Structure of the least square solutions and some problems in computer algebra

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Abstract

In this article, we study the structure of the least square solutions to over determined systems. We establish how to approximately solve overdetermined systems derived from practical requirement, which is our main result in this article. As applications of our main result, we take two examples. One is the rebar detection in RC structures and the other is GNSS positioning technique. We also pose some problems in computer algebra for further development of our main result.

1 Introduction

In this article, we first study the structure of solutions to the following overdetermined system

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n) = 0 \\ \cdots \\ F_m(x_1, x_2, \cdots, x_n) = 0 \end{cases} \tag{1}$$

where F_j's are continuous functions and $m \gg n$. Our main purpose in this article is to study the structure of the least square solutions to the system (1) with some errors.

The motivation of this study is based on the following system.

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r_1^2 \\ \cdots \\ (x - x_n)^2 + (y - y_n)^2 + (z - z_n)^2 = r_n^2 \end{cases} \tag{2}$$

In the system (2), we measure the distance between the object and the observation points, where (x, y, z) is the coordinates of the unknown object which we would like to reconstruct, (x_j, y_j, z_j)'s are the coordinates of the observation points and r_j's are measured distances. If the measurements are exact and they contain no errors, then the system must have the unique solution (x, y, z), however overdetermined the system is. It is, however, impossible for each equation to be exact. Therefore, the systems (1) and (2) must admit no solution if they are overdetermined. Even if the systems (1) and (2) admit no solution by errors and noises, the object really exists, where *the existence of the solution is proved by the phenomenon itself*. Therefore, we have to approximately reconstruct the object, which motivated us to study this problem.

In the next section, we study the solution to the system (1), whose application give the structure of the least square equations. In the third section, we introduce two application of the structure of the least square solutions; one is the rebar detection in RC structures and the other is GNSS positioning technique. In the fourth section, we pose some problems in computer algebra, for further development of our theory, from the viewpoint of both the theory and practice.

2 Structure of the least square solutions

In this section, we first study the structure of the least square solutions to the system (1), for which, we deform the system (1) by the procedures called semi-equivalent deformation.

Definition 1 (Semi-equivalent deformations)
In the system (1), we call the following procedures 'semi-equivalent deformations'.

(i) In the i-th equation, transpose some terms in the left hand side in $F_i(x_1, x_2, \cdots, x_n)$ to the right hand side.

(ii) Multiply both sides of the i-th equation and the counterparts of the j-th equation.

(iii) Add both sides of the i-th equation and the counterparts of the j-th equation multiplied by some constant ($\neq 0$).

Remark 1
Semi-equivalent deformations are procedures to solve the system (1). They are different from the procedures to generate the ideal by $F_1, F_2, \cdots F_n$, which is because of the procedure (i). For example, consider the system

$$\begin{cases} x^{30} + y - z^3 = 0 \\ x^{15} - y - z = 0 \end{cases} \tag{3}$$

of the three variables x, y, z. We transpose $-y$ and $-z$ in the second equation to the right hand side and take the square of the both hand sides, which is subtracted by first equation gives

$$z^3 - y^2 - z^2 - 2yz - y = 0. \tag{4}$$

The polynomial $z^3 - y^2 - z^2 - 2yz - y$ would not belong to the ideal generated by $x^{30} + y - z^3$ and $x^{15} - y - z$. By semi-equivalent deformations, a wider class is generated by $F_1, F_2, \cdots F_n$ than the ideal generated by $F_1, F_2, \cdots F_n$. If the semi-equivalent deformations are defined only by (ii) and (iii) then they are equivalent to the procedures to generate the ideal. The first author is very sorry for not mentioning the deformation (i) concretely in his oral representation.

The following properties are straightforward.

Proposition 2
If $\boldsymbol{x} \in \mathbb{R}^n$ is a solution to (1) then it is also a solution to a system derived from the system (1) by semi-equivalent deformations.

Proposition 3
If the solution to a system derived from the system (1) by equivalent deformations is unique then it must be the unique solution to (1). In this case, it is also proved that the system (1) has the unique solution.

It is not unusual in practice that the system (1) itself is an overdetermined system of linear equations, or an overdetermined system of linear equations is derived from the system (1) by semi-equivalent deformations. An an example of the former one is G. N. Hounsfield's overdetermined system of linear equations in the practicalization of the computerized tomography, for which confer [3, 4, 5]. As an example of the latter, we take (2), where if we subtract j-th equation from i-th equation then we obtain an overdetermined system of $\frac{m(m-1)}{2}$ linear equations. In this article, we focus on the case where the system (1) itself is an overdetermined system of linear equations, or an overdetermined system of linear equations is derived from the system (1) by semi-equivalent deformations.

Assume that a system of $k \geq m \gg n$ linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - s_1 &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - s_2 &= 0, \\ \cdots\cdots\cdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n - s_k &= 0, \end{cases} \tag{5}$$

or equivalently

$$A\boldsymbol{x} = \boldsymbol{s}, \tag{6}$$

is derived from the system (1) by equivalent deformations.

In practice, the system (1) itself, would not be obtained. By errors and noises, what we can obtain is the system (1) with a small error in each equation, consequently, we obtain a following overdetermined system (7) of linear equations;

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - s_1 &= \varepsilon_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - s_2 &= \varepsilon_2, \\ \cdots\cdots\cdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n - s_k &= \varepsilon_k \end{cases} \tag{7}$$

or equivalently

$$A\boldsymbol{x} = \widetilde{\boldsymbol{s}}, \tag{8}$$

where $\widetilde{s}_i = s_i + \varepsilon_i$ with small ε_i for $1 \leq i \leq k$.

If the derived overdetermined system (5) has the unique solution in the case where no errors and noises are included, then the original system (1) must also have the unique solution by Proposition 2, However, because of the errors and noises the system

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n) = \epsilon_1 \\ \cdots \\ F_m(x_1, x_2, \cdots, x_n) = \epsilon_m \end{cases} \tag{9}$$

with errors and noises becomes to allow no solution, consequently, neither does the derived linear system (7). In such cases, however, a suitable approximate solution to the system (7) must be an approximate solution to the system (9), which we shall study in terms of what is called *the least square solution*.

In this article, we assume the following;

Assumption 1
In the system (9), we assume the following; it is possible derive an overdetermined linear system (7) from the system (9) by semi-equivalent deformations, the rank of whose coefficient matrix is n.

By Assumption 1 we have the following proposition.

Proposition 4
The solution to the overdetermined system (9) is unique.

In order to construct an approximate solution, we study the overdetermined system (7). If the system (5) allows a solution $\boldsymbol{x}$ if and only if

$$A\boldsymbol{x} - \boldsymbol{s} = \boldsymbol{0}. \tag{10}$$

In view of (10), the least square solutions are defined.

Definition 5
A vector $\boldsymbol{x} \in \mathbb{R}^n$ is called as a least square solution to (5) (or (6)) if and only if it minimizes the norm

$$\|A\boldsymbol{x} - \boldsymbol{s}\| \tag{11}$$

In practice, it is usually possible for the system to satisfy Assumption 1 by taking observation number m to be very large. Therefore, the unique least square solution solution to the system (9) is obtained by the following theorem.

Theorem 6
If the rank of the coefficient matrix in the system (5) (or (6)), equivalently in the system (7) (or (8)), is the same as the number of the unknown variables n, then the unique least square solution to the system (9) is obtained by giving the unique solution to the system (7) (or (8))

$$ {}^tAA\boldsymbol{x} = {}^tA\widetilde{\boldsymbol{s}} \tag{12} $$

Confer [4, 5] for more in details about the least square solutions to the systems of linear equations. If we can deform the system (9) into the system (7) of linear equations, then we have a number of methods to analize and solve it, which is very convenient and helpful.

3 Applications

In this section, we take two application of our main results established in the previous section. One is the exact rebar probe in RC structures and the other is GNSS positioning technique.

3.1 Exact rebar probe

In this subsection, we discuss how to detect the exact position of the rebar in RC (reinforced concrete) structures. We first note that there being many researches to detect the rebar in RC structures, for example, [1, 6], there exists no non-destructive inspection technique to exactly probe the rebar. There are a number of advantages to probe the exact position of the rebar, among which, the best one may be that the knowledge of the exact position of the rebar enables us a non-destructive inspection of the concrete cover in RC structures by ultrasonic measurements, for which confer [2]. In the maintenance of RC structures, it is one of the most important tasks to keep the concrete cover in a sound state, in order to prevent the rebar from getting corroded.

For our exact rebar probe technique, we apply an apparatus to measure thickness of concrete cover for reinforcement by electromagnetic induction method. By this apparatus, we can measure the distance between the rebar and the observation point, however, the measurements by this apparatus are well known to be inexact because of the errors and the noises the apparatus contains. We take more data than the number of unknown variables and make an overdetermined system (2), whose least square solution equalize the errors and the noises to be almost zero. It is very unusual that the rebar imbedded in the structure is curved. Therefore, we assume that the rebar is straight, which yields that it is sufficient to reconstruct the exact position of the endpoints of the rebar.

Assume that a cuboid RC structure contains, in its interior, a rebar which locates parallelly or perpendicularly to each edge surface. Confer Figure 1 for its image.

In Figure 1, we set the axes as shown. For the reconstruction of the endpoint b in Figure 1, we take 6 to 8 observation points ($m = 6$ to 8)on the edge surface B and its around and measure the thickness of concrete cover, equivalently the distance between the observation and the endpoints. We let the endpoint $b = (x, y, z)$ and take n observation points $(x_1, y_1, z_1), \cdots, (x_n, y_n, z_n)$, whose distances from the endpoint b are measured as $r_1, \cdots, r_n$, respectively. We shall explain how to take the observation points in the real structure in the next section.

By measurement, we obtain an overdetermined system (2)of quadratic equations, however, it is well known that it is impossible to measure the thickness of concrete cover by the existing devices. Henceforth all measurements necessarily contain errors and noises. Therefore, we have make an approximate solution of the system (2) with errors and noises, where we can apply our main theorem, Theorem 6 since by subtracting i-th equation in system (2) by the j-th one and dividing the both hand sides of the difference

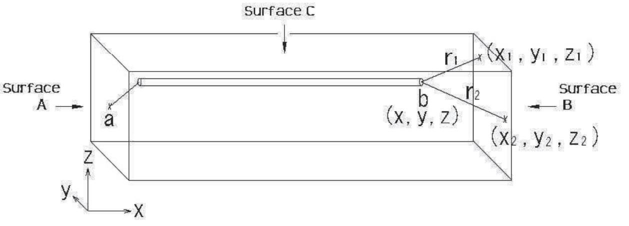

Figure 1: A cuboid test piece

by 2, we obtain the overdetermined system of $k = \frac{m(m-1)}{2}$ linear equations;

$$(x_i - x_j)x + (y_i - y_j)y + (z_i - z_j)z = \frac{1}{2}(x_i^2 + y_i^2 + z_i^2 - x_j^2 - y_j^2 - z_j^2 + r_j^2 - r_i^2). \tag{13}$$

For more detail of this analysis, as well as its practical background, confer the paper by Mr. Takabatake in these proceedings.

3.2 GNSS positioning technique

GNSS (global navigation satellite system) is a system that determines the geospatial location of GNSS receivers. The principle of GNSS is almost the same as the probe of the rebar. GNSS satellites are orbiting around the Earth and constantly transmit signals. A receiver on the ground measures the distance from each satellite using the signal and calculates its location, whose measurements are known not to be necessarily exact for several reasons. In order to determine the position of the receiver, we usually take 6 to 10 measurements by satellites, each of which measures the distance between the receiver and the satellite itself. Therefore, we have the same overdetermined system (13) of quadratic equations as the rebar probe. It is interesting that the number of observations are almost the same. The authors are studying to develop a simpler and cheaper-cost positioning technique with cheaper devices than the existing higher-cost precise techniques with expensive devices, by application of the least square equation to the system (7) or (8), which shall be discussed in our forthcoming paper.

4 Open problems in computer algebra

As we have studied in the second section, it is very convenient if the system (1) or (9) is deformed into a system of linear equation by semi-equivalent deformations. As a such example, we took the system (13) from the viewpoint of practice. As an interest from the theoretic viewpoint, it is interesting to determine the class of the system of polynomials which are deformed into systems of linear equations by semi-equivalent deformations.

Problem 1
Assume that the functions $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$ in the system (1) (or (9)) are polynomials. Give a necessary and sufficient condition on $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$ in order that the system (1) (or (9)) is deformed in the system of linear equations by semi-equivalent deformations.

We have a partial answer to Problem 1 as follows;

Proposition 7
If $k \geq m$ independent linear polynomials are included in the ideal generated by $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$ in the system (1) (or (9)), then the system (1) (or (9)) can be deformed in the system of linear equations by semi-equivalent deformations.

If the semi-equivalent deformations are defined by only (ii) and (iii) in Definition 1, then the semi-equivalent deformations are equivalent to the procedures to generate the ideal by the polynomials $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$, where Problem 1 is reduced to the problem whether the linear equations in the left hands side of the system (5) are contained in the ideal generated by the polynomials $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$. By this discussion, another problem arises.

Problem 2
Assume that the functions $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$ in the system (1) are polynomials. Is there any condition that makes the semi-equivalent deformations equivalent to the procedures to generate the ideal by the polynomials $F_1(x_1, x_2, \cdots, x_n), \cdots, F_m(x_1, x_2, \cdots, x_n)$?

Although giving a solution to Problem 1 is difficult, we also note that the solution to the following problem is of sufficient help for practical applications.

Problem 3
For a given overdetermined system of polynomials (1), judge whether it is deformed into a system of linear equations by the semi-equivalent deformations, in computer algebraic way.

Solutions to Problems 1, 2 and 3 will be of great help to generalize our main theory developed in the second section.

5 Summary

In this section, we make concluding remarks. The points in this paper are the followings.

- In order to obtain an approximate solution to an overdetermined system, it is very helpful to apply the idea of the least square solutions to. In application of the least square solutions, if the system deformed into the system of linear equations, then we can easily give the least square solution (Section 2).
- The least square solution to the overdetermined system (9) with errors and noises is obtained by giving the unique least square solution to the system (7) (or (8)), which is given by (12), under the condition that the rank the coefficient matrix of the derived linear system (7) (or (8)) from the system (9) by semi-equivalent deformation is equal to the number of unknown variables (Theorem 6).
- We introduced practical applications of our main theory (Section 3).
- In view of the first two conclusions, it is important to study whether the system is deformed into a linear one by semi-equivalent deformation. We have posed some problems in computer algebra concerning this deformation (Problems 1,2 and 3 in Section 4) and have given a partial answer (Proposition 7). Solutions of these problems are very helpful for further development.

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