Classification of ribbon 2-knots with ribbon crossing number up to four

Taizo Kanenobu
Department of Mathematics, Osaka City University

1 Introduction

A ribbon 2-knot is a knotted 2-sphere in $\mathbb{R}^4$ that bounds a ribbon 3-disk, which is an immersed 3-disk with only ribbon singularities. The ribbon crossing number of a ribbon 2-knot is the minimal number of the ribbon singularities of any ribbon 3-disk bounding the knot [14]. Yasuda has classified ribbon 2-knots with ribbon crossing number up to three in [13] and has enumerated those with ribbon crossing number four in [15]. In this paper we classify these ribbon 2-knots.

Theorem 1. *The number of mutually non-isotopic ribbon 2-knots with ribbon crossing number four is either 111 or 112. Amongst them 9 or 10 knots are positive-amphicheiral. So, if each chiral pair is counted as one knot, the number of ribbon 2-knots with ribbon crossing number four is either 60 or 61; see Table 1.*

Table 1: Numbers of the ribbon 2-knots with ribbon crossing number up to four.

Ribbon crossing number	0	1	2	3	4
(i) Number of ribbon 2-knots, each chiral pair is counted separetely	1	0	3	13	111/112
(ii) Number of ribbon 2-knots, each chiral pair is counted as one knot	1	0	2	7	60/61

The ribbon 2-knots with ribbon crossing number with up to three are completely classified by the Alexander polynomial. However, those with ribbon crossing number four listed in [15] have not been classified. Theorem 1 means that there is an indistinguishable pair of ribbon 2-knots, Y43 and Y46 in Table 3, which are positive-amphicheiral; they have isomorphic knot group. Also, there is one knot, Y112 (the ribbon handlebody is shown in Fig. 1), which had been missed in [15].

Satoh [8] introduced a virtual arc presentation for a ribbon 2-knot. If a ribbon 2-knot K is presented by a virtual arc with n classical crossings, then the ribbon crossing number of K is at most n. In [2] ribbon 2-knots presented by a virtual arc with up to four crossings are enumerated, and in [6] those ribbon 2-knots are classified. There are 24 ribbon 2-knots with ribbon crossing number up to four, which are not presented by a virtual arc with up to four crossings. So, we have only to consider these knots. We have 27 sets of ribbon 2-knots $\mathcal{A}_i$ ($i = 1, 2, \ldots, 17$) and $\mathcal{A}_j!$ ($j = 2, 3, 4, 7, 8, 10, 11, 12, 14, 16$), which consist of knots sharing the same Alexander polynomial; $\mathcal{A}_j!$ is the set consisting of the mirror images of the knots in $\mathcal{A}_j$. The knots in the sets $\mathcal{A}_i$ with $i \leq 13$ (and so $\mathcal{A}_j!$ with $j \leq 12$) have been classified in [6]. Thus, we classify the knots in $\mathcal{A}_i$ with $i = 14$, 15, 16, 17 (Sec. 5). The knots in these sets are ribbon 2-knots of 1-fusion. In order to classify the knots in these sets we use the trace set, or the twisted Alexander polynomial associated to the representations to $\mathrm{SL}(2, \mathbb{C})$. The *trace set* is an invariant defined for a ribbon 2-knot of 1-fusion from the representations of the knot group to $\mathrm{SL}(2, \mathbb{C})$; see Sec. 4 in [7]. For the twisted Alexander polynomial of a ribbon 2-knot, see [4].

This paper is organized as follows: In Secs. 2 and 3, we review a ribbon handlebody presentation of a ribbon 2-knot and the stable transformations for a ribbon handlebody presentation, which were introduced in [3]. In Sec. 4 we give Yasuda's table of the ribbon 2-knots with ribbon crossing number up to four (Tables 2 and 3), which contain the 1-fusion notation of the knots. In Sec. 5 we classify the knots in $\mathcal{A}_i$, $i = 14$, 15, 16, 17, which completes the proof of Theorem 1.

Acknowledgments

The author was partially supported by JSPS KAKENHI, Grant Number JS17K05259. This work was partially supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

2 Ribbon handlebody presentation of a ribbon 2-knot

In this section we review a ribbon handlebody presentation of a ribbon 2-knot introduced in [3]. A *ribbon handlebody* $\mathcal{H}$ is a ribbon 2-disk, which is a 2-dimensional handlebody in $\mathbb{R}^3$ consisting of $(m+1)$ 0-handles $D_0, D_1, \ldots, D_m$ and m 1-handles $B_1, B_2, \ldots, B_m$ such that the preimage of each ribbon singularity consists of an arc in the interior of a 0-handle and a cocore of a 1-handle. We set $\mathcal{H} = \mathcal{D} \cup \mathcal{B}$, where $\mathcal{D} = D_0 \cup D_1 \cup \cdots \cup D_m$ and $\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_m$. We associate to a ribbon handlebody $\mathcal{H}$ an immersed 3-disk $V_{\mathcal{H}}$ in $\mathbb{R}^4$ defined by

$$V_{\mathcal{H}} = \mathcal{D} \times [-2, 2] \cup \mathcal{B} \times [-1, 1]. \tag{1}$$

Then $V_{\mathcal{H}}$ is a ribbon 3-disk for the ribbon 2-knot $K_{\mathcal{H}} = \partial V_{\mathcal{H}}$ in $\mathbb{R}^4$. Conversely, for any ribbon 2-knot K in $\mathbb{R}^4$, there exists a ribbon handlebody $\mathcal{H}$ such that K is ambient isotopic to the associated 2-knot $K_{\mathcal{H}}$; see [1, 10, 12].

We suppose that each 1-handle B_q is the image of an embedding $b_q : I \times I \to \mathbb{R}^3$, $q = 1, 2, \ldots, m$. Let $\beta_q : I \to \mathbb{R}^3$ be the center line of the 1-handle B_p defined by $\beta_q(t) = b_q(1/2, t)$, which is an oriented path such that

$$\begin{aligned} \beta_q(I) \cap \mathcal{D} = \left\{\beta_q(0), \beta_q(t_{q,1}), \beta_q(t_{q,2}), \ldots, \beta_q(t_{q,\ell_q}), \beta_q(1)\right\}, \\ 0 < t_{q,1} < t_{q,2} < \cdots < t_{q,\ell_q} < 1. \end{aligned} \tag{2}$$

Let ι_q, τ_q, $\lambda(q, j)$ $(j = 1, 2, \ldots, \ell_q)$ be integers in $\{0, 1, \ldots, m\}$ determined by

$$\beta_q(0) \in \partial D_{\iota_q}, \quad \beta_q(1) \in \partial D_{\tau_q}, \quad \beta_q(t_{q,j}) \in \mathrm{Int} D_{\lambda(q,j)}, \quad j = 1, 2, \ldots, \ell_q. \tag{3}$$

Thus, β is an oriented path joining D_{ι_q} and D_{τ_q}. At the intersection $\beta_q(t_{q,j})$ if β_q passes from the negative side of $D_{\lambda(q,j)}$ through to the positive side we define $\epsilon(q,j)=+1$, and if it passes in the opposite direction we define $\epsilon(q,j)=-1$.

Then for a ribbon handlebody $\mathcal{H}$ we define a *ribbon handlebody presentation* $[\,X \mid R\,]$ consisting of:

- $X=\{x_0,x_1,\dots,x_m\}$, where each letter x_q corresponds to the 0-handle D_q,
- $R=\{\rho_1,\rho_2,\dots,\rho_m\}$, where each relation $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ (or $x_{\tau_q}=x_{\iota_q}^{w_q}$) corresponds to the 1-handle B_q that joins D_{ι_q} to D_{τ_q} passing through 0-handles according to the word w_q:
$$w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}. \tag{4}$$
In particular, if $\beta_q(I)\cap\mathcal{D}=\{\beta_q(0),\beta_q(1)\}$, then $\rho_q : x_{\iota_q}=x_{\tau_q}$.

For a ribbon handlebody presentation $P=[\,X\,|\,R\,]$, $X=\{x_0,x_1,\dots,x_m\}$ and $R=\{\rho_1,\rho_2,\dots,\rho_m\}$ with $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$, $w_q\in F[X]$, we can associate an oriented labelled tree $\tilde{P}=(X,E,\lambda)$, where X is a set of vertices, E is a set of oriented edges:

$$E=\left\{\overrightarrow{x_{\iota_q}x_{\tau_q}} \mid q=1,\dots,m\right\}, \tag{5}$$

and $\lambda : E\to F[X]$ is a labeling function defined by $\lambda(\overrightarrow{x_{\iota_q}x_{\tau_q}})=w_q$.

Conversely, for an oriented labeled tree (X,E,λ) as above, we obtain a unique ribbon handlebody presentation $P=[\,X\,|\,R\,]$ and also the associated ribbon 2-knot of m-fusion, which we denote by K_P; cf. Proposition 3.3 in [3]. Note that the knot group of K_P, $\pi_1(\mathbb{R}^4-K_P)$, is presented by $\langle\, X\,|\,\tilde{R}\,\rangle$, where $\tilde{R}$ is a set of relations $\{\tilde\rho_1,\tilde\rho_2,\dots,\tilde\rho_m\}$ with $\tilde\rho_q : w_q^{-1}x_{\iota_q}w_q=x_{\tau_q}$; see [11].

Therefore, any ribbon 2-knot with ribbon crossing number r is obtained from an oriented labeled tree (X,E,λ) as above such that $\sum_{q=1}^m \ell_q = r$, where ℓ_q is the word length of the word w_q as in Eq. (4).

3 Stable transformations of a ribbon handlebody presentation

Let $P=[\,X \mid R\,]$ be a ribbon handlebody presentation, where $X=\{x_0,x_1,\dots,x_m\}$ and $R=\{\rho_1,\dots,\rho_m\}$ with

$$\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}, \quad w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)}, \quad \epsilon(q,s)=\pm1. \tag{6}$$

We call the following transformations of a ribbon handlebody presentation *stable transformations*:

S1. Replace $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ by $x_{\iota_q}=x_{\tau_q}^{(w_q^{-1})}$.

S2. Replace $\rho_q : x_{\iota_q}^{w_q}=x_{\tau_q}$ by either $x_{\iota_q}^{x_{\iota_q}^{\epsilon}w_q}=x_{\tau_q}$ or $x_{\iota_q}^{w_q x_{\tau_q}^{\epsilon}}=x_{\tau_q}$, $\epsilon=\pm1$.

S3. Add a generator y and a relation $y=x_p^w$ or $x_p=y^w$, where w is a word in x_0, $x_1,\dots,x_m$.

S3′. Inverse transformation of S3.

S4. (i) Suppose $\tau_p = \iota_q$. Replace either $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ or $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ by $x_{\iota_p}^{w_p w_q} = x_{\tau_q}$.

(ii) Suppose $\iota_p = \iota_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\tau_q}^{{w_q}^{-1} w_p} = x_{\tau_p}$.

(iii) Suppose $\tau_p = \tau_q$. Replace $\rho_p : x_{\iota_p}^{w_p} = x_{\tau_p}$ by $x_{\iota_p}^{w_p {w_q}^{-1}} = x_{\iota_q}$.

S5. (i) Suppose $\lambda(p,s) = \tau_q$. Replace $x_{\lambda(p,s)} (= x_{\tau_q})$ in w_p in ρ_p by $w_q^{-1} x_{\iota_q} w_q$.

(ii) Suppose $\lambda(p,s) = \iota_q$. Replace $x_{\lambda(p,s)} (= x_{\iota_q})$ in w_p in ρ_p by $w_q x_{\tau_q} w_q^{-1}$.

Then we have the following (Proposition 4.1 in [3]):

Proposition 2. *Suppose that ribbon handlebody presentations P and P' are related by a finite sequence of stable transformations* S1–S5. *Then, the associated ribbon 2-knots K_P and $K_{P'}$ are ambient isotopic.*

We denote by

$$R(p_1, q_1, \ldots, p_n, q_n), \quad p_1, q_1, \ldots, p_n, q_n, \in \mathbb{Z}, \tag{7}$$

a ribbon 2-knot of 1-fusion, which is presented by the ribbon handlebody presentation

$$[\, x, y \mid x = y^w \ (w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}) \,] \,. \tag{8}$$

cf. [5, Sect. 2]. Then, by the transformation S1 we have:

$$R(p_1, q_1, \ldots, p_n, q_n) \approx R(-q_n, -p_n, \ldots, -q_1, -p_1) \tag{9}$$

$$R(p_1, q_1, \ldots, p_n, q_n)! \approx R(-p_1, -q_1, \ldots, -p_n, -q_n) \approx R(q_n, p_n, \ldots, q_1, p_1), \tag{10}$$

where $K \approx K'$ denotes that the two 2-knots K and K' are ambient isotopic and $K!$ the mirror image of K.

Example 3. The ribbon 2-knot Y43 presented by

$$P(\mathrm{Y43}) = [\, x_1, x_2, x_3 \,|\, \rho_1 : x_1^{x_2 x_1} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3 \,] \,. \tag{11}$$

is isotopic to the ribbon 2-knot of 1-fusion $R(1, 1, -1, -1, -1, -1, 1, 1)$. Thus, by Eqs. (9) and (10) Y43 is positive-amphicheiral.

Proof By the transformation S5(ii), we replace x_1 in the power of ρ_1 with $x_3 x_2 x_3 x_2^{-1} x_3^{-1}$ coming from ρ_2. Then $P(\mathrm{Y43})$ is deformed into

$$P(\mathrm{Y43})_1 = \left[x_1, x_2, x_3 \,\middle|\, \rho_1' : x_1^{x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3 \right] . \tag{12}$$

By the transformation S4(ii), we replace ρ_1' by $\rho_1'' : x_3^{(x_3 x_2)^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2$. Then $P(\mathrm{Y43})_1$ is deformed into

$$P(\mathrm{Y43})_2 = \left[x_1, x_2, x_3 \,\middle|\, \rho_1'' : x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2,\ \rho_2 : x_1^{x_3 x_2} = x_3 \right] . \tag{13}$$

By the transformation S3′, $P(\mathrm{Y43})_2$ is deformed into

$$P(\mathrm{Y43})_3 = \left[x_2, x_3 \,\middle|\, x_3^{x_2^{-1} x_3^{-1} x_2 x_3 x_2 x_3 x_2^{-1} x_3^{-1}} = x_2 \right] , \tag{14}$$

which presents $R(-1, -1, 1, 1, 1, 1, -1, -1) (\approx R(1, 1, -1, -1, -1, -1, 1, 1))$. □

4 Yasuda's Table

Yasuda enumerated ribbon 2-knots with ribbon crossing number up to three in [13] and ribbon 2-knots with ribbon crossing four in [15]. He claims that any ribbon 2-knots with ribbon crossing number up to four is presented by one of the following ribbon handlebody presentations:

$$P_1(w) = [\, x_1, x_2 \,|\, x_1^{w} = x_2 \,]\,; \tag{15}$$

$$P_2(w_1, w_2) = [\, x_1, x_2, x_3 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3 \,]\,; \tag{16}$$

$$P_3(w_1, w_2, w_3) = [\, x_1, x_2, x_3, x_4 \,|\, x_1^{w_1} = x_2,\ x_2^{w_2} = x_3,\ x_3^{w_3} = x_4 \,]\,; \tag{17}$$

$$P_4(w_1, w_2, w_3) = [\, x_1, x_2, x_3, x_4 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4 \,]\,; \tag{18}$$

$$P_5(w_1, w_2, w_3, w_4) = [\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4,\ x_2^{w_4} = x_5 \,]\,; \tag{19}$$

$$P_6(w_1, w_2, w_3, w_4) = [\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_1^{w_2} = x_3,\ x_1^{w_3} = x_4,\ x_1^{w_4} = x_5 \,]\,. \tag{20}$$

Remark 4. A ribbon 2-knot with ribbon crossing number up to four presented by the ribbon handlebody presentation

$$[\, x_1, x_2, x_3, x_4, x_5 \,|\, x_1^{w_1} = x_2,\ x_2^{w_2} = x_3,\ x_3^{w_3} = x_4,\ x_4^{w_4} = x_5 \,]\,, \tag{21}$$

$w_i \in F[x_1, x_2, x_3, x_4, x_5]$, is transformed into a ribbon 2-knot presented by one of the ribbon handlebody presentations (15)–(18).

In a similar way to Example 3, we can deform a ribbon 2-knot with up to four ribbon crossing by a finite sequence of stable transformations S1–S5 (Proposition 2) into one of the following two types:

- Type 1: a ribbon 2-knot of 1-fusion.
- Type 2: a composition of two ribbon 2-knots of 1-fusion.

In order to determine the type of a ribbon 2-knot we use the following proposition (Proposition 3.1 in [6]). Indeed, the fundamental group of a Type 2 ribbon 2-knot with ribbon crossing number up to four is isomorphic to the free product $\mathbb{Z}_3 * \mathbb{Z}_3$ (Proposition 3.2 in [6]).

Proposition 5. *The fundamental group of the 2-fold cover of S^4 branched over a ribbon 2-knot of 1-fusion K is the finite cyclic group whose order is the determinant of K, $|\Delta_K(-1)|$.*

Table 2 lists the ribbon 2-knots with ribbon crossing number up to three given by [13], and Table 3 lists the ribbon 2-knots with ribbon crossing four given by Yasuda [15]. Each column in Tables 2 and 3 shows as follows:

- The first column, Name, shows the names of the ribbon 2-knots:
  - (i) The names Ym_n, Ym_n^* ($m = 2$, 3) in Table 2 denote the knots m_n, m_n^* with ribbon crossing number m in [13]; Ym_n^* is the mirror image of Ym_n.
  - (ii) The name Yn ($1 \le n \le 111$) in Table 3 denotes the ribbon 2-knot K_n^2 with ribbon crossing number four in [15].

- The column, C, shows the chirality of the ribbon 2-knots:
  (i) The symbol "a" means that the ribbon 2-knot is positive-amphicheiral.
  (ii) In Table 3 the mirror image knot is listed.
- The column, Presentation, shows a ribbon handlebody presentation of the ribbon 2-knot: P_i is one of the ribbon handlebody presentations (15)–(20), and the symbols j and $\overline{j}$ ($j = 1, 2, 3, 4, 5$) denote the letters x_j and x_j^{-1}, respectively. For example, $P_2(21, 32)$ for the knot Y43 in Table 3 means the presentation Eq. (11) in Example 3.
- The column, Type, shows the type of the ribbon 2-knot:
  (i) A Type 1 ribbon 2-knot is presented by a 1-fusion notation $R(p_1, q_1, \ldots, p_m, q_m)$.
  (ii) A Type 2 ribbon 2-knot is presented by a composition $R(\epsilon_1, \epsilon_2)\#R(\epsilon_3, \epsilon_4)$, $\epsilon_i = \pm 1$.
- The column, $\Delta(t)$, shows the normalized Alexander polynomial of the ribbon 2-knot in the abbreviated form: $(c_{-m}\, c_{-m+1}\, \ldots\, c_{-1}\, [c_0]\, c_1\, \ldots\, c_{n-1}\, c_n) = \sum_{i=-m}^{n} c_i t^i$, $c_i \in \mathbb{Z}$. We normalize the Alexander polynomial of a ribbon 2-knot $\Delta(t) \in \mathbb{Z}[t^{\pm 1}]$, so that $\Delta(1) = 1$ and $(d/dt)\Delta(1) = 0$; cf. [1].
- The column, Det, shows the determinant of the ribbon 2-knot, which is given by $|\Delta(-1)|$.
- The column, Set, shows the name of the set of the ribbon 2-knots sharing the same Alexander polynomial; $\mathcal{A}_i!$ denotes the set of the mirror images of the knots in $\mathcal{A}_i$. For example, $\mathcal{A}_2 = \{\text{Y3\_1}^*, \text{Y27}\}$, $\mathcal{A}_2! = \{\text{Y3\_1}, \text{Y34}\}$, and the knots in the sets $\mathcal{A}_i$, $i = 1, 5, 6, 9, 13, 15, 17$, have reciprocal Alexander polynomials, and so we do not consider the set of mirror images. The sets $\mathcal{A}_i$ with $i \leq 13$ are the same sets as in [6].

The knot Y112 is missed in [15], which has the same Alexander polynomial as Y109; the ribbon handlebodies are shown as in Fig. 1.

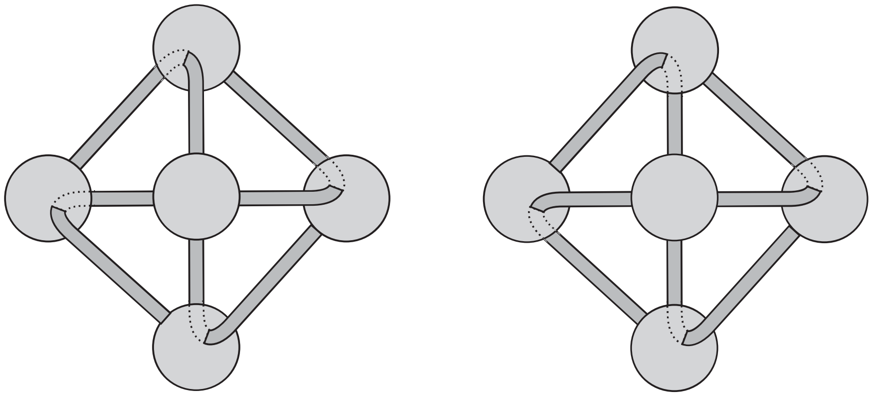

Figure 1: Ribbon handlebodies presenting Y109 and Y112.

Table 2: Ribbon 2-knots with up to three crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y0	a		Trivial knot	([1])	1	$\mathcal{A}_1$
Y2_1	a	$P_1(21)$	$R(1,1)$	(1 [−1] 1)	3	
Y2_2		$P_1(2\bar{1})$	$R(1,-1)$	([0] 2 −1)	3	$\mathcal{A}_3$!
Y2_2*		$P_1(\bar{2}1)$	$R(-1,1)$	(−1 2 [0])	3	$\mathcal{A}_3$
Y3_1		$P_1(211)$	$R(1,2)$	(1 −1 [0] 1)	1	$\mathcal{A}_2$!
Y3_1*		$P_1(221)$	$R(-1,-2)$	(1 [0] −1 1)	1	$\mathcal{A}_2$
Y3_2		$P_1(2\bar{1}\bar{1})$	$R(1,-2)$	([0] 1 1 −1)	1	
Y3_2*		$P_1(\bar{2}11)$	$R(-1,2)$	(−1 1 1 [0])	1	
Y3_3		$P_2(31,2)$	$R(-1,1,1,1)$	(−1 2 −1 [1])	5	
Y3_3*		$P_2(\bar{3}\bar{1},\bar{2})$	$R(1,-1,-1,-1)$	([1] −1 2 −1)	5	
Y3_4		$P_2(31,\bar{2})$	$R(1,1,-1,1)$	(1 −2 [2])	5	
Y3_4*		$P_2(3\bar{1},\bar{2})$	$R(-1,-1,1,-1)$	([2] −2 1)	5	
Y3_5	a	$P_2(3\bar{1},\bar{2})$	$R(1,-1,-1,1)$	(−1 [3] −1)	5	
Y3_6		$P_3(3,4,2)$	$R(-1,1,-1,-1,1,1)$	(1 −3 3 [0])	5	
Y3_6*		$P_3(\bar{3},\bar{4},\bar{2})$	$R(1,-1,1,1,-1,-1)$	([0] 3 −3 1)	5	
Y3_7		$P_3(3,4,\bar{2})$	$R(-1,1,1,-1,1,1)$	(−1 3 [−2] 1)	5	
Y3_7*		$P_3(3,\bar{4},\bar{2})$	$R(1,-1,-1,1,-1,-1)$	(1 [−2] 3 −1)	5	

Table 3: Ribbon 2-knots with four crossings.

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y1	Y7	$P_1(2111)$	$R(1,3)$	(1 −1 0 [0] 1)	3	
Y2	a	$P_1(2121)$	$R(1,1,1,1)$	(1 −1 [1] −1 1)	5	
Y3	Y8	$P_1(2\bar{1}\bar{1}\bar{1})$	$R(3,-1)$	([0] 1 0 1 −1)	3	$\mathcal{A}_4$!
Y4	Y9	$P_1(2\bar{1}2\bar{1})$	$R(1,-1,1,-1)$	([0] 0 3 −2)	5	
Y5	a	$P_1(2211)$	$R(2,2)$	(1 0 [−1] 0 1)	1	
Y6	Y10	$P_1(22\bar{1}\bar{1})$	$R(2,-2)$	([0] 0 2 0 -1)	1	
Y7	Y1	$P_1(2221)$	$R(3,1)$	(1 [0] 0 −1 1)	3	
Y8	Y3	$P_1(\bar{2}111)$	$R(-1,3)$	(−1 1 0 1 [0])	3	$\mathcal{A}_4$
Y9	Y4	$P_1(\bar{2}1\bar{2}1)$	$R(-1,1,-1,1)$	(−2 3 0 [0])	5	
Y10	Y6	$P_1(\bar{2}\bar{2}11)$	$R(-2,2)$	(−1 0 2 0 [0])	1	
Y11	Y18	$P_2(231,\bar{2})$	$R(2,1,-1,1)$	(1 0 [−2] 2)	3	
Y12	Y17	$P_2(23\bar{1},\bar{2})$	$R(2,1,-1,-1)$	([1] 1 −2 1)	3	
Y13	Y16	$P_2(2\bar{3}1,\bar{2})$	$R(2,-1,-1,1)$	([0] 2 −1)	3	$\mathcal{A}_3$!
Y14	Y15	$P_2(2\bar{3}\bar{1},\bar{2})$	$R(2,-1,-1,-1)$	([0] 1 0 1 −1)	3	$\mathcal{A}_4$!
Y15	Y14	$P_2(\bar{2}31,2)$	$R(-2,1,1,1)$	(−1 1 0 1 [0])	3	$\mathcal{A}_4$
Y16	Y13	$P_2(\bar{2}3\bar{1},2)$	$R(1,-1,-1,2)$	(−1 2 [0])	3	$\mathcal{A}_3$
Y17	Y12	$P_2(\bar{2}\bar{3}1,2)$	$R(-2,-1,1,1)$	(1 −2 1 [1])	3	
Y18	Y11	$P_2(\bar{2}\bar{3}\bar{1},2)$	$R(-2,-1,1,-1)$	(2 [−2] 0 1)	3	
Y19	Y31	$P_2(311,2)$	$R(-1,1,1,2)$	(−1 2 −1 0 [1])	3	
Y20	Y33	$P_2(313,2)$	$R(-1,1,1,1,-1,1)$	(−1 2 −2 2 [0])	7	
Y21	Y32	$P_2(313,\bar{2})$	$R(1,1,-1,1,1,1)$	(1 −2 2 [−1] 1)	7	
Y22	Y30	$P_2(3\bar{1}3,2)$	$R(1,-1,-1,1,-1,1)$	(−2 4 [−1])	7	
Y23	a	$P_2(3\bar{1}3,\bar{2})$	$R(1,1,-1,-1,1,1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y24	Y37	$P_2(321,2)$	$R(-1,1,2,1)$	(−1 2 0 [−1] 1)	1	
Y25	Y36	$P_2(32\bar{1},2)$	$R(1,-2,-1,1)$	(−1 [2] 1 −1)	1	
Y26	Y35	$P_2(3\bar{2}1,\bar{2})$	$R(1,1,-2,1)$	([1])	1	$\mathcal{A}_1$
Y27	Y34	$P_2(3\bar{2}\bar{1},\bar{2})$	$R(1,2,-1,-1)$	(1 [0] −1 1)	1	$\mathcal{A}_2$
Y28	Y39	$P_2(331,2)$	$R(-1,2,1,1)$	(−1 1 1 −1 [1])	1	
Y29	Y38	$P_2(331,\bar{2})$	$R(1,2,-1,1)$	(1 −1 −1 [2])	1	
Y30	Y22	$P_2(\bar{3}1\bar{3},\bar{2})$	$R(1,-1,-1,1,1,-1)$	([−1] 4 −2)	7	
Y31	Y19	$P_2(\bar{3}\bar{1}\bar{1},\bar{2})$	$R(2,1,1,-1)$	([1] 0 −1 2 −1)	3	
Y32	Y21	$P_2(\bar{3}\bar{1}\bar{3},2)$	$R(1,1,1,-1,1,1)$	(1 [−1] 2 −2 1)	7	
Y33	Y20	$P_2(\bar{3}\bar{1}\bar{3},\bar{2})$	$R(1,-1,1,1,1,-1)$	([0] 2 −2 2 −1)	7	
Y34	Y27	$P_2(\bar{3}21,2)$	$R(-1,-1,2,1)$	(1 −1 [0] 1)	1	$\mathcal{A}_2$!
Y35	Y26	$P_2(\bar{3}2\bar{1},2)$	$R(1,-2,1,1)$	([1])	1	$\mathcal{A}_1$
Y36	Y25	$P_2(\bar{3}\bar{2}1,\bar{2})$	$R(1,-1,-2,1)$	(−1 1 [2] −1)	1	
Y37	Y24	$P_2(\bar{3}\bar{2}\bar{1},\bar{2})$	$R(1,2,1,-1)$	(1 [−1] 0 2 −1)	1	
Y38	Y29	$P_2(\bar{3}\bar{3}\bar{1},2)$	$R(-1,-2,1,-1)$	([2] −1 −1 1)	1	
Y39	Y28	$P_2(\bar{3}\bar{3}\bar{1},\bar{2})$	$R(1,-2,-1,-1)$	([1] −1 1 1 −1)	1	

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y40	a	$P_2(21, 31)$	$R(1,1)\#R(1,1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y41	Y42	$P_2(21, 3\bar{1})$	$R(1,1)\#R(1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y42	Y41	$P_2(21, \bar{3}1)$	$R(1,1)\#R(-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7!$
Y43	a	$P_2(21, 32)$	$R(-1,-1,1,1,1,1,-1,-1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y44	Y45	$P_2(21, 3\bar{2})$	$R(1,-1,1,1,-1,1,1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y45	Y44	$P_2(21, \bar{3}2)$	$R(-1,1,1,-1,1,1,-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7!$
Y46	a	$P_2(21, \bar{3}\bar{2})$	$R(1,1,1,-1,-1,1,1,1)$	(1 −2 [3] −2 1)	9	$\mathcal{A}_6$
Y47	Y52	$P_2(2\bar{1}, 3\bar{1})$	$R(1,-1)\#R(1,-1)$	([0] 0 4 −4 1)	9	$\mathcal{A}_8$
Y48	a	$P_2(2\bar{1}, \bar{3}1)$	$R(1,-1)\#R(-1,1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y49	Y53	$P_2(2\bar{1}, 32)$	$R(-1,-1,1,1,1,-1,-1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y50	Y55	$P_2(2\bar{1}, 3\bar{2})$	$R(1,-1,1,1,-1,-1,1,-1)$	([0] 0 4 −4 1)	9	$\mathcal{A}_8$
Y51	Y54	$P_2(2\bar{1}, \bar{3}2)$	$R(-1,1,1,-1,1,-1,-1,1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y52	Y47	$P_2(\bar{2}1, \bar{3}1)$	$R(-1,1)\#R(-1,1)$	(1 −4 4 0 [0])	9	$\mathcal{A}_8!$
Y53	Y49	$P_2(\bar{2}1, 32)$	$R(-1,-1,-1,1,1,1,-1,-1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7!$
Y54	Y51	$P_2(\bar{2}1, 3\bar{2})$	$R(1,-1,-1,1,-1,1,1,-1)$	(−2 [5] −2)	9	$\mathcal{A}_9$
Y55	Y50	$P_2(\bar{2}1, \bar{3}2)$	$R(-1,1,-1,-1,1,1,-1,1)$	(1 −4 4 0 [0])	9	$\mathcal{A}_8!$
Y56	Y58	$P_3(3, 1\bar{4}, \bar{2})$	$R(1,1,-1,1,-1,-1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y57	a	$P_3(3, \bar{1}\bar{4}, 2)$	$R(1,-1,-1,-1,-1,1)$	(−1 2 [−1] 2 −1)	7	
Y58	Y56	$P_3(\bar{3}, \bar{1}4, 2)$	$R(-1,-1,1,-1,1,1)$	(2 [−3] 2)	7	$\mathcal{A}_5$
Y59	Y66	$P_4(31, 4, 2)$	$R(-1,-1,1,1,-1,1,1,1)$	(1 −3 3 −1 [1])	9	
Y60	Y64	$P_4(31, 4, \bar{2})$	$R(1,-1,-1,1,1,1,-1,1)$	(−1 3 −3 [2])	9	$\mathcal{A}_7!$
Y61	Y63	$P_4(31, \bar{4}, \bar{2})$	$R(1,1,-1,1,1,-1,-1,1)$	(1 −3 [4] −1)	9	$\mathcal{A}_{10}!$
Y62	Y65	$P_4(3\bar{1}, 4, 2)$	$R(-1,-1,1,1,-1,1,1,-1)$	(1 −3 [4] −1)	9	$\mathcal{A}_{10}!$
Y63	Y61	$P_4(3\bar{1}, 4, \bar{2})$	$R(1,-1,-1,1,1,1,-1,-1)$	(−1 [4] −3 1)	9	$\mathcal{A}_{10}$
Y64	Y60	$P_4(3\bar{1}, \bar{4}, \bar{2})$	$R(1,1,-1,1,1,-1,-1,-1)$	([2] −3 3 −1)	9	$\mathcal{A}_7$
Y65	Y62	$P_4(\bar{3}1, \bar{4}, \bar{2})$	$R(1,1,-1,-1,1,-1,-1,1)$	(−1 [4] −3 1)	9	$\mathcal{A}_{10}$
Y66	Y59	$P_4(\bar{3}1, \bar{4}, 2)$	$R(1,1,-1,-1,1,-1,-1,-1)$	([1] −1 3 −3 1)	9	
Y67	Y82	$P_3(23, 4, 1)$	$R(1,-1,-1,-1,1,1,1,1,-1,-1)$	(1 [−2] 4 −3 1)	11	$\mathcal{A}_{11}!$
Y68	Y81	$P_3(23, 4, \bar{1})$	$R(1,1,-1,-1,1,1,1,-1,-1,-1)$	([0] 3 −4 3 −1)	11	
Y69	Y80	$P_3(23, \bar{4}, 1)$	$R(1,-1,-1,-1,-1,1,1,1,-1,-1)$	(−1 3 [−3] 3 −1)	11	$\mathcal{A}_{13}$
Y70	Y79	$P_3(23, \bar{4}, \bar{1})$	$R(1,1,-1,-1,-1,1,1,-1,-1,-1)$	(1 [−2] 4 −3 1)	11	$\mathcal{A}_{11}!$
Y71	Y78	$P_3(2\bar{3}, 4, 1)$	$R(-1,-1,1,-1,1,1,-1,1,1,-1)$	(2 [−4] 4 -1)	11	$\mathcal{A}_{12}$
Y72	Y77	$P_3(2\bar{3}, 4, \bar{1})$	$R(-1,1,1,-1,1,1,-1,-1,1,-1)$	([-1] 5 −4 1)	11	
Y73	Y76	$P_3(2\bar{3}, \bar{4}, 1)$	$R(-1,-1,1,-1,-1,1,-1,1,1,-1)$	(−2 5 [−3] 1)	11	
Y74	Y75	$P_3(2\bar{3}, \bar{4}, \bar{1})$	$R(-1,1,1,-1,-1,1,-1,-1,1,-1)$	(2 [−4] 4 −1)	11	$\mathcal{A}_{12}$
Y75	Y74	$P_3(\bar{2}3, 4, 1)$	$R(1,-1,-1,1,1,-1,1,1,-1,1)$	(−1 4 [−4] 2)	11	$\mathcal{A}_{12}!$
Y76	Y73	$P_3(\bar{2}3, 4, \bar{1})$	$R(1,1,-1,1,1,-1,1,-1,-1,1)$	(1 [−3] 5 −2)	11	
Y77	Y72	$P_3(\bar{2}3, \bar{4}, 1)$	$R(1,-1,-1,1,-1,-1,1,1,-1,1)$	(1 −4 5 [-1])	11	
Y78	Y71	$P_3(\bar{2}3, \bar{4}, \bar{1})$	$R(1,1,-1,1,-1,-1,1,-1,-1,1)$	(−1 4 [−4] 2)	11	$\mathcal{A}_{12}!$
Y79	Y70	$P_3(\bar{2}\bar{3}, 4, 1)$	$R(-1,-1,1,1,1,-1,-1,1,1,1)$	(1 −3 4 [−2] 1)	11	$\mathcal{A}_{11}$
Y80	Y69	$P_3(\bar{2}\bar{3}, 4, \bar{1})$	$R(-1,1,1,1,1,-1,-1,-1,1,1)$	(−1 3 [−3] 3 −1)	11	$\mathcal{A}_{13}$
Y81	Y68	$P_3(\bar{2}\bar{3}, \bar{4}, 1)$	$R(-1,-1,1,1,-1,-1,-1,1,1,1)$	(−1 3 −4 3 [0])	11	
Y82	Y67	$P_3(\bar{2}\bar{3}, \bar{4}, \bar{1})$	$R(-1,1,1,1,-1,-1,-1,-1,1,1)$	(1 −3 4 [−2] 1)	11	$\mathcal{A}_{11}$

Table 3: Ribbon 2-knots with four crossings (cont'd).

Name	C	Presentation	Type	$\Delta(t)$	Det	Set
Y83	Y90	$P_3(43, 1, 2)$	$R(1, 1, 1, 1, -1, -1, -1, 1)$	$(1\ -2\ [3]\ -2\ 1)$	9	$\mathcal{A}_6$
Y84	Y89	$P_3(43, 1, \bar{2})$	$R(1, -1, 1, 1, -1, 1, -1, -1)$	$([3]\ -4\ 2)$	9	
Y85	Y88	$P_3(43, \bar{1}, 2)$	$R(1, 1, 1, -1, -1, -1, -1, 1)$	$(-1\ [3]\ -2\ 2\ -1)$	9	
Y86	Y87	$P_3(43, \bar{1}, \bar{2})$	$R(1, -1, 1, -1, -1, 1, -1, -1)$	$([1]\ -2\ 4\ -2)$	9	
Y87	Y86	$P_3(\overline{43}, 1, 2)$	$R(-1, 1, -1, 1, 1, -1, 1, 1)$	$(-2\ 4\ -2\ [1])$	9	
Y88	Y85	$P_3(\overline{43}, 1, \bar{2})$	$R(-1, -1, -1, 1, 1, 1, 1, -1)$	$(-1\ 2\ -2\ [3]\ -1)$	9	
Y89	Y84	$P_3(\overline{43}, \bar{1}, 2)$	$R(-1, 1, -1, -1, 1, -1, 1, 1)$	$(2\ -4\ [3])$	9	
Y90	Y83	$P_3(\overline{43}, \bar{1}, \bar{2})$	$R(-1, -1, -1, -1, 1, 1, 1, -1)$	$(1\ -2\ [3]\ -2\ 1)$	9	$\mathcal{A}_6$
Y91	Y106	$P_5(5, 2, 3, 4)$	$R(-1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1, 1)$	$(-1\ 4\ -5\ [3])$	13	$\mathcal{A}_{14}$
Y92	Y105	$P_5(5, 2, 3, \bar{4})$	$R(-1, -1, 1, 1, -1, 1, 1, -1, 1, -1, -1, 1)$	$(1\ -4\ [6]\ -2)$	13	
Y93	Y104	$P_5(5, 2, \bar{3}, 4)$	$R(-1, 1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1)$	$(1\ -3\ [5]\ -3\ 1)$	13	$\mathcal{A}_{15}$
Y94	Y103	$P_5(5, 2, \bar{3}, \bar{4})$	$R(-1, 1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1)$	$(-1\ [4]\ -4\ 3\ -1)$	13	$\mathcal{A}_{16}$
Y95	Y102	$P_5(5, \bar{2}, 3, 4)$	$R(1, -1, -1, -1, 1, 1, 1, -1, -1, 1, 1, 1)$	$(1\ -3\ [5]\ -3\ 1)$	13	$\mathcal{A}_{15}$
Y96	Y101	$P_5(5, \bar{2}, 3, \bar{4})$	$R(1, -1, -1, 1, 1, 1, 1, -1, -1, -1, 1, 1)$	$(-1\ [4]\ -4\ 3\ -1)$	13	$\mathcal{A}_{16}$
Y97	Y100	$P_5(5, \bar{2}, \bar{3}, 4)$	$R(1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1)$	$([3]\ -5\ 4\ -1)$	13	$\mathcal{A}_{14}!$
Y98	Y99	$P_5(5, \bar{2}, \bar{3}, \bar{4})$	$R(1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1)$	$([1]\ -2\ 5\ -4\ 1)$	13	
Y99	Y98	$P_5(\bar{5}, 2, 3, 4)$	$R(-1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1)$	$(1\ -4\ 5\ -2\ [1])$	13	
Y100	Y97	$P_5(\bar{5}, 2, 3, \bar{4})$	$R(-1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1)$	$(-1\ 4\ -5\ [3])$	13	$\mathcal{A}_{14}$
Y101	Y96	$P_5(\bar{5}, 2, \bar{3}, 4)$	$R(-1, 1, 1, -1, -1, -1, -1, 1, 1, 1, -1, -1)$	$(-1\ 3\ -4\ [4]\ -1)$	13	$\mathcal{A}_{16}!$
Y102	Y95	$P_5(\bar{5}, 2, \bar{3}, \bar{4})$	$R(-1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1)$	$(1\ -3\ [5]\ -3\ 1)$	13	$\mathcal{A}_{15}$
Y103	Y94	$P_5(\bar{5}, \bar{2}, 3, 4)$	$R(1, -1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1)$	$(-1\ 3\ -4\ [4]\ -1)$	13	$\mathcal{A}_{16}!$
Y104	Y93	$P_5(\bar{5}, \bar{2}, 3, \bar{4})$	$R(1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1)$	$(1\ -3\ [5]\ -3\ 1)$	13	$\mathcal{A}_{15}$
Y105	Y92	$P_5(\bar{5}, \bar{2}, \bar{3}, 4)$	$R(1, 1, -1, -1, 1, -1, -1, 1, -1, 1, 1, -1)$	$(-2\ [6]\ -4\ 1)$	13	
Y106	Y91	$P_5(\bar{5}, \bar{2}, \bar{3}, \bar{4})$	$R(1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1)$	$([3]\ -5\ 4\ -1)$	13	$\mathcal{A}_{14}!$
Y107	Y111	$P_6(3, 4, 5, 2)$	$R(-1, -1, 1, -1, -1, 1, 1, 1, -1, -1, 1, 1, -1, 1)$	$(-1\ 4\ -6\ 4\ [0])$	15	
Y108	Y110	$P_6(3, 4, 5, \bar{2})$	$R(1, -1, -1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, 1)$	$(1\ -4\ 6\ [-3]\ 1)$	15	
Y109	a	$P_6(3, 4, \bar{5}, \bar{2})$	$R(1, 1, -1, -1, 1, -1, -1, 1, 1, 1, -1, 1, 1, -1)$	$(-1\ 4\ [-5]\ 4\ -1)$	15	$\mathcal{A}_{17}$
Y110	Y108	$P_6(3, \bar{4}, \bar{5}, \bar{2})$	$R(-1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, -1, -1, -1)$	$(1\ [-3]\ 6\ -4\ 1)$	15	
Y111	Y107	$P_6(\bar{3}, \bar{4}, \bar{5}, \bar{2})$	$R(1, 1, -1, 1, 1, -1, -1, -1, 1, 1, -1, -1, 1, -1)$	$([0]\ 4\ -6\ 4\ -1)$	15	
Y112	a	$P_6(3, \bar{4}, 5, \bar{2})$	$R(1, -1, -1, 1, 1, 1, -1, 1, 1, -1, -1, -1, 1, 1)$	$(-1\ 4\ [-5]\ 4\ -1)$	15	$\mathcal{A}_{17}$

5 Classification of the knots

The ribbon 2-knots in the sets $\mathcal{A}_i$, $i = 1, 2, \ldots, 13$, have been classified in [6] except for the pair Y43 and Y46 in $\mathcal{A}_6$, which have isomorphic knot group; see Sect. 7 in [6], where $\mathrm{Y43} = R^8_{8,6}$ and $\mathrm{Y46} = R^8_{8,1}$. In this section we classify the ribbon 2-knots in each of the sets $\mathcal{A}_i$, $i = 14, 15, 16, 17$.

5.1 Classification of the knots in $\mathcal{A}_{14}$

The set $\mathcal{A}_{14}$ consists of the two knots Y91 and Y100, which share the same Alexander polynomial $-t^{-3}+4t^{-2}-5t^{-1}+3$. Since they have different trace sets as shown in Table 4, we obtain Y91 $\not\approx$ Y100.

Table 4: Trace sets of the knots in $\mathcal{A}_{14}$, $\mathcal{A}_{16}$, and $\mathcal{A}_{17}$.

Set	Knot	Trace set
$\mathcal{A}_{14}$	Y91	$\{0,0,0,0,0,0,(\delta+\epsilon\sqrt{5})/2 \mid \delta,\epsilon=\pm 1\}$
	Y100	$\left\{0,0,0,0,0,0,\delta\sqrt{5(3+\epsilon\sqrt{3})/6} \,\middle|\, \delta,\epsilon=\pm 1\right\}$
$\mathcal{A}_{16}$	Y94	$\{0,0,0,0,0,0\}$
	Y96	$\{0,0,0,0,0,0,\pm\alpha_1,\pm\alpha_2,\pm\alpha_3\}$
$\mathcal{A}_{17}$	Y109	$\left\{\begin{array}{l}\mathbb{C}-\{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0,\\ \mathbb{C}-\{\pm\sqrt{5}\},\ \mathbb{C}-\{\pm\sqrt{5}\},\ \pm 1,\\ (\delta+\epsilon\sqrt{13})/2,\ (\delta+\epsilon\sqrt{13})/2\ (\delta,\ \epsilon=\pm 1),\\ \pm\beta_1,\ \pm\beta_2,\ \pm\beta_3,\ \pm\beta_4\end{array}\right\}$
	Y112	$\left\{\begin{array}{l}\mathbb{C}-\{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0,\\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4\end{array}\right\}$

- The numbers α_k, $k = 1, 2, 3$, are the roots of the cubic equation $1 - x - 2x^2 + x^3 = 0$ with $-1 < \alpha_1 < 0 < \alpha_2 < 1$, $2 < \alpha_3 < 3$.
- The complex numbers β_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 2x - 4x^2 + x^3 + x^4 = 0$; $\beta_1, \beta_2 \risingdotseq 1.25 \pm 0.27i$, $\beta_3, \beta_4 \risingdotseq -1.75 \pm 0.17i$.
- The complex numbers γ_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 4x^2 + x^4 = 0$; $\gamma_k \risingdotseq \pm 1.46 \pm 0.34i$.

5.2 Classification of the knots in $\mathcal{A}_{15}$

The set $\mathcal{A}_{15}$ consists of the four knots Y93, Y95, Y102(= Y95!), and Y104(= Y93!), which share the same Alexander polynomial $t^{-2} - 3t^{-1} + 5 - 3t + t^2$. Table 5 lists the trace sets of the irreducible representations to $\mathrm{SL}(2,\mathbb{C})$ of the knot groups of Y93 and Y95, and the associated twisted Alexander polynomials, which show these four knots are mutually non-isotopic. In fact, since the twisted Alexander polynomials are not reciprocal, the knots Y93 and Y95 are not positive-amhicheiral.

Table 5: Twisted Alexander polynomials of Y93 and Y95 in $\mathcal{A}_{15}$.

Set	Knot	$(s+s^{-1},u)$	Twisted Alexander polynomial
$\mathcal{A}_{15}$	Y93	$\left(\frac{\epsilon}{\sqrt{2}},\frac{3}{2}\right)$ $(\epsilon=\pm1)$	$1-\epsilon\sqrt{2}t+\frac{5}{2}t^2-\epsilon\frac{3}{\sqrt{2}}t^3+\frac{5}{2}t^4-\epsilon\sqrt{2}t^5+t^6$
		$(0,\alpha_1)$	$1+\beta_1t^2+\gamma_2t^4+t^6$
		$(0,\alpha_2)$	$1+\beta_5t^2+\gamma_1t^4+t^6$
		$(0,\alpha_3)$	$1+\beta_4t^2+\gamma_1t^4+t^6$
		$(0,\alpha_4)$	$1+\beta_2t^2+\gamma_3t^4+t^6$
		$(0,\alpha_5)$	$1+\beta_3t^2+\gamma_2t^4+t^6$
		$(0,\alpha_6)$	$1+\beta_6t^2+\gamma_3t^4+t^6$
	Y95	$(0,\alpha_1)$	$1+\beta_1t^2+\beta_3t^4+t^6$
		$(0,\alpha_2)$	$1+\beta_2t^2+\beta_1t^4+t^6$
		$(0,\alpha_3)$	$1+\beta_5t^2+\beta_4t^4+t^6$
		$(0,\alpha_4)$	$1+\beta_6t^2+\beta_2t^4+t^6$
		$(0,\alpha_5)$	$1+\beta_6t^2+\beta_4t^4+t^6$
		$(0,\alpha_6)$	$1+\beta_5t^2+\beta_3t^4+t^6$

- The numbers α_k, $k=1,\ldots,6$, are the roots of the 6th order equation $13-91x+182x^2-156x^3+65x^4-13x^5+x^6=0$ with $0<\alpha_1<0.5<\alpha_2<1<\alpha_3<2<\alpha_4<3<\alpha_5<3.5<\alpha_6<4$.
- The numbers β_k, $k=1,\ldots,6$, are the roots of the 6th order equation $-1-81x+201x^2-178x^3+73x^4-14x^5+x^6=0$ with $-1<\beta_1<0<\beta_2<1$, $2<\beta_3<2.4<\beta_4<2.8<\beta_5<3$, $5<\beta_6<6$.
- The numbers γ_k, $k=1,2,3$, are the roots of the cubic equation $-5+12x-7x^2+x^3=0$ with $0<\gamma_1<1<\gamma_2<2$, $4<\gamma_3<5$.

5.3 Classification of the knots in $\mathcal{A}_{16}$

The set $\mathcal{A}_{16}$ consists of the two knots Y94 and Y96, which share the same Alexander polynomial $-t^{-1}+4-4t+3t^2-t^3$. Since they have different trace sets as shown in Table 4, we obtain Y94 $\not\approx$ Y96.

5.4 Classification of the knots in $\mathcal{A}_{17}$

The set $\mathcal{A}_{17}$ consists of the two knots Y109 and Y112, which share the same Alexander polynomial $-t^{-2}+4t^{-1}-5+4t-t^2$. Since they have different trace sets as shown in Table 4, we obtain Y109 $\not\approx$ Y112.

Remark 6. According to Toshio Sumi [9], we can also distinguish the knots Y109 and Y112 in the following ways:

(i) They have distinct twisted Alexander polynomials associated to the nonabelian representations to $\mathrm{SL}(2,2)$ as listed in Table 6.

(ii) They have distinct numbers of the irreducible representations to $\mathrm{SL}(2,7)$.

Table 6: Twisted Alexander polynomials of the knots in $\mathcal{A}_{17}$. .

Set	Knot	$\rho : \pi K \to \mathrm{SL}(2,2)$	$\Delta_{K,\rho}$
$\mathcal{A}_{17}$	Y109	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1 + t^6$
	Y112	$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$1 + t^2 + t^4 + t^6$

References

[1] Kazuo Habiro, Taizo Kanenobu, and Akiko Shima. Finite type invariants of ribbon 2-knots. In *Low-dimensional topology (Funchal, 1998)*, volume 233 of *Contemp. Math.*, pages 187–196. Amer. Math. Soc., Providence, RI, 1999.

[2] Taizo Kanenobu and Seiya Komatsu. Enumeration of ribbon 2-knots presented by virtual arcs with up to four crossings. *J. Knot Theory Ramifications*, 26(8):1750042, 41, 2017.

[3] Taizo Kanenobu and Masafumi Matsuda. Presentation of a ribbon 2-knot. *J. Knot Theory Ramifications* (to appear).

[4] Taizo Kanenobu and Toshio Sumi. Twisted Alexander polynomial of a ribbon 2-knot of 1-fusion. *Osaka J. Math.* (to appear).

[5] Taizo Kanenobu and Toshio Sumi. Classification of a family of ribbon 2-knots with trivial Alexander polynomial. *Commun. Korean Math. Soc.*, 33(2):591–604, 2018.

[6] Taizo Kanenobu and Toshio Sumi. Classification of ribbon 2-knots presented by virtual arcs with up to four crossings. *J. Knot Theory Ramifications*, 28(10):1950067, 18, 2019.

[7] Taizo Kanenobu and Kota Takahashi. Classification of ribbon 2-knots of 1-fusion with length up to six. Preprint, 2020.

[8] Shin Satoh. Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications*, 9(4):531–542, 2000.

[9] Toshio Sumi. E-mail to Kanenobu of Aug. 24, 2019.

[10] Takeshi Yajima. On simply knotted spheres in R^4. *Osaka J. Math.*, 1:133–152, 1964.

[11] Takeshi Yajima. On a characterization of knot groups of some spheres in R^4. *Osaka J. Math.*, 6:435–446, 1969.

[12] Takaaki Yanagawa. On ribbon 2-knots. The 3-manifold bounded by the 2-knots. *Osaka J. Math.*, 6:447–464, 1969.

[13] Tomoyuki Yasuda. Crossing and base numbers of ribbon 2-knots. *J. Knot Theory Ramifications*, 10(7):999–1003, 2001.

[14] Tomoyuki Yasuda. Ribbon 2-knots with distinct ribbon types. *J. Knot Theory Ramifications*, 18(11):1509–1523, 2009.

[15] Tomoyuki Yasuda. Ribbon 2-knots of ribbon crossing number four. *J. Knot Theory Ramifications*, 27(10):1850058, 20, 2018.

Department of Mathematics
Osaka City University University
Osaka 558-8585
JAPAN
E-mail address: kanenobu@sci.osaka-cu.ac.jp

大阪市立大学大学院・理学研究科　金信 泰造