ON CHARACTERISTIC p ALTERNATING MULTIZETA VALUES

RYOTARO HARADA

Abstract. We give positive characteristic analogues of alternating multizeta values which are generalizations of Thakur multizeta values. We also introduce their fundamental properties including non-vanishing, sum-shuffle relations, period interpretation and linear independence which is a direct sum result for these values.

0. Introduction

We introduce the *alternating multizeta vaues in positive characteristic* (AMZVs in short) which are defined in [H19] as the following infinite sums. For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$,

$$\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon}) = \sum_{\substack{a_1,\ldots,a_r \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{\epsilon_1^{\deg a_1} \cdots \epsilon_r^{\deg a_r}}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty. \tag{1}$$

For the definitions of $\mathbb{F}_q$, A, A_+, k, k_∞ and $\overline{k}$, see §1. We call $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ the weight and $\mathrm{dep}(\mathfrak{s}) := r$ the depth of the presentation of $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$. We also study their properties listed in below.

(A). Non-vanishing (Theorem2.1)
(B). Sum-shuffle relations (Theorem 2.6)
(C). Period interpretation (Theorem 3.4)
(D). Linear independence (Theorem 4.7)

We also note that $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ is generalization of Thakur multizeta values, positive characteristic analogue of multizeta values.

In characteristic p case, an analogue of multizeta values were invented by Thakur [T04]. First we recall the power sums. For $s \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$, power sums are defined by

$$S_d(s) := \sum_{a \in A_{d+}} \frac{1}{a^s} \in k.$$

For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $d \in \mathbb{Z}_{\geq 0}$, we define

$$S_d(\mathfrak{s}) := S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in k.$$

For $s \in \mathbb{Z}_{\geq 0}$, the *Carlitz zeta values* are defined by

$$\zeta_A(s) := \sum_{a \in A_+} \frac{1}{a^s} \in k_\infty.$$

Thakur generalized this definition to that of *multizeta values in positive characteristic* (MZVs in short). For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) := \sum_{d \geq 0} S_d(\mathfrak{s}) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum_{\substack{a_1, \dots, a_r \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty.$$

He and Anderson developed the following properties for MZVs:

(a). Non-vanishing ([T09, Theorem 4])
(b). Sum-shuffle relations ([T10, Theorem 3])
(c). Period interpretation ([AT09, Theorem 1])

About (b), Thakur showed that the product of two MZVs can be expressed as an $\mathbb{F}_p$-linear combination of some MZVs of the same weight and we follow Thakur's terminology to call these sum-shuffle relations. Thus the MZVs form an $\mathbb{F}_p$-algebra. By lacking of the integral expression, we do not know the existence of the integral-shuffle relation in characteristic p. By using the transcendence theory developed by Yu [Y97], Anderson-Brownawell-Papanikolas [ABP04] and Papanikolas [P08] together with period interpretations of MZVs by Anderson-Thakur [AT09], some advances on MZVs are established by Chang-Yu [CY07] (for Carlitz zeta values), Chang [C14, C16], Chang-Papanikolas-Yu [CPY19] and Mishiba [Mi15, Mi17]. For example, in [C14], a characteristic p analogue of Goncharov's Conjecture [Go97] (in stronger form) was proved as the following property:

(d). Linear independence ([C14, Theorem 2.2.1])

We emphasize that AMZVs have important and fundamental properties which generalize those of MZVs. Indeed, we show them consisting of non-vanishing (Theorem 2.1), sum-shuffle relations (Theorem 2.6) and period interpretation (Theorem 3.4) as an alternating analogue of (a), (b), and (c). By applying those properties, we will show the alternating analogue of (d) (Corollary 4.9) in §4 and thereby obtain the transcendence of each alternating multizeta value. These result give us the generalization of main theorems in [C14, T09, T10].

From these results, while we find differences in observing alternating analogue of (b) (cf. §2.2), (c) (cf. §3) and Corollary 4.9 (cf. §4) by units $\gamma \in \overline{\mathbb{F}}_q^\times$ and $\epsilon \in \mathbb{F}_q^\times$, their basic properties appeared in this paper are similar to those of MZVs. More precisely, for the property (A), it is immediately obtained by an inequality property of the absolute values of power sums proved by Thakur [T09]. For the property (B), we use Chen's formula [Chen15] and approach the higher depth case by induction method invented by Thakur [T10]. For the property (C), inspired by Anderson-Thakur polynomials ([AT90]) that can interpolate power sums, we use those polynomials to create suitable power series that their specialization are AMZVs and then we use these series to create suitable pre-t-motives to establish the period interpretation of (C). For the linear independence result of AMZVs, we use Chang's method [C14] by applying Anderson-Brownawell-Papanikolas criterion [ABP04] to establish the alternating analogue of MZ property for AMZVs. By the linear independence result, we have (D), that is, an alternating analogue of (d) (for the detail, see Corollary 4.9).

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1. Notations and Definitions

We put the following notations.

q	a power of a prime number p.
$\mathbb{F}_q$	a finite field with q elements.
θ, t	independent variables.
A	the polynomial ring $\mathbb{F}_q[\theta]$.
A_+	the set of monic polynomials in A.
A_{d+}	the set of elements in A_+ of degree d.
k	the rational function field $\mathbb{F}_q(\theta)$.
k_∞	the completion of k at the infinite place ∞, $\mathbb{F}_q((\frac{1}{\theta}))$.
$\overline{k_\infty}$	a fixed algebraic closure of k_∞.
$\mathbb{C}_\infty$	the completion of $\overline{k_\infty}$ at the infinite place ∞.
$\overline{k}$	a fixed algebraic closure of k in $\mathbb{C}_\infty$.
$\lvert\cdot\rvert_\infty$	a fixed absolute value for the completed field $\mathbb{C}_\infty$ so that $\lvert\theta\rvert_\infty = q$.
$\mathbb{T}$	the Tate algebra over $\mathbb{C}_\infty$, the subring of $\mathbb{C}_\infty[[t]]$ consisting of power series convergent on the closed unit disc $\lvert t\rvert_\infty \leq 1$.
D_i	$\prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j}) \in A_+$ where $D_0 := 1$.
Γ_{n+1}	the Carlitz gamma, $\prod_i D_i^{n_i}$ $(n = \sum_i n_i q^i \in \mathbb{Z}_{\geq 0}$ $(0 \leq n_i \leq q-1))$.

We define alternating power sums and alternating multizeta values in positive characteristic along the construction of MZVs by Thakur. For $s \in \mathbb{N}$, $\epsilon \in \mathbb{F}_q^\times$ and $d \in \mathbb{Z}_{\geq 0}$, we define the *alternating power sums* by

$$S_d(s;\epsilon) := \epsilon^d S_d(s) = \sum_{a \in A_{d+}} \frac{\epsilon^d}{a^s} \in k.$$

The above sums are extended inductively as follows.

For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ and $d \in \mathbb{Z}_{\geq 0}$, we define

$$S_{<d}(\mathfrak{s};\boldsymbol{\epsilon}) := \sum_{d > d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1;\epsilon_1)\cdots S_{d_r}(s_r;\epsilon_r) \in k$$

and

$$\begin{aligned} S_d(\mathfrak{s};\boldsymbol{\epsilon}) &:= S_d(s_1;\epsilon_1) S_{<d}(s_2,\dots,s_r;\epsilon_2,\dots,\epsilon_r) \\ &:= S_d(s_1;\epsilon_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2;\epsilon_2)\cdots S_{d_r}(s_r;\epsilon_r) \in k. \end{aligned} \tag{2}$$

When $r - 1 > d$, $S_d(\mathfrak{s};\boldsymbol{\epsilon}) = 0$ since it is empty sum. By using these alternating power sums, AMZVs (cf. (1)) are interpreted as follows:

$$\zeta_A(\mathfrak{s};\boldsymbol{\epsilon}) = \sum_{d \geq 0} S_d(\mathfrak{s};\boldsymbol{\epsilon}) \in k_\infty. \tag{3}$$

Remark 1.1. We remark that $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ specializes to $\zeta_A(\mathfrak{s})$ when $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) = (1, \dots, 1)$.

In this paper, we write n-fold Frobenius twisting as follows.

$$\begin{aligned}\mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t))\\ f := \sum_i a_i t^i &\mapsto \sum_i a_i^{q^n} t^i =: f^{(n)}.\end{aligned}$$

Moreover, we fix a fundamental period $\tilde{\pi}$ of the Carlitz module (see [Goss96, T04]). We define the following power series.

$$\Omega = \Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i}) \in \mathbb{C}_\infty[[t]]$$

where $(-\theta)^{1/(q-1)}$ is a fixed $(q-1)$st root of $-\theta$ so that $\frac{1}{\Omega(\theta)} = \tilde{\pi}$ ([ABP04, AT09]). Here, we also introduce another expression of $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ by using the following theorem of Anderson and Thakur.

Theorem 1.2 ([AT90]). *For each $s \in \mathbb{N}$, there exists a polynomial $H_s = H_s(t) \in A[t]$ such that*

$$(H_{s-1}\Omega^s)^{(d)}|_{t=\theta} = \Gamma_s S_d(s)/\tilde{\pi}^s \tag{4}$$

for all $d \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{N}$. Moreover, when we regard H_s as a polynomial of θ over $\mathbb{F}_q[t]$ by $A[t] = \mathbb{F}_q[t][\theta]$ then

$$\deg_\theta H_s \leq \frac{sq}{q-1}. \tag{5}$$

This polynomial H_s is called the *Anderson-Thakur polynomial.* From (3) and (4), we obtain the following expression of $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$;

$$\zeta_A(\mathfrak{s};\boldsymbol{\epsilon}) = \frac{\tilde{\pi}^{s_1+\cdots s_r}}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{d_1>\cdots>d_r\geq 0} \epsilon_1^{d_1}(H_{s_1-1}\Omega^{s_1})^{(d_1)}|_{t=\theta}\cdots\epsilon_r^{d_r}(H_{s_r-1}\Omega^{s_r})^{(d_r)}|_{t=\theta}. \tag{6}$$

2. Fundamental properties

We introduce the non-vanishing and sum-shuffle relation of AMZVs.

2.1. Non-vanishing property of AMZVs.

We show the non-vanishing property as the following theorem by using valuation of power sums which evaluated by Thakur [T09].

Theorem 2.1 ([H19] Theorem 2.1). *For any $\mathfrak{s} = (s_1,\ldots,s_r) \in \mathbb{N}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1,\ldots,\epsilon_r) \in (\mathbb{F}_q^\times)^r$, $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are non-vanishing.*

Proof. From (3), we can write $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ as follows.

$$\zeta_A(\mathfrak{s};\boldsymbol{\epsilon}) = \sum_{d_1>d_2>\cdots>d_r\geq 0} \epsilon_1^{d_1}\epsilon_2^{d_2}\cdots\epsilon_r^{d_r} S_{d_1}(s_1)S_{d_2}(s_2)\cdots S_{d_r}(s_r)$$

On the other hand, in [T09], Thakur showed that

$$\deg_\theta S_d(k) > \deg_\theta S_{d+1}(k).$$

Therefore we have

$$\begin{aligned}|\zeta_A(s_1,\dots,s_r;\epsilon_1,\dots,\epsilon_r)|_\infty &= |\sum_{d_1>d_2>\cdots>d_r\geq 0}\epsilon_1^{d_1}\epsilon_2^{d_2}\cdots\epsilon_r^{d_r}S_{d_1}(s_1)S_{d_2}(s_2)\cdots S_{d_r}(s_r)|_\infty\\ &= |S_{r-1}(s_1)S_{r-2}(s_2)\cdots S_0(s_r)|_\infty\\ &\neq 0\end{aligned}$$

by using $\deg_\theta S_0(k)=0$ and $\deg_\theta S_d(k)<0$ $(k>0,d>0)$ in [T09, §2.2.3.]. Thus $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are non-vanishing. □

2.2. **Sum-shuffle relations for AMZVs.** We introduce the sum-shuffle relations for our AMZVs. This kind of relations show that products of two AMZVs are expressed by $\mathbb{F}_p$-linear combination of AMZVs with preserving their weights. From the relation, $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ form an $\mathbb{F}_p$-algebra.

For the products of power sums $S_d(s)$, the following formula was shown by Chen [Chen15].

Proposition 2.2 ([Chen15, Theorem 3.1]). $s_1,s_2\in\mathbb{N}$, *we have*

$$S_d(s_1)S_d(s_2)-S_d(s_1+s_2)=\sum_{\substack{0<j<s_1+s_2\\ q-1|j}}\Delta^j_{s_1,s_2}S_d(s_1+s_2-j,j)$$

where

$$\Delta^j_{s_1,s_2}=(-1)^{s_1-1}\binom{j-1}{s_1-1}+(-1)^{s_2-1}\binom{j-1}{s_2-1}.$$

The key idea to prove Proposition 2.2 is the following partial fraction decomposition

$$\frac{1}{a^{s_1}b^{s_2}}=\sum_{0<j<s_1+s_2}\left\{\frac{(-1)^{s_1-1}\binom{j-1}{s_1-1}}{a^{s_1+s_2-j}(a-b)^j}+\frac{(-1)^{s_2-1}\binom{j-1}{s_2-1}}{b^{s_1+s_2-j}(a-b)^j}\right\}$$

for $a,b\in A\backslash\{0\}$. By using the above decomposition, she obtained the following formula in the proof of [Chen15, Theorem3.1]:

$$\sum_{\substack{a\neq b\in A_+\\ \deg a=\deg b}}\frac{1}{a^{s_1}b^{s_2}}=\sum_{\substack{0<j<s_1+s_2\\ (q-1)|j}}\left\{\sum_{\substack{a,b\in A_+\\ \deg a>\deg b}}\frac{(-1)^{s_1-1}\binom{j-1}{s_1-1}}{a^{s_1+s_2-j}b^j}+\sum_{\substack{a,b\in A_+\\ \deg b>\deg a}}\frac{(-1)^{s_2-1}\binom{j-1}{s_2-1}}{b^{s_1+s_2-j}a^j}\right\}.$$

By using the above Chen's method, we obtain the following lemma.

Lemma 2.3 ([H19, Lemma 2.3]). *For* $s_1,s_2\in\mathbb{N}$, $\epsilon_1,\epsilon_2\in\mathbb{F}_q^\times$ *and* $d\in\mathbb{Z}_{\geq 0}$, *we have*

$$S_d(s_1;\epsilon_1)S_d(s_2;\epsilon_2)-S_d(s_1+s_2;\epsilon_1\epsilon_2)=\sum_{\substack{0<j<s_1+s_2\\ q-1|j}}\Delta^j_{s_1,s_2}S_d(s_1+s_2-j,j;\epsilon_1\epsilon_2,1) \tag{7}$$

We remark that the coefficients $\Delta^j_{s_1,s_2}$ are independent of d.

Preceding to the next argument, we introduce an expression which is used in the rest of this section. For any index $\mathfrak{s}=(s_1,s_2,\dots,s_r)\in\mathbb{N}^r$, we can write $\mathfrak{s}=(s_1,\mathfrak{s}')$ where $\mathfrak{s}'=(s_2,\dots,s_r)\in\mathbb{N}^r$ (resp. $\boldsymbol{\epsilon}=(\epsilon_1,\dots,\epsilon_r)\in(\mathbb{F}_q^\times)^r$) and when $r=1$, we set $\mathfrak{s}'=\phi$ (resp. $\boldsymbol{\epsilon}'=\phi$) and further $S_d(\mathfrak{s}';\boldsymbol{\epsilon}'):=1$.

Next we prepare the following lemma to show sum-shuffle relations for alternating power sums in general depth.

Lemma 2.4 ([H19, Lemma 2.4]). *For* $\mathfrak{a} := (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$, $\mathfrak{b} := (b_1, b_2, \dots, b_s) \in \mathbb{N}^s$, $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_s) \in (\mathbb{F}_q^\times)^s$, *we may express the product* $S_{<d}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d}(\mathfrak{b}; \boldsymbol{\lambda})$ *as follows:*

$$S_{<d}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d}(\mathfrak{b}; \boldsymbol{\lambda}) = \sum_i f_i S_{<d}(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) \tag{8}$$

for some $c_{ij} \in \mathbb{N}$, $\mu_{ij} \in \mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m + \sum_{n=1}^s b_n = \sum_{h=1}^{l_i} c_h$, $\prod_{m=1}^r \epsilon_m \prod_{n=1}^s \lambda_n = \prod_{h=1}^{l_i} \mu_h$, $l_i \leq r + s$ *and* $f_i \in \mathbb{F}_p$ *which are independent of* d *for each* i.

Sketch of the proof. We can prove the lemma by induction on $\mathrm{dep}(\mathfrak{a}) + \mathrm{dep}(\mathfrak{b}) (= r + s) > 2$ and the relation (7). □

To prove the sum-shuffle result in Theorem 2.6 the following is the key ingredient.

Theorem 2.5 ([H19], Theorem 2.6). *For* $\mathfrak{a} := (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$, $\mathfrak{b} := (b_1, b_2, \dots, b_s) \in \mathbb{N}^s$, $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_s) \in (\mathbb{F}_q^\times)^s$, *we may express the product* $S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda})$ *as follows:*

$$S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda}) = \sum_i f_i' S_d(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) \tag{9}$$

for some $c_{ij} \in \mathbb{N}$, $\mu_{ij} \in \mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m + \sum_{n=1}^s b_n = \sum_{h=1}^{l_i} c_{ih}$, $\prod_{m=1}^r \epsilon_m \prod_{n=1}^s \lambda_n = \prod_{h=1}^{l_i} \mu_{ih}$, $l_i \leq r + s$ *and* $f_i' \in \mathbb{F}_p$ *for each* i.

Sketch of the proof. The theorem follows from the decomposition which is described in (2) and the relation (8). □

We remark that the coefficients $f_i \in \mathbb{F}_p$ are independent of d by Lamma 2.4.

By summing over d in the relation (9), we obtain the following sum-shuffle relations for AMZVs:

Theorem 2.6 ([H19, Theorem 2.7]). *For* $\mathfrak{a} := (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$, $\mathfrak{b} := (b_1, b_2, \dots, b_s) \in \mathbb{N}^s$, $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_s) \in (\mathbb{F}_q^\times)^s$, *we may express the product* $\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon}) \zeta_A(\mathfrak{b}; \boldsymbol{\lambda})$ *as follows:*

$$\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon}) \zeta_A(\mathfrak{b}; \boldsymbol{\lambda}) = \sum_i f_i'' \zeta_A(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) \tag{10}$$

for some $c_{ij} \in \mathbb{N}$ *and* $\mu_{ij} \in \mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m + \sum_{n=1}^s b_n = \sum_{h=1}^{l_i} c_{ih}$, $\prod_{m=1}^r \epsilon_m \prod_{n=1}^s \lambda_n = \prod_{h=1}^{l_i} \mu_{ih}$, $l_i \leq r + s$ *and* $f_i'' \in \mathbb{F}_p$ *for each* i.

Therefore the $\mathbb{F}_p$-linear span of our AMZVs form an $\mathbb{F}_p$-algebra by Theorem 2.6.

3. Period interpretation of AMZVs

In this section, we show that each $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ appears as a period of certain pre-t-motive basing on the idea in [AT09].

We denote the ring $\overline{k}(t)[\sigma, \sigma^{-1}]$ the non-commutative Laurent polynomial ring in σ with coefficients in $\overline{k}(t)$ subject to the following relations,

$$\sigma f = f^{(-1)} \sigma, \quad f \in \overline{k}(t).$$

We denote $\mathbb{E}$ to be the ring consisting of formal power series

$$\sum_{n=0}^{\infty} a_n t^n \in \overline{k}[[t]]$$

such that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|_\infty} = 0, \quad [k_\infty(a_0, a_1, \dots) : k_\infty] < \infty.$$

We note that

$$\Omega \in \mathbb{E} \tag{11}$$

and thus $\Omega \in \mathbb{T}$.

3.1. **Review of pre-t-motive.** First we recall the notions of pre-t-motives. For more detail, see [P08].

Definition 3.1 ([P08, §3.2.1])**.** A *pre-t-motive* M is a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module that is finite dimensional over $\overline{k}(t)$.

Let M be a pre-t-motive dimension r over $\overline{k}(t)$ and Φ be representing matrix of multiplication by σ on M with respect to a given basis of $\mathbf{m}$ of M, then M is rigid analytically trivial if and only if there is a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ ($\mathbb{L}$ is a quotient field of $\mathbb{T}$) satisfying

$$\Psi^{(-1)} = \Phi\Psi.$$

Here we define $\Psi^{(-1)}$ by $(\Psi^{(-1)})_{ij} := (\Psi_{ij})^{(-1)}$. Such matrix Ψ is called *rigid analytic trivialization* of Φ and if all the entries of Ψ are convergent at $t = \theta$, and entries of $\Psi|_{t=\theta}$ are called *periods* of M.

Anderson and Thakur obtained the period interpretation of MZVs with certain pre t-motive in [AT09].

3.2. **AMZVs related to pre-t-motive.** Next we give a period interpretation of AMZVs .

Definition 3.2 ([H19, Definition 3.2])**.** Given $\gamma_i \in \overline{\mathbb{F}_q}^{\times}$ $(i = 1, \dots, r)$, a fixed $(q-1)$st root of $\epsilon_i \in \mathbb{F}_q^{\times}$, we let M be the pre-t-motive such that $\dim_{\overline{k}(t)} M = r + 1$ and for the fixed $\overline{k}(t)$-basis $\mathbf{m}$ of M which satisfy

$$\sigma\mathbf{m} = \Phi\mathbf{m}$$

where Φ is the following matirix:

$$\begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0 \\ \gamma_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r} H_{s_1-1}^{(-1)} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0 \\ 0 & \gamma_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} H_{s_2-1}^{(-1)} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \cdots & 0 & \gamma_r^{(-1)}(t-\theta)^{s_r} H_{s_r-1}^{(-1)} & 1 \end{pmatrix}.$$

Here $H_{s_i-1} \in A[t]$ is the Anderson-Thakur polynomial.

For the above pre-t-motive, we can give a matrix Ψ which has a relation with Φ as the following proposition.

Proposition 3.3 ([H19, Proposition 3.3])**.** *For the pre-t-motive M defined in Definition 3.2, there exists $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ so that it satisfies $\Psi^{(-1)} = \Phi\Psi$.*

Sketch of the proof. For the matrix Φ in Definition 3.2, we show that the matrix $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ satisfies $\Psi^{(-1)} = \Phi\Psi$ and which is given as follows:

$$\Psi := \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & & & & & \\ \gamma_1 L(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & & & & \\ \vdots & \gamma_2 L(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \gamma_1\cdots\gamma_{r-1}L(s_1,\dots,s_{r-1})\Omega^{s_r} & \gamma_2\cdots\gamma_{r-1}L(s_2,\dots.s_{r-1})\Omega^{s_r} & & \ddots & \Omega^{s_r} & \\ \gamma_1\cdots\gamma_r L(s_1,\dots,s_r) & \gamma_2\cdots\gamma_r L(s_2,\dots,s_r) & \cdots & \cdots & \gamma_r L(s_r) & 1 \end{pmatrix}. \tag{12}$$

Here we define the following:

$$\begin{aligned} L(s_1,\dots,s_r) &:= L(s_1,\dots,s_r;\epsilon_1,\dots,\epsilon_r) \\ &:= \sum_{d_1>\cdots>d_r\ge 0} \epsilon_1^{d_1}(H_{s_1-1}\Omega^{s_1})^{(d_1)}\cdots\epsilon_r^{d_r}(H_{s_r-1}\Omega^{s_r})^{(d_r)}. \end{aligned}$$

We also note that the following equations hold by their definitions.

$$\begin{aligned} \gamma_i^{(-1)} &= \epsilon_i^{-1}\gamma_i, \\ \Omega^{(-1)} &= (t-\theta)\Omega, \\ L(s_1,\dots,s_r)^{(-1)} &= \left(\prod_{n=0}^{r}\epsilon_n\right)L(s_1,\dots,s_r) \\ &\quad + \left(\prod_{n=0}^{r-1}\epsilon_n\right)(\Omega^{s_r}H_{s_r-1})^{(-1)}L(s_1,\dots,s_{r-1}). \end{aligned}$$

By using these three equations, we obtain the following desired formula.

$$\Psi^{(-1)} = \Phi\Psi$$

□

As we will see in Lemma 4.3, the matrix Ψ in (12) belongs to $\mathrm{Mat}_{r+1}(\mathbb{T})$ then we obtain that $\Psi \in \mathrm{GL}_{r+1}(\mathbb{L})$ from (11) and $\det\Psi = \Omega^{\sum_{i=1}^{r} d_i} \neq 0$ where $d_i = s_i + \cdots + s_r$. Therefore Ψ is a rigid analytic trivialization of Φ and we call each entry of $\Psi|_{t=\theta}$ a period of M. Thus we have the following result by the above proposition.

Theorem 3.4 ([H19, Theorem 3.4])**.** *For $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1,\dots,\epsilon_r) \in (\mathbb{F}_q^\times)^r$, $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are periods of the pre-t-motive M in Definition 3.2.*

Proof. By using $\Omega|_{t=\theta} = \tilde{\pi}^{-1}$ and (4), we obtain

$$\begin{aligned} L(\mathfrak{s})|_{t=\theta} &= \frac{\Gamma_{s_1}\cdots\Gamma_{s_r}}{\tilde{\pi}^{s_1+\cdots+s_r}} \sum_{d_1>\cdots>d_r\ge 0} \epsilon_1^{d_1}\cdots\epsilon_r^{d_r} S_{d_1}(s_1)\cdots S_{d_r}(s_r) \\ &= \frac{\Gamma_{s_1}\cdots\Gamma_{s_r}}{\tilde{\pi}^{s_1+\cdots+s_r}}\zeta_A(\mathfrak{s};\boldsymbol{\epsilon}). \end{aligned} \tag{13}$$

Therefore by the matrix Ψ in (12), $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are periods of the pre-t-motive M in Definition 3.2. □

4. Linear independence property of AMZVs

We show that AMZVs form an weight-graded algebra as an application of their period interpretation in the former section. The proof is shown along the method which was invented by Chang [C14].

4.1. **Anderson-Brownawell-Papanikolas's criterion.** In our proof, we need to use the following ABP criterion (ABP stands for Anderson-Brownawell-Papanikolas) which was introduced in [ABP04].

Theorem 4.1 ([ABP04, Theorem 3.1.1])**.** *Fix $\Phi \in \mathrm{Mat}_d(\overline{k}[t])$ so that $\det\Phi = c(t-\theta)^s$ for some $c \in \overline{k}^{\times}$ and some $s \in \mathbb{Z}_{\geq 0}$. Suppose that there exists a vector $\psi \in \mathrm{Mat}_{d\times 1}(\mathbb{E})$ satisfies*

$$\psi^{(-1)} = \Phi\psi.$$

For every $\rho \in \mathrm{Mat}_{1\times d}(\overline{k})$ such that $\rho\psi(\theta) = 0$, there is a $P \in \mathrm{Mat}_{1\times d}(\overline{k}[t])$ so that $P(\theta) = \rho$ and $P\psi = 0$.

From Proposition 3.3, it is enough to verify that the matrices Φ in Definition 3.2 satisfy the conditions. We thus only need to show that $\Psi \in \mathrm{Mat}_{r+1}(\mathbb{E})$. This is necessary in applying Theorem 4.1 to our AMZVs. We use the following proposition which was given in [ABP04].

Proposition 4.2 ([ABP04, Proposition 3.1.3])**.** *Suppose*

$$\Phi \in \mathrm{Mat}_l(\overline{k}[t]), \quad \psi \in \mathrm{Mat}_{l\times 1}(\mathbb{T})$$

such that

$$\det\Phi|_{t=0} \neq 0, \quad \psi^{(-1)} = \Phi\psi.$$

Then

$$\psi \in \mathrm{Mat}_{l\times 1}(\mathbb{E}).$$

By using this proposition, we obtain the following lemma.

Lemma 4.3 ([H19, Lemma 4.3])**.** *Let $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ be as in Proposition 3.3. Then the following holds:*

$$\Psi \in \mathrm{Mat}_{r+1}(\mathbb{E}).$$

4.2. **MZ property for AMZVs.** First we verify that AMZVs satisfy the following lemma which is alternating analogue of MZ property in [C14].

Lemma 4.4 ([H19, Lemma 4.4])**.** *For a given AMZV $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ with $\mathrm{wt}(\mathfrak{s}) = w$ and $\mathrm{dep}(\mathfrak{s}) = r$, there exists $\Phi \in \mathrm{Mat}_{r+1}(\overline{k}[t])$ and $\psi \in \mathrm{Mat}_{(r+1)\times 1}(\mathbb{E})$ with $r \geq 1$ such that:*

- (i) *$\psi^{(-1)} = \Phi\psi$ and Φ satisfies the condition of Theorem 4.1;*
- (ii) *the last column of Φ is of the form $(0,\dots,0,1)^{\mathrm{tr}}$;*
- (iii) *for some $a \in \overline{\mathbb{F}}_q^{\times}$ and $b \in k^{\times}$, $\psi(\theta)$ is of the form with specific first and last entries*

$$\psi(\theta) = \left(\frac{1}{\tilde{\pi}^w}, \dots, a\frac{b\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})}{\tilde{\pi}^w}\right)^{\mathrm{tr}};$$

(iv) *for any* $N \in \mathbb{N}$ *and some* $c \in \mathbb{F}_q^{\times}$, $\psi(\theta^{q^N})$ *is of the form*

$$\psi(\theta^{q^N}) = \left(0, \ldots, 0, ac^N \left(\frac{b\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})}{\tilde{\pi}^w}\right)^{q^N}\right)^{\mathrm{tr}}.$$

Proof. See the proof of Lemma 4.4 in [H19]. □

Next we show that monomials of AMZVs also satisfy Lemma 4.4 by using the same method in proof of [C14, Proposition 3.4.4].

Proposition 4.5 ([H19, Proposition 4.5]). *Let* $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1), \ldots, \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)$ *be AMZVs with weights* $w_1, \ldots, w_n$ *respectively and let* $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$. *Then there exist matrices* $\Phi \in \mathrm{Mat}_d(\overline{k}[t])$ *and* $\psi \in \mathrm{Mat}_{d\times 1}(\mathbb{E})$ *with* $d \geq 2$ *so that the triple* $(\Phi, \psi, \zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n})$ *satisfies (i)-(iv) in Lemma 4.4.*

Proof. We take triple $(\Phi_i, \psi_i, \zeta_A(\mathfrak{s}_i; \boldsymbol{\epsilon}_i))$ which satisfies Lemma 4.4 for each i. Then we consider the Kronecker product $\otimes$ (See [S05, Chapter 8]) of Φ_i and ψ_i respectively as followings:

$$\Phi := \Phi_1^{\otimes m_1} \otimes \cdots \otimes \Phi_n^{\otimes m_n}, \quad \psi := \psi_1^{\otimes m_1} \otimes \cdots \otimes \psi_n^{m_n}.$$

By our assumption, $(\Phi_i, \psi_i, \zeta_A(\mathfrak{s}_i; \boldsymbol{\epsilon}_i))$ satisfy Lemma 4.4 and thus by using the property of Kronecker product which involves matrix multiplication (cf. [S05, Theorem 7.7]), the triple $(\Phi, \psi, \zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n})$ does so. □

Definition 4.6 ([H19, Definition 4.6]). Let $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1), \ldots, \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)$ be AMZVs of $\mathrm{wt}(\mathfrak{s}_i) = w_i$ $(i = 1, \ldots, n)$. For $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ not all zero, we define the total weight of the monomial $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n}$ as

$$\sum_{i=1}^{n} m_i w_i.$$

For $w \in \mathbb{N}$, we denote AZ_w the set of monomials of AMZVs with total weight w.

We note that AZ_w is finite set.

Now we prove the linear independence of monomials of AMZVs.

Theorem 4.7 ([H19, Theorem 4.7]). *Let* $w_1, \ldots, w_l \in \mathbb{N}$ *be distinct. We suppose that* V_i *is a* k*-linearly independent subset of* AZ_{w_i} *for* $i = 1, \ldots, l$. *Then the following union*

$$\{1\} \cup \bigcup_{i=1}^{l} V_i$$

is a linearly independent set over $\overline{k}$, *that is, there are no nontrivial* $\overline{k}$*-linear relation among elements of* $\{1\} \cup \bigcup_{i=1}^{l} V_i$.

Proof. We may assume that $w_l > \cdots > w_1$ without loss of generality. For each $i = 1, \ldots, l$, AZ_{w_i} is a finite set by definition and thus its subset V_i is also finite. Let V_i consist of $\{Z_{i1}, \ldots, Z_{im_i}\}$ where $Z_{ij} \in AZ_{w_i} (j = 1, \ldots, m_i)$ are the same total weight w_i. The proof is by induction on weight w_l.

We require on the contrary that $\{1\} \cup \bigcup_{i=1}^{l} V_i$ is $\overline{k}$-linearly dependent set. Then we may also assume that there are nontrivial $\overline{k}$-linear relations

$$a_0 \cdot 1 + a_{11} Z_{11} + \cdots + a_{1m_1} Z_{1m_1} + \cdots + a_{l1} Z_{l1} + \cdots + a_{lm_l} Z_{lm_l} = 0,$$

we may take $a_0, a_{11}, \dots, a_{lm_l} \in \overline{k}$ with $a_{li} \neq 0$ for some $i = 1, \dots, m_l$.

We proceed our proof by assuming the existence of nontrivial $\overline{k}$-linear relations between V_l and $\{1\} \cup \bigcup_{i=1}^{l-1} V_i$.

For $1 \leq i \leq l$ and $1 \leq j \leq m_l$, by combining Proposition 3.3 with Proposition 4.5, there exist the matrices

$$\Phi_{ij} \in \mathrm{Mat}_{d_{ij}}(\overline{k}[t]) \quad \text{and} \quad \psi_{ij} \in \mathrm{Mat}_{d_{ij} \times 1}(\mathbb{E}) \tag{14}$$

so that $d_{ij} \geq 2$ and each $(\Phi_{ij}, \psi_{ij}, Z_{ij})$ satisfy Lemma 4.4.

For the matrix Φ_{ij} and the column vector ψ_{ij}, we define the following block diagonal matrix and the column vector

$$\tilde{\Phi} := \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{m_i} (t-\theta)^{w_l - w_i} \Phi_{ij} \right) \quad \text{and} \quad \tilde{\psi} := \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{m_i} \Omega^{w_l - w_i} \psi_{ij} \right).$$

In this proof, we define the direct sum of any column vectors $\mathbf{v}_1, \dots, \mathbf{v}_\mathbf{m}$ whose entries belong to $\mathbb{C}_\infty((t))$ by $\bigoplus_{i=1}^{m} \mathbf{v}_i := (\mathbf{v}_1^{\mathrm{tr}}, \dots, \mathbf{v}_m^{\mathrm{tr}})^{\mathrm{tr}}$.

By the requirement, $\{1\} \cup \bigcup_{i=1}^{l} V_i$ is a linearly dependent over $\overline{k}$. Thus there exists a nonzero vector

$$\rho = (\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l})$$

such that

$$\begin{aligned} \rho \cdot (\tilde{\psi}|_{t=\theta}) &= \rho \cdot \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m_i} \left(\frac{1}{\tilde{\pi}^{w_l}}, \dots, a\frac{bZ_{ij}}{\tilde{\pi}^{w_l}} \right)^{\mathrm{tr}} \\ &= \frac{1}{\tilde{\pi}^{w_l}} (\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l}) \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m_i} \left(1, \dots, abZ_{ij} \right)^{\mathrm{tr}} = 0, \end{aligned}$$

where $\mathbf{v}_{ij} \in \mathrm{Mat}_{1 \times d_{ij}}(\overline{k})$ for $1 \leq i \leq l$ and $1 \leq j \leq m_i$. Then we have nontrivial $\overline{k}$-linear relation

$$(\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l}) \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m_i} \left(1, \dots, abZ_{ij} \right)^{\mathrm{tr}} = 0.$$

In the beginning of this proof, we assumed that there exists nontrivial $\overline{k}$-linear relations between V_l and $\{1\} \cup \bigcup_{i=1}^{l-1} V_i$ and then for some $1 \leq s \leq m_l$, the last entry of $\mathbf{v}_{ls}$ is nonzero. Since the last entry in $\mathbf{v}_{li}$ is coefficient of abZ_{li} for $1 \leq i \leq m_l$ in the above relation. By using Theorem 4.1, we have $\mathbf{F} := (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})$ where $\mathbf{f}_{ij} = (f_{i1}, \dots, f_{id_{ij}}) \in \mathrm{Mat}_{1 \times d_{ij}}(\overline{k}[t])$ for $1 \leq i \leq l$, $1 \leq j \leq m_i$ and it satisfies

$$\mathbf{F}\tilde{\psi} = 0 \quad \text{and} \quad \mathbf{F}|_{t=\theta} = \rho.$$

The last entry of $\mathbf{f}_{ls}$ is a nontrivial polynomial because the last entry of $\mathbf{v}_{ls}$ is not zero. We choose a sufficiently large $N \in \mathbb{Z}$ so that $\mathbf{f}_{ls}|_{t=\theta^{q^N}} \neq 0$. We rewrite the equation $(\mathbf{F}\tilde{\psi})|_{t=\theta^{q^N}} = 0$

by using $\Omega|_{t=\theta^{q^N}} = 0$, Lemma 4.4 (iv) and the definition of $\tilde{\psi}$ as follows:

$$
\begin{aligned}
&(\mathbf{F}\tilde{\psi})|_{t=\theta^{q^N}} \\
&= (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})|_{t=\theta^{q^N}} \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m_i} \Omega^{w_l - w_i}|_{t=\theta^{q^N}} \left(0, \dots, 0, a_j c_j^N \left(\frac{b_j Z_{ij}}{\tilde{\pi}^{w_i}}\right)^{q^N}\right)^{\mathrm{tr}} \\
&= (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})|_{t=\theta^{q^N}} \bigoplus_{j=1}^{m_l} \left(0, \dots, 0, a_j c_j^N \left(\frac{b_j Z_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N}\right)^{\mathrm{tr}} \\
&= (f_{11}, \dots, f_{1d_{11}}, \dots, f_{l1}, \dots, f_{ld_{lm_l}})|_{t=\theta^{q^N}} \bigoplus_{j=1}^{m_l} \left(0, \dots, 0, a_j c_j^N \left(\frac{b_j Z_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N}\right)^{\mathrm{tr}} \\
&= \sum_{j=1}^{m_l} (f_{ld_{lj}}|_{t=\theta^{q^N}}) a_j c_j^N \left(\frac{b_j Z_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N} = 0.
\end{aligned}
$$

Thus we obtain the following nontrivial $\overline{k}$-linear relation with some $f_{ld_{ls}} \neq 0$:

$$\sum_{j=1}^{m_l} (f_{ld_{lj}}|_{t=\theta^{q^N}}) a_j c_j^N \left(b_j Z_{lj}\right)^{q^N} = 0.$$

Therefore by taking q^Nth root of the above $\overline{k}$-linear relation, we get a nontrivial relation for $\{Z_{l1}, \dots, Z_{lm_l}\}$ as follows.

$$\sum_{j=1}^{m_l} \left\{(f_{ld_{lj}}|_{t=\theta^{q^N}}) a_j c_j^N\right\}^{\frac{1}{q^N}} b_j Z_{lj} = 0.$$

This shows that V_l is a $\overline{k}$-linearly dependent set. Then by using Lemma 4.8 in [H19], we can show that V_l is a k-linearly dependent subset. However, it contradicts the condition saying that V_l is the k-linearly independent set. Therefore we obtain the claim. □

The above theorem provides an alternating analogue of Theorem 2.2.1 in [C14]. Finally, we define the following notations and state the results.

Notation 4.8. We denote $\overline{\mathcal{AZ}}_w$ (resp. $\mathcal{AZ}_w$) be the $\overline{k}$-vector space (resp. k-vector space) spanned by weight w AMZVs. By Theorem 2.6, we derive $\mathcal{AZ}_w \cdot \mathcal{AZ}_{w'} \subseteq \mathcal{AZ}_{w+w'}$. We also note the $\overline{k}$-algebra $\overline{\mathcal{AZ}}$ (resp. k-algebra $\mathcal{AZ}$) generated by AMZVs.

Corollary 4.9 ([H19, Theorem 4.10]). *We have the following:*

(i) $\overline{\mathcal{AZ}}$ *forms an weight-graded algebra, that is,* $\overline{\mathcal{AZ}} = \overline{k} \bigoplus_{w \in \mathbb{N}} \overline{\mathcal{AZ}}_w$,

(ii) $\overline{\mathcal{AZ}}$ *is defined over* k, *that is, we have the canonical map* $\overline{k} \otimes_k \mathcal{AZ} \to \overline{\mathcal{AZ}}$ *which is bijective.*

As a direct consequence of Corollary 4.9, we have the following transcendence result.

Corollary 4.10. *Each AMZV* $\zeta_A(\mathfrak{a}; \epsilon)$ *is transcendental over* $\overline{k}$.

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Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan

E-mail address: `m15039r@math.nagoya-u.ac.jp`