_3)\sigma_2 + 12\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^3 + (2\overline{\sigma_3\sigma_2\sigma_1}^2 + (-2\overline{\sigma_2}^3 + 18\overline{\sigma_3}^2)\overline{\sigma_1} + 6\overline{\sigma_3\sigma_2}^2)\sigma_3^2 + (-8\overline{\sigma_2\sigma_1}^2 +
36\overline{\sigma_3\sigma_1})\sigma_3)z + (-4\sigma_3\sigma_1^3 + \sigma_2^2\sigma_1^2 + 18\sigma_3\sigma_2\sigma_1 - 4\sigma_2^3 - 27\sigma_3^2)\overline{z}^6 + ((-2\sigma_2 - 12\overline{\sigma_1}\sigma_3)\sigma_1^3 + (4\overline{\sigma_1}\sigma_2^2 - 2\overline{\sigma_2}\sigma_3\sigma_2 +
12\overline{\sigma_3}\sigma_3^2 + 6\sigma_3)\sigma_1^2 + ((-2\overline{\sigma_3}\sigma_3 + 8)\sigma_2^2 + 54\overline{\sigma_1}\sigma_3\sigma_2 - 18\overline{\sigma_2}\sigma_3^2)\sigma_1 - 16\overline{\sigma_1}\sigma_2^3 + 12\overline{\sigma_2}\sigma_3\sigma_2^2 + (-18\overline{\sigma_3}\sigma_3^2 - 36\sigma_3)\sigma_2 -
54\overline{\sigma_1}\sigma_3^2)\overline{z}^5 + (\sigma_1^4 + (-6\overline{\sigma_1}\sigma_2 + (-12\overline{\sigma_1}^2 - 10\overline{\sigma_2})\sigma_3)\sigma_1^3 + ((6\overline{\sigma_1}^2 + 2\overline{\sigma_2})\sigma_2^2 + ((-6\overline{\sigma_2\sigma_1} + 16\overline{\sigma_3})\sigma_3 - 4)\sigma_2 +
(24\overline{\sigma_3\sigma_1} + \overline{\sigma_2}^2)\sigma_3^2 + 24\overline{\sigma_1}\sigma_3)\sigma_1^2 + (-2\overline{\sigma_3}\sigma_2^3 + (-6\overline{\sigma_3\sigma_1}\sigma_3 + 22\overline{\sigma_1})\sigma_2^2 + (4\overline{\sigma_3\sigma_2}\sigma_3^2 + (54\overline{\sigma_1}^2 + 20\overline{\sigma_2})\sigma_3)\sigma_2 -
12\overline{\sigma_3}^2\sigma_3^3 + (-36\overline{\sigma_2\sigma_1} - 30\overline{\sigma_3})\sigma_3^2 - 12\sigma_3)\sigma_1 + (-24\overline{\sigma_1}^2 - 4\overline{\sigma_2})\sigma_2^3 + (\overline{\sigma_3}^2\sigma_3^2 + (36\overline{\sigma_2\sigma_1} - 14\overline{\sigma_3})\sigma_3 + 4)\sigma_2^2 +
((-36\overline{\sigma_3\sigma_1} - 12\overline{\sigma_2}^2)\sigma_3^2 - 90\overline{\sigma_1}\sigma_3)\sigma_2 + 18\overline{\sigma_3\sigma_2}\sigma_3^3 + (-27\overline{\sigma_1}^2 - 18\overline{\sigma_2})\sigma_3^2)\overline{z}^4 + (2\overline{\sigma_1}\sigma_1^4 + ((-6\overline{\sigma_1}^2 - 4\overline{\sigma_2})\sigma_2 +
(-4\overline{\sigma_1}^3 - 20\overline{\sigma_2\sigma_1} - 2\overline{\sigma_3})\sigma_3)\sigma_1^3 + ((4\overline{\sigma_1}^3 + 6\overline{\sigma_2\sigma_1} + 4\overline{\sigma_3})\sigma_2^2 + ((-6\overline{\sigma_2\sigma_1}^2 + 32\overline{\sigma_3\sigma_1} - 4\overline{\sigma_2}^2)\sigma_3 - 4\overline{\sigma_1})\sigma_2 +
(12\overline{\sigma_3\sigma_1}^2 + 2\overline{\sigma_2}^2\overline{\sigma_1} + 20\overline{\sigma_3\sigma_2})\sigma_3^2 + (30\overline{\sigma_1}^2 + 16\overline{\sigma_2})\sigma_3)\sigma_1^2 + (-6\overline{\sigma_3\sigma_1}\sigma_2^3 + ((-6\overline{\sigma_3\sigma_1}^2 + 2\overline{\sigma_3\sigma_2})\sigma_3 + 18\overline{\sigma_1}^2 +
8\overline{\sigma_2})\sigma_2^2 + ((8\overline{\sigma_3\sigma_2\sigma_1} - 26\overline{\sigma_3}^2)\sigma_3^2 + (18\overline{\sigma_1}^3 + 44\overline{\sigma_2\sigma_1} - 20\overline{\sigma_3})\sigma_3 - 8)\sigma_2 + (-12\overline{\sigma_3}^2\overline{\sigma_1} - 2\overline{\sigma_3\sigma_2}^2)\sigma_3^3 + (-18\overline{\sigma_2\sigma_1}^2 -
54\overline{\sigma_3\sigma_1} - 28\overline{\sigma_2}^2)\sigma_3^2 - 42\overline{\sigma_1}\sigma_3)\sigma_1 + (2\overline{\sigma_3}^2\sigma_3 - 16\overline{\sigma_1}^3 - 12\overline{\sigma_2\sigma_1})\sigma_2^3 + (2\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3^2 + (36\overline{\sigma_2\sigma_1}^2 - 26\overline{\sigma_3\sigma_1} + 12\overline{\sigma_2}^2)\sigma_3 +
12\overline{\sigma_1})\sigma_2^2 + (-2\overline{\sigma_3}^2\overline{\sigma_2}\sigma_3^3 + (-18\overline{\sigma_3\sigma_1}^2 - 24\overline{\sigma_2}^2\overline{\sigma_1} - 8\overline{\sigma_3\sigma_2})\sigma_3^2 + (-72\overline{\sigma_1}^2 - 44\overline{\sigma_2})\sigma_3)\sigma_2 + 4\overline{\sigma_3}^3\sigma_3^4 + (18\overline{\sigma_3\sigma_2\sigma_1} +
4\overline{\sigma_2}^3 + 24\overline{\sigma_3}^2)\sigma_3^3 - 6\overline{\sigma_3}\sigma_3^2 + 32\sigma_3)\overline{z}^3 + ((\overline{\sigma_1}^2 + 2\overline{\sigma_2})\sigma_1^4 + ((-2\overline{\sigma_1}^3 - 8\overline{\sigma_2\sigma_1} - 2\overline{\sigma_3})\sigma_2 + (-10\overline{\sigma_2\sigma_1}^2 - 2\overline{\sigma_3\sigma_1} -
8\overline{\sigma_2}^2)\sigma_3 - 2\overline{\sigma_1})\sigma_1^3 + ((\overline{\sigma_1}^4 + 6\overline{\sigma_2\sigma_1}^2 + 8\overline{\sigma_3\sigma_1} + \overline{\sigma_2}^2)\sigma_2^2 + ((-2\overline{\sigma_2\sigma_1}^3 + 16\overline{\sigma_3\sigma_1}^2 - 8\overline{\sigma_2}^2\overline{\sigma_1} + 24\overline{\sigma_3\sigma_2})\sigma_3 + 4\overline{\sigma_1}^2 -
4\overline{\sigma_2})\sigma_2 + (\overline{\sigma_2}^2\overline{\sigma_1}^2 + 20\overline{\sigma_3\sigma_2\sigma_1} + 2\overline{\sigma_2}^3 + \overline{\sigma_3}^2)\sigma_3^2 + (12\overline{\sigma_1}^3 + 36\overline{\sigma_2\sigma_1} - 14\overline{\sigma_3})\sigma_3 + 4)\sigma_1^2 + ((-6\overline{\sigma_3\sigma_1}^2 - 2\overline{\sigma_3\sigma_2})\sigma_2^3 +
((-2\overline{\sigma_3\sigma_1}^3 + 4\overline{\sigma_3\sigma_2\sigma_1} - 16\overline{\sigma_3}^2)\sigma_3 + 2\overline{\sigma_1}^3 + 14\overline{\sigma_2\sigma_1} + 4\overline{\sigma_3})\sigma_2^2 + ((4\overline{\sigma_3\sigma_2\sigma_1}^2 - 26\overline{\sigma_3}^2\overline{\sigma_1} + 2\overline{\sigma_3\sigma_2}^2)\sigma_3^2 + (28\overline{\sigma_2\sigma_1}^2 -
30\overline{\sigma_3\sigma_1} + 2\overline{\sigma_2}^2)\sigma_3 - 16\overline{\sigma_1})\sigma_2 + (-2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 10\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^3 + (-24\overline{\sigma_3\sigma_1}^2 - 30\overline{\sigma_2}^2\overline{\sigma_1} - 40\overline{\sigma_3\sigma_2})\sigma_3^2 + (-42\overline{\sigma_1}^2 -
4\overline{\sigma_2})\sigma_3)\sigma_1 + \overline{\sigma_3}^2\sigma_2^4 + (4\overline{\sigma_3}^2\overline{\sigma_1}\sigma_3 - 4\overline{\sigma_1}^4 - 12\overline{\sigma_2\sigma_1}^2 - 2\overline{\sigma_3\sigma_1})\sigma_2^3 + ((\overline{\sigma_3}^2\overline{\sigma_1}^2 - 4\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (12\overline{\sigma_2\sigma_1}^3 - 10\overline{\sigma_3\sigma_1}^2 +
24\overline{\sigma_2}^2\overline{\sigma_1} - 14\overline{\sigma_3\sigma_2})\sigma_3 + 13\overline{\sigma_1}^2)\sigma_2^2 + ((-2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 12\overline{\sigma_3}^3)\sigma_3^3 + (-12\overline{\sigma_2}^2\overline{\sigma_1}^2 - 12\overline{\sigma_3\sigma_2\sigma_1} - 12\overline{\sigma_2}^3 + 30\overline{\sigma_3}^2)\sigma_3^2 +
(-18\overline{\sigma_1}^3 - 70\overline{\sigma_2\sigma_1} + 12\overline{\sigma_3})\sigma_3)\sigma_2 + \overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^4 + ((4\overline{\sigma_2}^3 + 12\overline{\sigma_3}^2)\overline{\sigma_1} + 22\overline{\sigma_3\sigma_2}^2)\sigma_3^3 + (18\overline{\sigma_2\sigma_1}^2 - 6\overline{\sigma_3\sigma_1} + 13\overline{\sigma_2}^2)\sigma_3^2 +
48\overline{\sigma_1}\sigma_3)\overline{z}^2 + (2\overline{\sigma_2\sigma_1}\sigma_1^4 + ((-4\overline{\sigma_2\sigma_1}^2 - 2\overline{\sigma_3\sigma_1} - 2\overline{\sigma_2}^2)\sigma_2 + (-8\overline{\sigma_2}^2\overline{\sigma_1} - 2\overline{\sigma_3\sigma_2})\sigma_3 - 2\overline{\sigma_1}^2)\sigma_1^3 + ((2\overline{\sigma_2\sigma_1}^3 +
4\overline{\sigma_3\sigma_1}^2 + 2\overline{\sigma_2}^2\overline{\sigma_1} + 4\overline{\sigma_3\sigma_2})\sigma_2^2 + ((-4\overline{\sigma_2}^2\overline{\sigma_1}^2 + 24\overline{\sigma_3\sigma_2\sigma_1} - 2\overline{\sigma_2}^3 + 2\overline{\sigma_3}^2)\sigma_3 + 4\overline{\sigma_1}^3 - 8\overline{\sigma_3})\sigma_2 + (2\overline{\sigma_2}^3\overline{\sigma_1} +
8\overline{\sigma_3\sigma_2}^2)\sigma_3^2 + (20\overline{\sigma_2\sigma_1}^2 - 16\overline{\sigma_3\sigma_1} + 10\overline{\sigma_2}^2)\sigma_3 + 4\overline{\sigma_1})\sigma_1^2 + ((-2\overline{\sigma_3\sigma_1}^3 - 4\overline{\sigma_3\sigma_2\sigma_1} - 2\overline{\sigma_3}^2)\sigma_2^3 + ((2\overline{\sigma_3\sigma_2\sigma_1}^2 -
16\overline{\sigma_3}^2\overline{\sigma_1} + 4\overline{\sigma_3\sigma_2}^2)\sigma_3 - 2\overline{\sigma_1}^4 + 4\overline{\sigma_2\sigma_1}^2 + 12\overline{\sigma_3\sigma_1})\sigma_2^2 + ((2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 20\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (4\overline{\sigma_2\sigma_1}^3 - 10\overline{\sigma_3\sigma_1}^2 + 6\overline{\sigma_2}^2\overline{\sigma_1} -
28\overline{\sigma_3\sigma_2})\sigma_3 - 10\overline{\sigma_1}^2)\sigma_2 - 2\overline{\sigma_3\sigma_2}^3\sigma_3^3 + (-2\overline{\sigma_2}^2\overline{\sigma_1}^2 - 38\overline{\sigma_3\sigma_2\sigma_1} - 10\overline{\sigma_2}^3 + 18\overline{\sigma_3}^2)\sigma_3^2 + (-12\overline{\sigma_1}^3 - 22\overline{\sigma_2\sigma_1} +
36\overline{\sigma_3})\sigma_3)\sigma_1 + 2\overline{\sigma_3}^2\overline{\sigma_1}\sigma_2^4 + ((2\overline{\sigma_3}^2\overline{\sigma_1}^2 - 2\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2\sigma_1}^3 - 4\overline{\sigma_3\sigma_1}^2)\sigma_2^3 + ((-4\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 12\overline{\sigma_3}^3)\sigma_3^2 + (2\overline{\sigma_3\sigma_1}^3 +
12\overline{\sigma_2}^2\overline{\sigma_1}^2 - 6\overline{\sigma_3\sigma_2\sigma_1} + 6\overline{\sigma_3}^2)\sigma_3 + 6\overline{\sigma_1}^3)\sigma_2^2 + (2\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^3 + (-4\overline{\sigma_3\sigma_2\sigma_1}^2 + (-12\overline{\sigma_2}^3 + 6\overline{\sigma_3}^2)\overline{\sigma_1} + 10\overline{\sigma_3\sigma_2}^2)\sigma_3^2 +
(-28\overline{\sigma_2\sigma_1}^2 - 6\overline{\sigma_3\sigma_1})\sigma_3)\sigma_2 + (2\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} + 4\overline{\sigma_2}^4 + 18\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^3 + (12\overline{\sigma_3\sigma_1}^2 + 22\overline{\sigma_2}^2\overline{\sigma_1} - 18\overline{\sigma_3\sigma_2})\sigma_3^2 + 24\overline{\sigma_1}^2\sigma_3)\overline{z} +
\overline{\sigma_2}^2\sigma_1^4 + ((-2\overline{\sigma_2}^2\overline{\sigma_1} - 2\overline{\sigma_3\sigma_2})\sigma_2 - 2\overline{\sigma_2}^3\sigma_3 - 2\overline{\sigma_2\sigma_1} + 4\overline{\sigma_3})\sigma_1^3 + ((\overline{\sigma_2}^2\overline{\sigma_1}^2 + 4\overline{\sigma_3\sigma_2\sigma_1} + \overline{\sigma_3}^2)\sigma_2^2 + ((-2\overline{\sigma_2}^3\overline{\sigma_1} +
8\overline{\sigma_3\sigma_2}^2)\sigma_3 + 4\overline{\sigma_2\sigma_1}^2 - 10\overline{\sigma_3\sigma_1})\sigma_2 + \overline{\sigma_2}^4\sigma_3^2 + (8\overline{\sigma_2}^2\overline{\sigma_1} - 6\overline{\sigma_3\sigma_2})\sigma_3 + \overline{\sigma_1}^2)\sigma_1^2 + ((-2\overline{\sigma_3\sigma_2\sigma_1}^2 - 2\overline{\sigma_3}^2\overline{\sigma_1})\sigma_2^3 +
((4\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} - 10\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_2\sigma_1}^3 + 8\overline{\sigma_3\sigma_1}^2)\sigma_2^2 + (-2\overline{\sigma_3\sigma_2}^3\sigma_3^2 + (4\overline{\sigma_2}^2\overline{\sigma_1}^2 - 26\overline{\sigma_3\sigma_2\sigma_1} + 18\overline{\sigma_3}^2)\sigma_3 - 2\overline{\sigma_1}^3)\sigma_2 +
(-2\overline{\sigma_2}^3\overline{\sigma_1} - 6\overline{\sigma_3\sigma_2}^2)\sigma_3^2 + (-10\overline{\sigma_2\sigma_1}^2 + 18\overline{\sigma_3\sigma_1})\sigma_3)\sigma_1 + \overline{\sigma_3}^2\overline{\sigma_1}^2\sigma_2^4 + ((-2\overline{\sigma_3}^2\overline{\sigma_2\sigma_1} + 4\overline{\sigma_3}^3)\sigma_3 - 2\overline{\sigma_3\sigma_1}^3)\sigma_2^3 +
(\overline{\sigma_3}^2\overline{\sigma_2}^2\sigma_3^2 + (8\overline{\sigma_3\sigma_2\sigma_1}^2 - 6\overline{\sigma_3}^2\overline{\sigma_1})\sigma_3 + \overline{\sigma_1}^4)\sigma_2^2 + ((-10\overline{\sigma_3\sigma_2}^2\overline{\sigma_1} + 18\overline{\sigma_3}^2\overline{\sigma_2})\sigma_3^2 + (-2\overline{\sigma_2\sigma_1}^3 - 6\overline{\sigma_3\sigma_1}^2)\sigma_3)\sigma_2 +
4\overline{\sigma_3\sigma_2}^3\sigma_3^3 + (\overline{\sigma_2}^2\overline{\sigma_1}^2 + 18\overline{\sigma_3\sigma_2\sigma_1} - 27\overline{\sigma_3}^2)\sigma_3^2 + 4\overline{\sigma_1}^3\sigma_3 = 0.
$$

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