Some generalizations of harmonic numbers and their applications

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1 Introduction

Harmonic numbers
$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
have direct generalizations. One is the *higher-order* harmonic number,
$$\mathfrak{H}_n^{(r)} = \sum_{j=1}^n \frac{1}{j^r}\,.$$

When $r = 1$, $H_n = \mathfrak{H}_n^{(1)}$ is the original harmonic number. The third type is related to this generalized harmonic number. Another is the *hyperharmonic* number,
$$H_n^{(r)} := \sum_{j=1}^n H_j^{(r-1)} \quad (r \ge 1)$$
with $H_n^{(1)} = H_n$ and $H_n^{(0)} = 1/n$. The second type is related to this generalized harmonic number.

Harmonic numbers also have several different q-generalizations. Some keep good relations in extensive ways, and some do not. Any generalization has each advantages and disadvantages. We consider three different kinds of q-generalizations with their applications.

One type of q-harmonic numbers [9] are defined by
$$\mathcal{H}_n = \mathcal{H}_n(q) := \sum_{k=1}^n \frac{q^k}{1 - q^k}\,.$$

Another type of q-harmonic numbers [4] are defined by
$$H_n(q) = \sum_{k=1}^n \frac{q^{k-1}}{[k]_q}\,,$$

where $[k]_q = \dfrac{1-q^k}{1-q}$. Still other q-harmonic numbers [2] are given by

$$h_n^{(s)}(x) = \sum_{k=1}^{n} \frac{1}{[k-1+x]_{q^r}^{s}}\,.$$

2 First type of q-harmonic numbers

The first type of q-harmonic numbers are related to the generating function of the sum of the jth powers of the divisors of n. If $\sigma_j(n) = \sum_{d|n} d^j$, then for $q \in \mathbb{C}$ and $|q| < 1$,

$$\sum_{n=1}^{\infty} \sigma_j(n) q^n = \sum_{n=1}^{\infty} \frac{n^j q^n}{1-q^n}\,.$$

Van Hamme [7] gave the following identity.

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}_q \frac{q^{\binom{k+1}{2}}}{1-q^k} = \sum_{k=1}^{n} \frac{q^k}{1-q^k} = \mathcal{H}_n, \tag{1}$$

where the q-binomial is defined as

$$\binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \quad \text{with} \quad (a;q)_n := \prod_{j=0}^{n-1} (1-aq^j)\,.$$

There exist several generalizations of identity (1).

The generalized q-harmonic numbers $\mathcal{H}_n^{(m)}$ are defined by

$$\mathcal{H}_n^{(m)} := \sum_{k=1}^{n} \frac{q^k}{(1-q^k)^m} \quad (n = 1, 2, \dots)\,,$$

When $m = 1$, $\mathcal{H}_n^{(1)} = \mathcal{H}_n$ is the q-harmonic number.

We give a continued fraction expansion of the generaing funciton of generalized q-harmonic numbers, given by

$$H_m(x) := \sum_{n=1}^{\infty} \mathcal{H}_n^{(m)} x^n. \tag{2}$$

Theorem 1.

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}x^n$$
$$=\cfrac{qx}{(1-q)^m(1-x)-\cfrac{(1-q)^{2m}qx(1-x)}{(1-q^2)^m+(1-q)^mqx-\cfrac{(1-q^2)^{2m}qx}{(1-q^3)^m+(1-q^2)^mqx-\cfrac{(1-q^3)^{2m}qx}{(1-q^4)^m+(1-q^3)^mqx-\ddots}}}}.$$

We study the summations

$$\sum_{k=1}^{n}q^{pk}\mathcal{H}_k^{(m_1,\dots,m_\ell)},\quad \sum_{k=1}^{n}q^{p(n-k)}\mathcal{H}_k^{(m_1,\dots,m_\ell)},$$
$$\sum_{k=1}^{n}q^{pk}\mathcal{H}_{s+k}^{(m_1,\dots,m_\ell)},\quad \sum_{k=1}^{n}q^{p(n-k)}\mathcal{H}_{s+k}^{(m_1,\dots,m_\ell)},$$

where $\mathcal{H}_k^{(m_1,\dots,m_\ell)}$ are the multiple generalized q-harmonic numbers defined by

$$\mathcal{H}_n^{(m_1,\dots,m_\ell)}:=\sum_{1\le k_1\le\cdots\le k_\ell\le n}\frac{q^{k_1}+\cdots+q^{k_\ell}}{(1-q^{k_1})^{m_1}\cdots(1-q^{k_\ell})^{m_\ell}},\quad (n=1,2,\dots).$$

If $m_1=m_2=\cdots=m_\ell=m$ and $\ell=1$, the generalized q-harmonic numbers $\mathcal{H}_n^{(m)}$ are studied in [8]. Note that $\mathcal{H}_n^{(m_1,\dots,m_\ell)}$ can be considered as a q-analogue of the multiple generalized harmonic numbers of order m

$$H_n^{(m_1,\dots,m_\ell)}=\sum_{1\le k_1\le\cdots\le k_\ell\le n}\frac{1}{k_1^{m_1}k_2^{m_2}\cdots k_\ell^{m_\ell}}\quad (n=1,2,\dots).$$

We give some finite summation identities of generalized q-harmonic numbers.

Theorem 2. *If p and n are positive integers and $m_1,m_2,\dots,m_\ell$ are complex numbers, the*

$$\sum_{k=1}^{n}q^{pk}\mathcal{H}_k^{(m_1,\dots,m_\ell)}=\frac{1}{1-q^p}\Bigg\{\sum_{s=1}^{m_\ell}(-1)^{m_\ell-s}\binom{p}{m_\ell-s}\mathcal{H}_n^{(m_1,\dots,m_{\ell-1},s)}+$$
$$(-1)^{m_\ell}\sum_{t=1}^{n}\sum_{j=1}^{p-m_\ell+1}\binom{p-j}{m_\ell-1}q^{jt}\mathcal{H}_t^{(m_1,\dots,m_{\ell-1})}-q^{p(n+1)}\mathcal{H}_n^{(m_1,\dots,m_\ell)}\Bigg\}.\quad (3)$$

Some particular cases of Theorem 2 can be seen in simple forms. For example,

- When $m_1 = \cdots = m_\ell = 1$ and $p = 1$:

$$\sum_{k=1}^{n} q^k \mathcal{H}_k^{(1,\ell)} = \frac{1-q^{n+1}}{1-q}\mathcal{H}_n^{(1,\ell)} - \frac{1}{1-q}\sum_{t=1}^{n} q^t \mathcal{H}_t^{(1,\ell-1)}$$
$$= \sum_{i=0}^{\ell-1}(-1)^i \frac{1-q^{n+1}}{(1-q)^{i+1}}\mathcal{H}_n^{(1,\ell-i)} + (-1)^\ell \frac{q(1-q^n)}{(1-q)^2}.$$

Corollary 1. *If p, n and s are positive integers, then*

$$\sum_{k=1}^{n} q^{pk} \mathcal{H}_{k+s}^{(m_1,\ldots,m_\ell)} = q^{-ps}\left(F(n+s,p,\overrightarrow{m_\ell}) - F(s,p,\overrightarrow{m_\ell})\right),$$

where

$$F(n,p,\overrightarrow{m_\ell}) := \sum_{k=1}^{n} q^{pk}\mathcal{H}_k^{(m_1,\ldots,m_\ell)}$$

with $\overrightarrow{m_\ell} = (m_1, \ldots, m_\ell)$.

Some particular cases of Corollary 1 can be seen in simple forms. For example,

- When $s = n$, $m_1 = \cdots = m_\ell = 1$ and $p = 1$:

$$\sum_{k=1}^{n} q^k \mathcal{H}_{n+k}^{(1,\ell)} = q^{-n}\left(\sum_{i=0}^{\ell-1}\frac{(-1)^i}{(1-q)^{i+1}}\right.$$
$$\left.\times\left((1-q^{2n+1})\mathcal{H}_{2n}^{(1,\ell-i)} - (1-q^{n+1})\mathcal{H}_n^{(1,\ell-i)}\right) + (-1)^\ell\frac{q^{n+1}(1-q^n)}{(1-q)^2}\right).$$

Theorem 3. *If p and n are positive integers and $m_1, m_2, \ldots, m_\ell$ are complex numbers, then*

$$\sum_{k=1}^{n} q^{p(n-k)}\mathcal{H}_k^{(m_1,\ldots,m_\ell)} = \frac{1-q^{p(n+1)}}{1-q^p}\mathcal{H}_n^{(m_1,\ldots,m_\ell)} - \frac{q^{p(n+1)}}{1-q^p}$$
$$\times\left\{\sum_{s=1}^{m_\ell-1}\binom{p+s-1}{s}\mathcal{H}_n^{(m_1,\ldots,m_{\ell-1},m_\ell-s)} + \sum_{\ell=1}^{n}\sum_{j=1}^{p}\binom{m_\ell+j-2}{j-1}q^{\ell(j-p)}\mathcal{H}_\ell^{(m_1,\ldots,m_{\ell-1})}\right\}.$$

Corollary 2. *If p, n and s are positive integers then*

$$\sum_{k=1}^{n} q^{p(n-k)} \mathcal{H}_{k+s}^{(m_1,\ldots,m_\ell)} = G(n+s, p, \overrightarrow{m_\ell}) - q^{pn} G(s, p, \overrightarrow{m_\ell}),$$

where

$$G(n, p, \overrightarrow{m_\ell}) := \sum_{k=1}^{n} q^{p(n-k)} \mathcal{H}_{k}^{(m_1,\ldots,m_\ell)}$$

with $\overrightarrow{m_\ell} = (m_1, \ldots, m_\ell)$.

3 Second type of q-harmonic numbers

In [4], a q-hyperharmonic number $H_n^{(r)}(q)$ is defined by

$$H_n^{(r)}(q) = \sum_{j=1}^{n} q^j H_j^{(r-1)}(q) \quad (r, n \geq 1) \tag{4}$$

with

$$H_n^{(0)}(q) = \frac{1}{q[n]_q}.$$

In this q-generalization,

$$H_n(q) = H_n^{(1)}(q) = \sum_{j=1}^{n} \frac{q^{j-1}}{[j]_q} \tag{5}$$

is a q-harmonic number. When $q \to 1$, $H_n^{(r)} = \lim_{q\to 1} H_n^{(r)}(q)$ is the hyperharmonic number and $H_n = \lim_{q\to 1} H_n(q)$ is the original harmonic number.

Weighted sums of this kind of q-hyperharmonic numbers can be expressed in terms of several types of q-analogue of the sum of consecutive integers.

Theorem 4. *For positive integers n and r,*

$$\begin{aligned}\sum_{l=1}^{n} q^{l-1} [l]_q H_l^{(r)}(q) &= \frac{[n]_q [n+r]_q}{[r+1]_q} H_n^{(r)}(q) - \frac{q^r [n-1]_q [n]_q}{([r+1]_q)^2} \binom{n+r-1}{r-1}_q \\ &= \frac{[n]_q [r]_q}{[r+1]_q} H_n^{(r+1)}(q) + \frac{q^{r-1}}{[r+1]_q} \binom{n+r}{r+1}_q .\end{aligned}$$

Remark. When $q \to 1$, we have for $n, r \geq 1$,

$$\sum_{l=1}^{n} l H_l^{(r)} = \frac{n(n+r)}{r+1} H_n^{(r)} - \frac{(n-1)^{(r+1)}}{(r-1)!(r+1)^2}$$
$$= \frac{nr}{r+1} H_n^{(r+1)} + \frac{1}{r+1}\binom{n+r}{r+1},$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \geq 1)$ denotes the rising factorial with $(x)^{(0)} = 1$.

Next, we show a square weighted summation formula, which is yielded from the following identity.

Theorem 5. *For positive integers n and r,*

$$\sum_{\ell=1}^{n} q^{\ell-1}[\ell]_q[\ell+1]_q H_\ell^{(r)}(q)$$
$$= \frac{[n]_q[n+r]_q([2]_q[n+2]_q + q^3[r-1]_q[n+1]_q)}{[r+1]_q[r+2]_q} H_n^{(r)}(q)$$
$$- q^r[n-1]_q[n]_q \binom{n+r-1}{r-1}_q \frac{[2]_q[r+2]_q^2 + q^4[r+1]_q^2[n-2]_q}{[r+1]_q^2[r+2]_q^2}.$$

Remark. When $q \to 1$, we have

$$\sum_{\ell=1}^{n} \ell(\ell+1) H_\ell^{(r)} = \frac{n(n+r)\big((r+1)n + (r+3)\big)}{(r+1)(r+2)} H_n^{(r)}$$
$$- (n-1)n\binom{n+r-1}{r-1}\frac{(r+1)^2 n + 2(2r+3)}{(r+1)^2(r+2)^2}. \tag{6}$$

Combining Theorem 4 and Theorem 5, we can obtain the square weighted summation formula.

Corollary 3. *For positive integers n and r,*

$$\sum_{\ell=1}^{n} q^{\ell-1}([\ell]_q)^2 H_\ell^{(r)}(q)$$
$$= \frac{[n]_q[n+r]_q(1 + q[r+1]_q[n]_q)}{[r+1]_q[r+2]_q} H_n^{(r)}(q)$$
$$- q^r[n-1]_q[n]_q \binom{n+r-1}{r-1}_q \frac{q[r+1]_q^2[n]_q - q^3[r]_q^2 + [2]_q}{[r+1]_q^2[r+2]_q^2}.$$

Remark. When $q \to 1$, we have for $n, r \geq 1$,

$$\sum_{l=1}^{n} l^2 H_l^{(r)} = \frac{n(n+r)\big((r+1)n+1\big)}{(r+1)(r+2)} H_n^{(r)} - \frac{(n-1)^{(r+1)}\big((r+1)^2 n - (r^2-2)\big)}{(r-1)!(r+1)^2(r+2)^2}.$$

We can obtain the following summation of the cubic powers, but no q-generalization has been found yet. For $n, r \geq 1$,

$$\begin{aligned}\sum_{l=1}^{n} l^3 H_l^{(r)} &= \frac{n(n+r)\big((r+1)(r+2)n^2 + 3(r+1)n - r + 1\big)}{(r+1)(r+2)(r+3)} H_n^{(r)} \\ &\quad - \frac{(n-1)^{(r+1)}\big((r+1)^2(r+2)^2 n^2 - (r+1)^2(2r^2+2r-7)n + (r^4 - 2r^3 - 17r^2 - 12r + 6)\big)}{(r-1)!(r+1)^2(r+2)^2(r+3)^2}.\end{aligned}$$

Nevertheless, when $r = 1$, we can get more general summation formulas. Fpr example,

Theorem 6. *For $n, N \geq 1$, we have*

$$\sum_{\ell=1}^{n} q^{\ell-1}[\ell]_q[\ell+1]_q \dots [\ell+N-1]_q H_\ell(q) = \frac{[n]_q[n+1]_q \cdots [n+N]_q}{[N+1]_q} H_n(q) - \frac{[N-1]_q!}{[N+1]_q} \sum_{l=1}^{N} q^l [l]_q \binom{n+l-1}{l+1}_q.$$

Remark. When $q \to 1$, we have the ordinary relation

$$\sum_{\ell=1}^{n} \ell(\ell+1)\dots(\ell+N-1) H_\ell = \frac{n(n+1)\cdots(n+N)}{N+1} H_n - \frac{(N-1)!}{N+1} \sum_{l=1}^{N} l \binom{n+l-1}{l+1}.$$

4 Third type of q-harmonic numbers

The results for the third type of q-harmonic numbers are yielded from Abel's Lemma on summation by parts [1], for two sequences $\{f_k\}$ and $\{g_k\}$:

$$\sum_{k=m}^{n} f_k \Delta g_k = f_n g_{n+1} - f_m g_m - \sum_{k=m}^{n-1} g_{k+1} \Delta f_k \,,$$

where $\Delta \tau_k = \tau_{k+1} - \tau_k$ is a forward difference of an arbitrary complex sequence $\{\tau_k\}$.

There are many definitions for q-zeta functions. For $0 < x \leq 1$, $s \in \mathbb{C}$, and $\Re e(s) > 1$, define the Hurwitz q-zeta function ([5]) as

$$\zeta_q(s, x) = \sum_{n=0}^{\infty} \frac{q^{(n+x)r}}{[n+x]_{q^r}^s} \,.$$

When $x = 1$, $\zeta_q(s) = \zeta_q(s, 1)$ is the q-zeta function. Following [2], define a generalized q-harmonic number $h_n^{(s)}(x)$ by

$$h_n^{(s)}(x) = \sum_{k=1}^{n} \frac{1}{[k-1+x]_{q^r}^s} \,.$$

The main result can be stated as follows.

Theorem 7. *For $r \in \mathbb{N}$, $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re e(s) > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{q^{(n-1+x)r} h_n^{(s)}(x)}{[n+x]_{q^r} [n-1+x]_{q^r}} = \zeta_q(s+1, x) \,.$$

When $x = 1$ in Theorem 7, we have the following corollary.

Corollary 4.

$$\sum_{n=1}^{\infty} \frac{q^{nr} h_n^{(s)}(1)}{[n]_{q^r} [n+1]_{q^r}} = \zeta_q(s+1) \,.$$

Remark. When $q \to 1$, Corollary 4 is reduced to

$$\sum_{n=1}^{\infty} \frac{\mathfrak{H}_n^{(s)}}{n(n+1)} = \zeta(s+1)$$

([6]).

In order to get more q-generalization results, we introduce different two q-binomial coefficients. We define the shifted q-binomial coefficients by

$$\binom{n+x}{k}_q = \frac{[n+x]_q[n+x-1]_q \cdots [n+x-k+1]_q}{[k]_q!}.$$

We define the (q_1, q_2)-binomial coefficients $\binom{n}{k}_{q_1,q_2}$ by

$$\binom{n}{k}_{(q_1,q_2)} = \frac{[n]_{q_2}!}{[k]_{q_1}![n-k]_{q_2}!}.$$

We define more generalized Hurwitz q-zeta functions $\zeta_{t,q}(s,x)$ by

$$\zeta_{t,q}(s,x) = \sum_{n=0}^{\infty} \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\,, \tag{7}$$

where $r, t \in \mathbb{N}$, $0 < x \leq 1$ and $s \in \mathbb{C}, \Re e(s) > 1$. We define the generalized q-harmonic numbers $h_{t,n}^{(s)}(x)$ by

$$h_{t,n}^{(s)}(x) = h_{t,n,q}^{(s)}(x) := \sum_{k=1}^{n} \frac{q^{rt(k-1+x)}}{[k-1+x]_{q^r}^s}\,.$$

It is clear that the right hand side of (7) is absolutely convergent as $0 < q \leq 1$. When $s = 1$ and $0 < q < 1$, the right hand side of (7) is also absolutely convergent. We have $h_{t,n}^{(s)}(x) \to \zeta_{t,q}(s,x)$ $(n \to \infty)$.

With the help of Abel's Lemma on summation by parts, we show that infinite sums involving the generalized q-harmonic numbers $h_{t,n}^{(s)}(x)$ in terms of linear combinations of the generalized Hurwitz q-zeta values $\zeta_{t,q}(s,x)$.

Theorem 8. *For $r, s, t, a \in \mathbb{N}$ with $t \geq s$, $a > 1$ and $0 < x \leq 1$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}[a-1]_{q^r}}{[n+a-1+x]_{q^r}[n+x]_{q^r}} \\
&= \sum_{b=1}^{a-1} \frac{(-1)^{s-1} q^{rb}}{[b]_{q^r}^s} (\zeta_{t-s+1,q}(1,x) - q^{-rb(t-s+1)} \zeta_{t-s+1,q}(1,x+b)) \\
&\quad + \sum_{m=2}^{s} (-1)^{s-m} \zeta_{t-s+m,q}(m,x) h_{1,a-1}^{(s-m+1)}(1)\,.
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n-1+x)} [a]_{q^r}}{[n+a-1+x]_{q^r} [n-1+x]_{q^r}}$$
$$= \zeta_{t+1,q}(s+1,x) + \sum_{m=2}^{s} (-1)^{s-m} \zeta_{t-s+m,q}(m,x) h_{1,a-1}^{(s-m+1)}(1)$$
$$+ \sum_{b=1}^{a-1} \frac{(-1)^{s-1} q^{rb}}{[b]_{q^r}^s} (\zeta_{t-s+1,q}(1,x) - q^{-rb(t-s+1)} \zeta_{t-s+1,q}(1,x+b)) .$$

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{\binom{n-1+x+k}{k}_{q^r}} = \sum_{a=2}^{k} \frac{[-1]_{q^{-r}}^{a-2} [a]_{q^{-r}} [a-1]_{q^{-r}}}{[a-1]_{q^r}} \binom{k}{a}_{(q^{-r},q^r)} A_a ,$$

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{[n+x-1]_{q^r} \binom{n-1+x+k}{k}_{q^r}}$$
$$= \sum_{a=2}^{k} \frac{[-1]_{q^{-r}}^{a-1} [a]_{q^{-r}}}{[a]_{q^r}} \binom{k}{a}_{(q^{-r},q^r)} (\zeta_{t+1,q}(s+1,x) + A_a) ,$$

where

$$A_a = \sum_{m=2}^{s} (-1)^{s-m} \zeta_{t-s+m,q}(m,x) h_{1,a-1}^{(s-m+1)}(1)$$
$$+ \sum_{b=1}^{a-1} \frac{(-1)^{s-1} q^{rb}}{[b]_{q^r}^s} (\zeta_{t-s+1,q}(1,x) - q^{-rb(t-s+1)} \zeta_{t-s+1,q}(1,x+b)) .$$

When $q \to 1$, $\zeta_{t-s+1,1}(1,x) - \zeta_{t-s+1,1}(1,x+b)$ *is interpreted as*

$$h_b^{<1>}(1,b) = \sum_{k=1}^{b} \frac{1}{k-1+x} . \tag{8}$$

When $q \to 1$ and $x = 1$ in Theorem 8, we get the following formulas, which are given in [6].

Corollary 5.

$$\sum_{n=1}^{\infty} \frac{(a-1)\mathfrak{H}_n^{(s)}}{(n+1)(n+a)} = \sum_{j=1}^{a-1} \frac{(-1)^{s+1} \mathfrak{H}_j}{j^s} + \sum_{i=2}^{s} (-1)^{s-i} \mathfrak{H}_{a-1}^{(s-i+1)} \zeta(i) .$$

$$\sum_{n=1}^{\infty} \frac{a\mathfrak{H}_n^{(s)}}{n(n+a)}$$
$$= \zeta(s+1) + \sum_{j=1}^{a-1} \frac{(-1)^{s+1}\mathfrak{H}_j}{j^s} + \sum_{i=2}^{s} (-1)^{s-i} \mathfrak{H}_{a-1}^{(s-i+1)} \zeta(i) \,.$$

$$\sum_{n=1}^{\infty} \frac{\mathfrak{H}_n^{(s)}}{\binom{n+k}{k}}$$
$$= \sum_{r=2}^{k} (-1)^r r \binom{k}{r} \left(\sum_{j=1}^{r-1} \frac{(-1)^{s+1}\mathfrak{H}_j}{j^s} + \sum_{i=2}^{s} (-1)^{s-i} \mathfrak{H}_{r-1}^{(s-i+1)} \zeta(i) \right) \,.$$

$$\sum_{n=1}^{\infty} \frac{\mathfrak{H}_n^{(s)}}{n\binom{n+k}{k}} = \zeta(s+1)$$
$$+ \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \left(\sum_{j=1}^{r-1} \frac{(-1)^{s+1}\mathfrak{H}_j}{j^s} + \sum_{i=2}^{s} (-1)^{s-i} \mathfrak{H}_{r-1}^{(s-i+1)} \zeta(i) \right) \,.$$

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