On v-adic multiple zeta values in positive characteristic

YEN-TSUNG CHEN

Abstract. This is a survey article for the recent progress on the integrality of v-adic multiple zeta values which plays a characteristic p counterpart of Furusho's p-adic multiple zeta values.

1. Introduction

1.1. Furusho's p-adic multizeta values. Let $r \in \mathbb{N}$. An r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ is called an index, and is called *admissible* if $s_1 \geq 2$. Classical multiple zeta values, abbreviated as MZVs, are real numbers defined by

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}^{\times},$$

where $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ is an admissible index. We define $\operatorname{dep}(\mathfrak{s}) := r$ to be the *depth* of $\mathfrak{s}$, $\operatorname{wt}(\mathfrak{s}) := \sum_{i=1}^{r} s_i$ to be the *weight* of $\mathfrak{s}$ and $\operatorname{ht}(\mathfrak{s}) :=$ the cardinality of $\{i \mid s_i \neq 1\}$ to be the *height* of $\mathfrak{s}$.

In what follows, we briefly review the p-adic MZVs introduced by Furusho in [F04]. Consider the one-variable multiple polylogarithm

$$\mathrm{Li}_{\mathfrak{s}}(z) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{Q}[\![z]\!],$$

for admissible index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. We have

$$\zeta(\mathfrak{s}) = \mathrm{Li}_{\mathfrak{s}}(z) \mid_{z=1} .$$

We write $\mathrm{Li}_{\mathfrak{s}}(z)_p$ for the p-adic function defined by the same series as $\mathrm{Li}_{\mathfrak{s}}(z)$, but regarded p-adically. Then $\mathrm{Li}_{\mathfrak{s}}(z)_p$ converges on the open unit disk centered at 0 but does not be defined at $z = 1$. Furusho [F04] applied Coleman's integration theory [Col82] to p-adically analytically continue $\mathrm{Li}_{\mathfrak{s}}(z)_p$ to $\mathbb{C}_p \setminus \{1\}$, and then took a certain limit $z \to 1$ to define the p-adic MZV $\zeta(\mathfrak{s})_p$.

Furusho predicted that his p-adic MZVs are all p-adic integers and this prediction was independently proved by Akagi-Hirose-Yasuda and Chatzistamatiou.

Theorem 1.1.1 ([AHY, Cha17]). *Every p-adic MZV is a p-adic integer. Moreover, fix an admissible index $\mathfrak{s} \in \mathbb{N}^r$; then for all but finitely many primes p, the p-adic valuation of p-adic MZVs is greater or equal to the weight* $\operatorname{wt}(\mathfrak{s})$.

Date: February 27, 2020.

2010 *Mathematics Subject Classification.* Primary 11R58, 11M32.

Key words and phrases. Carlitz multiple star polylogarithms, multizeta values, integrality.

1.2. **Multizeta values in positive characteristic.** In what follows, we recall the basic setting of function fields in positive characteristic. For the purpose of this article, our basic arithmetic object is the polynomial ring $A := \mathbb{F}_q[\theta]$, where $\mathbb{F}_q$ is the finite field of q elements with characteristic p and θ is a variable. The field of fraction of A is denoted by $k := \mathbb{F}_q(\theta)$ and the completion of k at the infinite place is denoted by k_∞. For a finite place v, we set k_v to be the completion of k at v. Throughout this article, we fix an algebraic closure $\overline{k}$ together with fixed embeddings into $\mathbb{C}_\infty$ and $\mathbb{C}_v$ respectively, where $\mathbb{C}_\infty$ is the completion of a fixed algebraic closure of k_∞ and $\mathbb{C}_v$ is the completion of a fixed algebraic closure of k_v.

The function field analogue of real-valued MZVs is defined by Thakur in [T04], generalizing Carlitz zeta values [Ca35]. For *any* index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, the ∞-adic MZV is defined by the series

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty^\times, \tag{1.2.1}$$

where $(a_1, \ldots, a_r) \in A^r$ with a_i monic and $\deg_\theta a_i$ strictly decreasing. Since there are no natural orders on the set of monic polynomials in A, the non-vanishing property of ∞-adic MZVs is nontrivial although the classical real-valued counterparts are immediate consequence from the defining series. This non-vanishing property was proved in [T09].

Inspired by Furusho's p-adic MZVs, Chang and Mishiba considered the Carlitz multiple star polylogarithms, abbreviated as CMSPLs, as follows

$$\mathrm{Li}^\star_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[\![z_1, \cdots, z_r]\!], \tag{1.2.2}$$

where $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \geq 1$. It is known by [C14, Thm. 5.5.2], [CM19b, Thm. 5.2.5] that any ∞-adic MZV can be written as a k-linear combinations of CMSPLs at some precise integral points with explicit coefficients. Let v be a fixed finite place of k and we write $\mathrm{Li}^\star_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ for the v-adic function defined by the same series, but regarded v-adically. Then $\mathrm{Li}^\star_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ converges on a small region inside $\mathbb{C}_v^r$ but does not be defined at arbitrary integral point. Chang and Mishiba [CM19a, Prop. 4.1.1] used the logarithmic interpretation of CMSPLs to do the analytic continuation of CMSPLs v-adically and then defined v-adic MZVs in [CM19b, Def. 6.1.1] by using the same k-linear combinations of CMSPLs. We refer the reader to Definition 2.3.2 for details.

Inspired by Theorem 1.1.1, it is natural to ask the integrality question of v-adic MZVs. Recently, the author establishes a function field analogue of Theorem 1.1.1 and in this article we give a survey on this result as well as the essential ideas of the proof.

1.3. **Overview.** In Section 2, we first briefly review Anderson's theory of t-modules [A86]. Then we recall Chang-Mishiba's construction of v-adic MZVs [CM19b]. In Section 3, we state a function field analogue of Theorem 1.1.1 [Chen20, Thm. 4.2.1] and then sketch the main strategy of the proof.

Acknowledgement

The author would like to thank the organizer Hidezaku Furusho for the invitation to talk in RIMS. Also the author would like to thank C. Y. Chang and Y. Mishiba for many helpful comments and discussion. It was partially supported by MOST Grant 107-2628-M-007-002-MY4.

2. Reviews of characteristic p multizeta values

2.1. **Anderson's t-modules.** In this section, we quickly review the theory of t-modules introduced by Anderson [A86]. Let L/k be a field extension and τ be the Frobenius q-th power operator

$$\tau := (x \mapsto x^q) : L \to L.$$

This τ-action naturally extends to matrices by componentwise action. Let $L[\tau]$ be the non-commutative polynomial ring generated by τ subject to the relation

$$\tau\alpha = \alpha^q \tau \text{ for } \alpha \in L.$$

For any d-dimensional additive algebraic group $\mathbb{G}^d_{a/L}$ defined over L, one may identify the ring of $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}^d_{a/L}$ with $\mathrm{Mat}_d(L[\tau])$. We then define the operator

$$\partial := (\sum_{i \geq 0} \alpha_i \tau^i \mapsto \alpha_0) : \mathrm{Mat}_d(L[\tau]) \to \mathrm{Mat}_d(L).$$

The definition of t-modules is given as follows.

Definition 2.1.1. *Let $d \in \mathbb{N}$. A d-dimensional t-module over L is a pair $G = (\mathbb{G}^d_{a/L}, \rho)$, where*

$$\rho : \mathbb{F}_q[t] \to Mat_d(L[\tau])$$

is an $\mathbb{F}_q$-linear ring homomorphism so that $\partial\rho_t - \theta I_d$ is a nilpotent matrix.

The *exponential function* of G is an $\mathbb{F}_q$-linear power series of the form

$$\exp_G := I_d + \sum_{i \geq 1} Q_i \tau^i, Q_i \in \mathrm{Mat}_d(L).$$

It is the unique power series satisfying the property that

$$\exp_G \circ \partial\rho_a = \rho_a \circ \exp_G \text{ for all } a \in \mathbb{F}_q[t].$$

The *logarithm* of G denoted by $\log_G$, is defined to be the formal inverse of $\exp_G$. It is a $\mathbb{F}_q$-linear power series of the form

$$\log_G := I_d + \sum_{i \geq 1} P_i \tau^i, P_i \in \mathrm{Mat}_d(L),$$

satisfy that

$$\log_G \circ \rho_a = \partial\rho_a \circ \log_G \text{ for all } a \in \mathbb{F}_q[t].$$

2.2. **Formulae for ∞-adic MZVs.** In what follows, we introduce the connection between MZVs and CMSPLs in positive characteristic. Recall the MZVs are defined in (1.2.1) and the CMSPLs are defined in (1.2.2). Chang and Mishiba proved the following theorem [CM19b], generalizing the result of Anderson and Thakur [AT90] in the case $r = 1$.

Theorem 2.2.1 ([C14, Thm. 5.5.2], [CM19b, Thm. 5.2.5])**.** *For any depth $r \in \mathbb{N}$ and any index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, there are explicit tuples $\mathfrak{s}_\ell \in \mathbb{N}^{\mathrm{dep}(\mathfrak{s}_\ell)}$ with $\mathrm{wt}(\mathfrak{s}_\ell) = \mathrm{wt}(\mathfrak{s})$, $\mathrm{dep}(\mathfrak{s}_\ell) \leq \mathrm{dep}(\mathfrak{s}) = r$, explicit coefficients $c_\ell \in k$ and integral vectors $\mathbf{u}_\ell \in A^{\mathrm{dep}(\mathfrak{s}_\ell)}$ so that*

$$\zeta_A(\mathfrak{s}) = \sum_\ell c_\ell \cdot \mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell).$$

2.3. **Chang-Mishiba's v-adic multizeta values.** In what follows, we collect Chang and Mishiba's results on the logarithmic interpretation of CMSPLs and its v-adic analytic continuation.

Theorem 2.3.1 (cf. [CM19a, Thm. 3.3.3], [CM19a, Thm. 4.1.1]). *We set $|\cdot|_v$ to be the normalized v-adic absolute value so that $|v|_v = q^{-\deg_\theta v}$. For any depth $r \in \mathbb{N}$, index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in \overline{k}^r$, the following assertions hold.*

(1) *If $|u_1|_v < 1$ and $|u_j|_v \leq 1$ for $2 \leq j \leq r$, then there exists an explicit t-module $G_{\mathfrak{s},\mathbf{u}}$ defined over $\overline{k}$ and an explicit vector $\mathbf{v}_{\mathfrak{s},\mathbf{u}} \in G_{\mathfrak{s},\mathbf{u}}(\overline{k})$ such that $(-1)^{r-1} \cdot \mathrm{Li}^\star_{(s_r,\dots,s_1)}(u_r, \dots, u_1)_v$ appears in the* $\mathrm{wt}(\mathfrak{s})$*-th coordinate of* $\log_{G_{\mathfrak{s},\mathbf{u}}}(\mathbf{v}_{\mathfrak{s},\mathbf{u}})$.

(2) *If $|u_j|_v \leq 1$ for $1 \leq j \leq r$, then there exists $a(t) \in \mathbb{F}_q[t]$, depending on $\mathfrak{s}$ and $\mathbf{u}$, such that* $\log_{G_{\mathfrak{s},\mathbf{u}}}(\rho_a(\mathbf{v}_{\mathfrak{s},\mathbf{u}}))$ *converges v-adically, where $G_{\mathfrak{s},\mathbf{u}}$, $\mathbf{v}_{\mathfrak{s},\mathbf{u}}$ are the same as (1) and ρ_a is the image of $a(t)$ under the associated $\mathbb{F}_q$-linear ring homomorphism ρ of the t-module $G_{\mathfrak{s},\mathbf{u}}$.*

Theorem 2.3.1 implies that the following definition makes sense.

Definition 2.3.2 (cf. [CM19a, Def. 4.1.2], [CM19b, Def. 6.1.1]). *For any depth $r \in \mathbb{N}$, index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $(u_1, \dots, u_r) \in \overline{k}^r$, we define the following:*

(1) *If $|u_j|_v \leq 1$ for $1 \leq j \leq r$. Let $G_{\mathfrak{s},\mathbf{u}}$ and $\mathbf{v}_{\mathfrak{s},\mathbf{u}}$ be given in Theorem 2.3.1. Let $a(t) \in \mathbb{F}_q[t]$ satisfy that* $\log_{G_{\mathfrak{s},\mathbf{u}}}(\rho_a(\mathbf{v}_{\mathfrak{s},\mathbf{u}}))$ *converges v-adically. We define the v-adic Carlitz multiple star logarithm* $\mathrm{Li}^\star_{(s_r,\dots,s_1)}(u_r, \dots, u_1)_v$ *to be the value*
$$\frac{(-1)^r}{a(\theta)} \times \mathrm{wt}(\mathfrak{s})\text{-}th\ coordinate\ of\ \log_{G_{\mathfrak{s},\mathbf{u}}}(\rho_a(\mathbf{v}_{\mathfrak{s},\mathbf{u}})).$$

(2) *Let c_ℓ, $\mathfrak{s}_\ell$ and $\mathbf{u}_\ell$ be given in Theorem 2.2.1. We define the v-adic MZV*
$$\zeta_A(\mathfrak{s})_v := \sum_\ell c_\ell \cdot \mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v.$$

Remark 2.3.3. Theorem 2.3.1 guarantees the existence of $a(t) \in \mathbb{F}_q[t]$ in Definition 2.3.2. Moreover, Definition 2.3.2 is independent of the choices of $a(t) \in \mathbb{F}_q[t]$. We refer the reader to [CM19a, Rem. 4.1.3] for details.

3. Integrality of v-adic multizeta values

The aim of this section is to state a function field analogue of Theorem 1.1.1 and give a sketch of the ideas of the proof.

3.1. **Statement and example.**

Theorem 3.1.1 ([Chen20, Thm. 4.2.1]). *Fix a monic irreducible polynomial v of A. Let $\mathfrak{s} = (s_1, \cdots, s_r) \in \mathbb{N}^r$, $q_v := \#(A/vA)$, A_v be the valuation ring of k_v and $\mathrm{ord}_v(\cdot)$ be the associated valuation of k_v. If we set*
$$B_{w,v} := \min_{n \geq 0}\{q_v^n - n \cdot w\},$$
then we have
$$\mathrm{ord}_v(\zeta_A(\mathfrak{s})_v) \geq B_{\mathrm{wt}(\mathfrak{s}),v} - \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1}.$$
In particular,
$$\zeta_A(\mathfrak{s})_v \in A_v \ \ if\ q_v \geq \mathrm{wt}(\mathfrak{s}).$$

We provide a non-integral example when the restriction in the theorem is omitted. The example was found by using the computer algebra system SageMath. The author is grateful to Yoshinori Mishiba for providing the example.

Example 3.1.2. *Consider $q = 2$, $v = \theta$ and $\mathfrak{s} = (4, 1)$. In this case, we have*

$$\zeta_A(4,1)_\theta = \mathrm{Li}^\star_{(4,1)}(1,1)_\theta + \mathrm{Li}^\star_{(5)}(1)_\theta.$$

We will provide a strategy on calculation of v-adic CMSPLs in the next section. Consequently, we obtain

$$\zeta_A(4,1)_\theta = \theta^{-3} + \theta^2 + O(\theta^7) \notin A_\theta.$$

See [Chen20, Ex. 4.2.3] *for details.*

3.2. **Sketch of the proof.** In what follows, we provide the key ingredients of proof of Theorem 3.1.1. Let $w \in \mathbb{N}$ and

$$S_{w,v} := \{\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \mid r \in \mathbb{N},\ \mathfrak{s} \in \mathbb{N}^r,\ \mathrm{wt}(\mathfrak{s}) = w,\ \mathbf{u} \in A^r\}.$$

We define $\mathscr{L}_{w,v}$ to be the k-vector space spanned by elements in $S_{w,v}$. When $\mathrm{wt}(\mathfrak{s}) = w$, it is clear that

$$\zeta_A(\mathfrak{s})_v \in \mathscr{L}_{w,v}.$$

Due to the v-adic analytic continuation, some elements in $S_{w,v}$ do not coincide with the original power series expansion and thus it is not easy to study linear combinations of elements in $S_{w,v}$ even though we have already known the explicit coefficients.

Consider the subset of $S_{w,v}$, which is defined by

$$S^{(o)}_{w,v} := \{\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \mid r \in \mathbb{N},\ \mathfrak{s} \in \mathbb{N}^r,\ \mathrm{wt}(\mathfrak{s}) = w,\ \mathbf{u} \in vA \times A^{r-1}\} \subset S_{w,v}.$$

Let $\mathscr{L}^{(o)}_{w,v} \subset \mathscr{L}_{w,v}$ be the k-vector subspace spanned by elements in $S^{(o)}_{w,v}$. Since the elements in $S^{(o)}_{w,v}$ coincide with the original power series expansion, we can estimate the v-adic valuation of elements in $S^{(o)}_{w,v}$. Consequently, we are able to study v-adic valuation of elements in $\mathscr{L}^{(o)}_{w,v}$ if the coefficients are explicitly described.

Therefore, to prove Theorem 3.1.1 it suffices to show that $\zeta_A(\mathfrak{s})_v \in \mathscr{L}^{(o)}_{w,v}$. In fact, we can prove more about it.

Theorem 3.2.1 (cf. [Chen20, Cor. 3.2.11])**.** *We adopt the same notations as above. Then*

$$\mathscr{L}^{(o)}_{w,v} = \mathscr{L}_{w,v}.$$

Moreover, let

$$B_{w,v} := \min_{n \geq 0}\{q_v^n - n \cdot w\}.$$

Then

$$\mathrm{ord}_v(\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v) \geq B_{w,v} \text{ for every } \mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \in S_{w,v}.$$

In particular,

$$\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \in A_v \text{ if } q_v \geq \mathrm{wt}(\mathfrak{s}).$$

The key idea of the proof comes from certain kind of algebraic functional equations of CMSPLs arising from the logarithmic interpretation [Chen20, Thm. 3.2.9]. More precisely, we adopt some techniques in [AT90] to improve [CM19a, Prop. 3.2.1] about the explicit formula for the certain entries of the coefficient matrix of logarithms of iterated extensions of Carlitz tensor powers, which is a generalization of [Pp, Cor.4.1.5] in the case $r = 1$. Consequently, we derive the following

Proposition 3.2.2 ([Chen20, Prop. 3.2.2]). *Let* $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{u} := (u_1, \ldots, u_r) \in (\bar{k}^\times)^r$, $d_m := s_m + \cdots + s_r$, $d := d_1 + \cdots + d_r$, *and let* $G_{\mathfrak{s},\mathbf{u}}$ *be the t-module given in Theorem 2.3.1. If we write*

$$\log_{G_{\mathfrak{s},\mathbf{u}}} := \sum_{i \geq 0} P_i \tau^i, \; P_0 = I_d,$$

and put

$$\mathrm{wt}(\mathfrak{s})\text{-th row of } P_i := (Y_1^{<i>}, \cdots, Y_r^{<i>}), \text{ where } Y_m^{<i>} \in \bar{k}^{d_m} \text{ for } 1 \leq m \leq r.$$

If we set

$$Y_m^{<i>} = (y_{m,1}^{<i>}, \cdots, y_{m,d_m}^{<i>}),$$

then for $1 \leq j \leq d_1$ *we have*

$$y_{1,j}^{<i>} = \frac{(\theta - \theta^{q^i})^{d_1 - j}}{L_i^{d_1}} \tag{3.2.3}$$

and for $m \geq 2$, $1 \leq j \leq d_m$ *we have*

$$y_{m,j}^{<i>} = (-1)^{m-1} (\theta - \theta^{q^i})^{d_m - j} \sum_{0 \leq i_1 \leq \cdots \leq i_{m-1} < i} \frac{u_1^{q^{i_1}} \cdots u_{m-1}^{q^{i_{m-1}}}}{L_{i_1}^{s_1} \cdots L_{i_{m-1}}^{s_{m-1}} L_i^{d_m}}. \tag{3.2.4}$$

To introduce the strategy of the proof of Theorem 3.1.1, we define the Carlitz difference operators $\triangle_1$, which acts on $f \in k[\![z_1 \ldots, z_r]\!]$ by

$$(\triangle_1 f)(z_1 \cdots, z_r) := f(\theta z_1, z_2, \ldots, z_r) - \theta f(z_1, \ldots, z_r).$$

A simple application of binomial theorem derives the following

Lemma 3.2.5. *Let* $\triangle_1^j := \triangle_1 \circ \cdots \circ \triangle_1$ *be the j-fold composition of the Carlitz difference operator. Then we have*

$$(\triangle_1^j f)(z_1, \ldots, z_r) = \sum_{\ell=0}^{j} (-1)^\ell \binom{j}{\ell} \theta^\ell f(\theta^{j-\ell} z_1, z_2, \ldots, z_r).$$

for each $f \in k[\![z_1 \ldots, z_r]\!]$.

On the other hand, if we have an $\mathbb{F}_q$-linear power series in the variable z_1

$$f(z_1, \ldots, z_r) = \sum_{i=0}^{\infty} c_i z_1^{q^i} \in k[\![z_1 \ldots, z_r]\!],$$

where $c_i \in k[\![z_2, \ldots, z_r]\!]$. Then we have

$$(\triangle_1^j f)(z_1, \ldots, z_r) = \sum_{i=0}^{\infty} c_i [i]^j z_1^{q^i}.$$

Since $\mathrm{Li}_{\mathfrak{s}}^\star$ is $\mathbb{F}_q$-multilinear, the observations above combined with Proposition 3.2.2 lead to the desired result in Theorem 3.2.1. We refer the reader to [Chen20, Thm. 3.2.9] for details.

Instead of giving detailed proof, we give an example [Chen20, Ex. 3.2.12].

Example 3.2.6. *Consider $r=1$, $v=\theta$, $u\in A$ and $s\in\mathbb{N}$. In this case, we have*

$$\mathrm{Li}^{\star}_{s}(u)_v=\frac{1}{\theta^s-1}\left(\mathrm{Li}^{\star}_{s}(\theta^s u+u^q-u)_v+\sum_{j=1}^{s-1}\sum_{k=0}^{j}(-1)^{j+k}\binom{j}{k}\theta^k\,\mathrm{Li}^{\star}_{s}(\binom{s}{j}\theta^{s-k}u)_v\right).$$

In particular, if $s=p^\ell$ for some $\ell\in\mathbb{Z}_{\geq 0}$, then

$$\mathrm{Li}^{\star}_{s}(u)_v=\frac{1}{\theta^s-1}\,\mathrm{Li}^{\star}_{s}(\theta^s u+u^q-u)_v.$$

3.3. **Some remarks.** We now describe an application of p-adic integrality Theorem 1.1.1, which is given in [AHY]. Consider the $\mathbb{Q}$-algebra

$$\mathscr{A}:=(\prod_p\mathbb{Z}/p\mathbb{Z})/(\bigoplus_p\mathbb{Z}/p\mathbb{Z})$$

where p runs over all prime numbers. Kaneko and Zagier defined the *finite multiple zeta values* by

$$\zeta_{\mathscr{A}}(s_1,\dots,s_r):=(\zeta_{\mathscr{A}}(s_1,\dots,s_r)_p)_p\in\mathscr{A}$$

where the p-th component $\zeta_{\mathscr{A}}(s_1,\dots,s_r)_p$ is defined by the following truncated sum

$$\sum_{p>n_1>\cdots>n_r>0}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}}\mod p\in\mathbb{Z}/p\mathbb{Z}.$$

For any weight $w\in\mathbb{N}$ with $w\geq 2$, we consider the $\mathbb{Q}$-vector space $\mathfrak{Z}^{\mathscr{A}}_w$ which is spanned by all finite multiple zeta values of weight w and the $\mathbb{Q}$-vector space $\mathfrak{Z}_w$ which is spanned by all real-valued multiple zeta values of weight w. We further set $\mathfrak{Z}_0:=\mathbb{Q}$ and $\mathfrak{Z}_1:=\{0\}$. The following is the celebrated dimension conjecture of Zagier:

Conjecture 3.3.1 (Zagier). *The following equality holds*

$$\frac{1}{1-X^2-X^3}=\sum_{w\geq 0}(\dim_{\mathbb{Q}}\mathfrak{Z}_w)X^w.$$

Kaneko and Zagier predict the following assertions:

Conjecture 3.3.2 (Kaneko-Zagier). *For each $w\geq 2$, if we set $d_w:=\dim_{\mathbb{Q}}\mathfrak{Z}_w$, then we have*

$$\dim_{\mathbb{Q}}\mathfrak{Z}^{\mathscr{A}}_w=d_w-d_{w-2}.$$

As an application of Theorem 1.1.1, Akagi, Hirose and Yasuda combine with a special case of Jarossay's result [J18, (0.3.8)] to establish the upper bound for $\dim_{\mathbb{Q}}\mathfrak{Z}^{\mathscr{A}}_w$.

Theorem 3.3.3 ([AHY]). *For each $w\geq 2$, we have $\dim_{\mathbb{Q}}\mathfrak{Z}^{\mathscr{A}}_w\leq d_w-d_{w-2}$.*

Now let us turn back to the function field side. Consider the k-algebra

$$\mathscr{A}_k:=(\prod_v A/vA)/(\bigoplus_v A/vA)$$

where v runs over all monic irreducible polynomials of A. An analogue of Kaneko-Zagier's finite multiple zeta values over function fields is defined by

$$\zeta_{\mathscr{A}_k}(s_1,\dots,s_r):=(\zeta_{\mathscr{A}_k}(s_1,\dots,s_r)_v)_v\in\mathscr{A}_k$$

where the v-th component $\zeta_{\mathscr{A}_k}(s_1,\ldots,s_r)_v$ is defined by the following truncated sum

$$\sum_{\deg_\theta v>\deg_\theta a_1>\cdots>\deg_\theta a_r\geq 0}\frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \bmod v\in A/vA.$$

For any weight $w\in\mathbb{N}$, we consider the k-vector space $\mathfrak{Z}_w^{\mathscr{A}_k}$ spanned by all finite MZVs of weight w and the k-vector space $\mathfrak{Z}_w^{A}$ spanned by all Thakur's ∞-adic MZVs of weight w. In [To18], Todd discovered some linear relations among the same weight ∞-adic MZVs and he predicted the following dimension conjecture.

Conjecture 3.3.4 (Todd)**.** *We have*

$$\dim_k \mathfrak{Z}_w^A=\begin{cases} 2^{w-1} & \text{if } 1\leq w<q,\\ 2^{w-1}-1 & \text{if } w=q,\\ \sum_{i=1}^{q}\dim_k\mathfrak{Z}_{w-i}^A & \text{if } w>q.\end{cases}$$

There is a natural question arsing: can we apply Theorem 3.1.1 to prove the analogue of Theorem 3.3.3 in our setting of the function field? At present this question is still unclear and we will work this problem in a future project.

References

[A86] G. W. Anderson, *t-motives*, Duke Math. J. **53** (1986), no. 2, 457–502.

[AHY] K. Akagi, M. Hirose and S. Yasuda *Integrality of p-adic multiple zeta values and a bound for the space of finite multiple zeta values*, In preparation.

[AT90] G. W. Anderson and D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. (2) **132** (1990), no. 1, 159–191.

[Ca35] L. Carlitz, *On certain functions connected with polynomials in a Galois field*, Duke Math. J. **1** (1935), no. 2, 137-168.

[C14] C.-Y. Chang, *Linear independence of monomials of multizeta values in positive characteristic*, Compos. Math. **150** (2014), 1789-1808.

[Cha17] A. Chatzistamatiou, *On integrality of p-adic iterated integrals*, J. Algebra, 474 (2017), 240–270.

[Chen20] Y.-T. Chen, *Integrality of v-adic multiple zeta values*, arXiv:2001.01855

[CM19a] C.-Y. Chang and Y. Mishiba, *On multiple polylogarithms in characteristic p: v-adic vanishing versus ∞-adic Eulerianness*, Int. Math. Res. Not. (2019), no. 3, 923–947

[CM19b] C.-Y. Chang and Y. Mishiba, *On a conjucture of Furusho over function fields*, `http://www.math.nthu.edu.tw/~cychang/FC_Final.pdf`.

[Col82] R. F. Coleman, *Dilogarithms, regulators and p-adic L-functions*, Invent. Math. 69 (1982), no. 2, 171-208.

[F04] H. Furusho, *p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation*, Invent. Math. 155 (2004), no. 2, 253-286.

[Go96] D. Goss, *Basic structures of function field arithmetic*, Springer-Verlag, Berlin, 1996. Compositio Math. **142** (2006), 307–338.

[J18] D. Jarossay, *Pro-unipotent harmonic actions and a computation of p-adic cyclotomic multiple zeta values*, arXiv:1501.04893.

[Pp] M. A. Papanikolas, *Log-algebraicity on tensor powers of the Carlitz module and special values of Goss L-functions*, In preparation.

[T04] D. S. Thakur, *Function field arithmetic*, World Scientific Publishing, River Edge NJ, 2004.

[T09] D. S. Thakur, *Power sums with applications to multizeta and zeta zero distribution for $\mathbb{F}_q[t]$*, Finite Fields Appl. **15** (2009), no. 4, 534–552.

[To18] G. Todd, *A Conjectural Characterization for Fq(t)-Linear Relations between Multizeta Values*, J. Number Theory **187**, 264-287 (2018).

Department of Mathematics, National Tsing Hua University, Hsinchu City 30042, Taiwan R.O.C.
Email address: `s107021901@m107.nthu.edu.tw`