# Khovanov–Rozansky HOMFLY homology and general diagrams: a survey

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In this note, we will review the definition of the HOMFLY homology, and we will see that it is not well-defined for general diagrams, by computing the graded Euler characteristic of the lowest $q$-degree part of the homology.

## 1 Overview

The HOMFLY polynomial $P(L)(v, q)$ [3, 10] is a polynomial link invariant, which has been widely studied since its discovery. The polynomial $P(L)(v, q)$ is recursively computed from a diagram of $L$, by the skein relation

$$v^{-1}P(\diagup\!\!\!\!\diagdown) - vP(\diagdown\!\!\!\!\diagup) = (q^{-1} - q)P(\uparrow\uparrow),$$

and the normalization $P(\text{unknot}) = 1$.

The invariant is specialized to other skein polynomials, namely the Alexander polynomial ($v = 1$) and the $sl(N)$ polynomials ($v = q^N$). We note that the $sl(2)$ polynomial is the Jones polynomial, which is the first discovered skein polynomial [4].

At the end of the 20th century, Khovanov introduced the concept of "categorification" of link invariants. He defined a homological link invariant $\mathrm{Kh}(L)$ called Khovanov homology [6], which is thought as a categorification of the Jones polynomial. It has two main properties as a categorification:

- its graded Euler characteristic is the Jones polynomial,
- it is a topological quantum field theory (TQFT), that is, a cobordism $D_1 \to D_2$ between two link diagrams is associated with a homomorphism $\mathrm{Kh}(D_1) \to \mathrm{Kh}(D_2)$ between their homologies.

The concept of categorification is now extremely common in topology, for example, the knot Floer homology is a categorification of the Alexander polynomial.

A few years after Khovanov's breakthrough, Khovanov and Rozansky [7, 8] defined the $sl(N)$ homologies $H_N(L)$ and the HOMFLY homology $H(L)$. Dunfield, Gukov and Rasmussen [2] proposed one conjecture concerning these homology theories, stated vaguely below.

A homological knot invariant $\mathcal{H}(K)$ is expected to exist here, along with differentials $\{d_N\}_{N\in\mathbb{Z}}$ on it. The conjecture claims that the homologies $H(\mathcal{H}(K), d_N)$ are isomorphic

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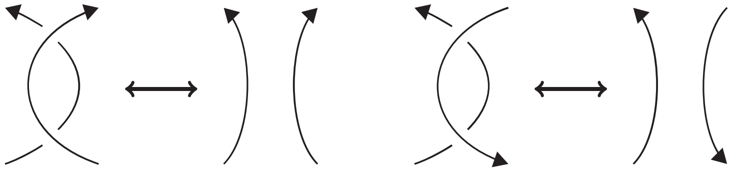
to the $sl(N)$ homologies $H_N(K)$ for $N > 0$, and the homology $H(\mathcal{H}(K), d_0)$ is isomorphic to the knot Floer homology of $K$. Although more properties on differentials $d_N$ are required in the conjecture, we can roughly summarize the conjecture as follows: $\mathcal{H}(K)$ is an inclusive categorified invariant of skein polynomials.

The Khovanov–Rozansky HOMFLY homology $H(K)$ is a reasonable candidate for $\mathcal{H}(K)$. As another evidence for it, Rasmussen [11] constructed a spectral sequence from $H(K)$ converging to $H_N(K)$, but the conjecture is still unsettled right now.

As Rasmussen's spectral sequence tells us, $H(K)$ "contains" all the homologies $H_N(K)$. However, there is a big philosophical mystery between them: well-definedness for general diagrams.

First of all, Khovanov and Rozansky defined the $sl(N)$ homology $H_N(D)$ for all link diagrams $D$. Its invariance under the Reidemeister moves is proven, therefore, $H_N(D)$ is well-defined for all diagrams. They also defined the HOMFLY homology $H(D)$ for all diagrams $D$, however, they failed to show its invariance under certain Reidemeister moves. The missing move here is the RIIb move (see figure 1), and it turns out that $H(D)$ cannot be invariant under this move [1]. At least $H(D)$ is invariant under braid-like Reidemeister moves, so $H(D)$ is well-defined for *closed braid* diagrams.



(a) RIIa move

(b) RIIb move

Figure 1: The Reidemeister II moves

Considering the above, now we have a natural question below.

**Question 1.** *Is there any definition of the HOMFLY homology for general diagrams? If not, why doesn't it exist?*

In this note, we will see how the homology $H(D)$ fails to be an invariant for general diagrams, by observing the lowest $q$-degree part of $H(D)$. We hope that it gives some hints on the question. (Note that it gives us a similar construction with the chromatic homology defined by Stošić [12].)

The organization of the note is as follows. In section 2, the definition of the HOMFLY homology will be explained. In section 3, we review Koszul complexes and how to compute their homologies in simple cases, and prepare some lemmas for section 4. In section 4, we observe the lowest $q$-degree part of the homology and we compute its graded Euler characteristic. The result shows that it cannot be an invariant for general diagrams.

## 2 Definition of the HOMFLY homology

We review the definition of the HOMFLY homology, following Rasmussen [11].

*Remark* 2. We will define the homology $H(D)$ for general diagrams $D$ first, however, this is *not* a link invariant. When we restrict $D$ to closed braid diagrams, $H(D)$ induces a link invariant and it is called the HOMFLY homology. (In [11], it is rather called the reduced HOMFLY homology.)

Let $D$ be an oriented link diagram. We assume that $D$ is connected, in the sense that the underlying projection $G(D)$ of $D$ is connected. We will define the *diagram ring* $R(D)$ of $D$ below.

We consider the projection $G(D)$ as an oriented 4-valent graph. The set of edges in $G(D)$ is denoted by $E(D)$. Let $V'(D)$ be the $\mathbb{Q}$-vector space $\mathbb{Q}\langle e\rangle_{e\in E(D)}$ generated by the edge set $E(D)$. Let $\pi \in V'(D)$ be the sum of all edges in $G(D)$. For each crossing $p$ of $D$, we define a vector $v_p \in V'(D)$ by $v_p = a + b - c - d$, where $a, b$ are outgoing edges at $p$ and $c, d$ are incoming edges at $p$ (see figure 2). Let $W'(D)$ be a subspace of $V'(D)$ spanned by $\pi$ and $v_p$ for all crossings $p$. We define the vector space $V(D)$ as the quotient $V'(D)/W'(D)$.

**Lemma 3.** $\dim V(D) = c(D)$*, where* $c(D)$ *is the number of crossings in* $D$*.*

*Proof.* We have $\dim V'(D) = 2c(D)$. The number of generators of $W'(D)$ is $1+c(D)$, but these generators have unique relation $\sum_p v_p = 0$. Therefore $\dim W'(D) = c(D)$. ☐

The ring $R(D)$ is defined as the symmetric algebra of $V(D)$; if we fix a basis $\{x_i\}$ of $V(D)$, then $R(D)$ is a polynomial ring $\mathbb{Q}[x_i]$ generated by the vectors $x_i$. We introduce the *q-grading* on $R(D)$ by setting every element in $V(D)$ on degree 2.

*Remark* 4. We can also regard that $R(D)$ is the quotient of the polynomial ring $\mathbb{Q}[E(D)]$, by the relations $\pi = 0$ and $v_p = 0$.

We wish to define the double chain complex $C(D)$ over $R = R(D)$ for a diagram $D$, but we start with some conventions and terminologies on $C(D)$.

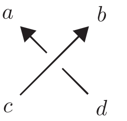
The complex $C(D)$ has two homological gradings, and we write $C(D) = \oplus C^{j,k}(D)$. The first degree is called the *horizontal degree*, and the second degree is called the *vertical degree*.

The complex $C(D)$ has two differentials: the *horizontal differential* $d_h$ and the *vertical differential* $d_v$. Each of them corresponds to one of the above gradings, and they have degree 2; $\deg(d_h) = (2, 0)$, and $\deg(d_v) = (0, 2)$. The $R$-module $C^{j,k}(D)$ will be trivial if $j$ or $k$ is odd. Summarizing the above, $C(D)$ is a double complex with "doubled" cohomological gradings.

Now it is time to construct $C(D)$. The complex $C(D)$ is partitioned into pieces $C_p(D)$, where $p$ runs over all crossings in $D$. The double complex $C_p(D)$ is defined by the diagrams in figure 2, according to the sign of $p$. In the figure, the horizontal arrows define $d_h$ and the vertical arrows define $d_v$. Each arrow is labeled by an element $x$ in $R$, and it means that the arrow represents the multiplication map by $x$. Here, $a, b, c, d$ are edges in $G(D)$ near $p$ as shown in the right figure. The underline specifies the module on bidegree $(0, 0)$, and $\{n\}$ means a $q$-grading upper-shift by $n$; for example, in the complex for a positive crossing, the $R$ on the lower-right corner has bidegree $(0, -2)$.

It is easy to check that two differentials commute and that $\deg_q(d_h) = 2$, $\deg_q(d_v) = 0$ in $C_p(D)$. Finally $C(D)$ is defined as the tensor product (over $R$) of $C_p(D)$ for all crossings $p$ of the diagram $D$.

$$C_p(D) \;=\; \begin{tikzcd} R \arrow[r, "a-c"] & \underline{R} \\ R\{2\} \arrow[u, "b-c"] \arrow[r, "(a-c)(b-c)"'] & R \arrow[u, "1"'] \end{tikzcd} \qquad \text{for} \qquad p \;=$$




$$C_p(D) \;=\; \begin{tikzcd} R \arrow[r, "(a-c)(b-c)"] & R\{-2\} \\ R \arrow[u, "1"] \arrow[r, "a-c"'] & \underline{R} \arrow[u, "b-c"'] \end{tikzcd} \qquad \text{for} \qquad p \;=$$

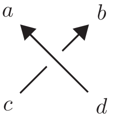



Figure 2: The definition of $C_p(D)$

In order to obtain $H(D)$ from $C(D)$, we need to take homology twice. First we take homology $H(C(D), d_h)$ by the horizontal differential $d_h$, and then we take homology $H(H(C(D), d_h), d_v^*)$ by the induced differential $d_v^*$. The resulting homology will be denoted by $\overline{H}(D)$. By applying some grading shifts on $\overline{H}(D)$, we obtain $H(D)$.

**Definition 5** ([8, 11])**.** Let $w$ be the writhe of $D$, and let $s$ be the number of Seifert circles of $D$. The homology $H(D)$ of the diagram $D$ is given by

$$H(D) = \overline{H}(D)\{-w+s-1\}(w+s-1, w-s+1),$$

where $\{\cdot\}$ means $q$-grading upper-shift, and $(\cdot,\cdot)$ means homological bigrading upper-shift. When $D$ is a closed braid diagram, then $H(D)$ is called the *HOMFLY homology* of $D$, and it induces a link invariant.

The graded Euler characteristic of the HOMFLY homology $H(D)$ is the HOMFLY polynomial $P(D)$.

**Theorem 6** ([11])**.** *For a closed braid diagram $D$, we have*

$$P(D) = \sum_{i,j,k} (-1)^{(k-j)/2} q^i v^j \dim H^{i,j,k}(D),$$

*where $H^{i,j,k}(D)$ is the part of $H(D)$ with q-degree $i$ and homological bidegree $(j,k)$.*

## 3 Koszul complexes

Let $R$ be an algebra over $\mathbb{Q}$. For an element $a$ in $R$, we define a cochain complex $[a]$ by

$$[a] = (\underline{R} \xrightarrow{a} R),$$

where the underline specifies the module on cohomological degree 0.

Let $a = (a_1, \dots, a_n)$ be a sequence of elements in $R$. The *Koszul complex* for $a$ is the tensor product $[a_1] \otimes \cdots \otimes [a_n]$ of complexes over $R$. We denote this complex by $[a_1, \dots, a_n]$.

We give an equivalent definition of the Koszul complex $[a_1, \dots, a_n]$ in the following. Let $I$ be the finite set $\{1, \dots, n\}$. For each subset $A \subset I$, we assign one copy of $R$ and denote it by $R_A$. For $A \subset I$ and $i \in I - A$, we define a map $d_{A,i}\colon R_A \to R_{A \cup \{i\}}$ as the multiplication map by $(-1)^s a_i$ where $s = |\{j \in A : j < i\}|$.

We define the $k$-th cochain $R$-module $C^k$ by

$$C^k = \bigoplus_{|A|=k} R_A,$$

and we define the differential $d\colon C^k \to C^{k+1}$ as the sum

$$d = \sum_{|A|=k,\ i \in I-A} d_{A,i}.$$

Now the cochain complex $(C^*, d)$ is isomorphic to $[a_1, \dots, a_n]$. The complex $(C^*, d)$ is called a cube complex of $R$'s, because of its shape (figure 3 shows it for $n = 3$).

$$\begin{tikzcd}
 & R_{\{1\}} \arrow[r, "-a_2"] \arrow[dr, "-a_3"] & R_{\{1,2\}} \arrow[dr, "a_3"] & \\
R_{\emptyset} \arrow[ur, "a_1"] \arrow[r, "a_2"] \arrow[dr, "a_3"'] & R_{\{2\}} \arrow[ur, "a_1"] \arrow[dr, "-a_3"'] & R_{\{1,3\}} \arrow[r, "-a_2"] & R_{\{1,2,3\}} \\
 & R_{\{3\}} \arrow[ur, "a_1"] \arrow[r, "a_2"'] & R_{\{2,3\}} \arrow[ur, "a_1"'] &
\end{tikzcd}$$

Figure 3: A 3-dimensional cube complex

The complex $[a_1, \dots, a_n]$ has another description. As a graded $R$-module, it is isomorphic to the exterior algebra (over $R$) $\bigwedge^* M$ of the free $R$-module $M = R\langle e_1, \dots, e_n\rangle$. The differential on $\bigwedge^* M$ can be defined by $(\sum_i a_i e_i)\wedge$. It is not hard to verify that the complex $\bigwedge^* M$ is isomorphic to $[a_1, \dots, a_n]$. With this identification, we will sometimes write

$$[a_1, \dots, a_n] = \begin{bmatrix} a_1, \dots, a_n \\ e_1, \dots, e_n \end{bmatrix}.$$

This description gives us elementary operations on Koszul complexes. For $\lambda \in R$ and $1 \le i, j \le n$ with $i \ne j$, we have

$$\begin{aligned} &a_1 e_1 + \cdots + a_i e_i + \cdots + a_j e_j + \cdots + a_n e_n \\ &\qquad = a_1 e_1 + \cdots + (a_i + \lambda a_j) e_i + \cdots + a_j (e_j - \lambda e_i) + \cdots + a_n e_n. \end{aligned}$$

Therefore, we have the following isomorphism:

$$\begin{bmatrix} a_1, \dots, a_n \\ e_1, \dots, e_n \end{bmatrix} = \begin{bmatrix} a_1, & \dots & a_i + \lambda a_j, & \dots & a_j, & \dots & a_n \\ e_1, & \dots & e_i, & \dots & e_j - \lambda e_j, & \dots & e_n \end{bmatrix}.$$

In general, it is quite difficult to compute the homology of Koszul complexes. Fortunately, however, the complexes we will work with are relatively easy to treat.

Now we additionally assume that $R$ is the symmetric product of a $\mathbb{Q}$-vector space $V$. Let $x = (x_1, \dots, x_n)$ be a sequence of vectors in $V$, and we wish to compute the homology of $[x_1, \dots, x_n]$.

**Lemma 7.** *If $x_1, \dots, x_n$ are linearly independent in $V$, then*

$$H\left(\begin{bmatrix} x_1, \dots, x_n \\ e_1, \dots, e_n \end{bmatrix}\right) = R/(x_1, \dots, x_n) \cdot (e_1 \wedge \cdots \wedge e_n).$$

*Proof.* This is a special case of the well-known result for regular sequences. Briefly, the lemma can be proven by induction on $n$, using short exact sequences

$$0 \to [x_1, \dots, x_{n-1}] \xrightarrow{\wedge x_n} [x_1, \dots, x_n] \xrightarrow{\pi} [x_1, \dots, x_{n-1}] \to 0.$$

$\square$

**Lemma 8.** *Suppose that $x_1, \dots, x_k$ are linearly independent, and that $x_{k+1}, \dots, x_n$ can be written as a linear combination of $x_1, \dots, x_k$ as follows:*

$$x_j = \sum_{i=1}^{k} r_{ij} x_i \quad (j = k+1, \dots, n).$$

*Then we have*

$$H\left(\begin{bmatrix} x_1, \dots, x_n \\ e_1, \dots, e_n \end{bmatrix}\right) = (e_1' \wedge \cdots \wedge e_k') \wedge \bigwedge\nolimits^{*} R'\langle e_{k+1}, \dots, e_n \rangle,$$

*where $R' = R/(x_1, \dots, x_k)$ and*

$$e_i' = e_i + \sum_{j=k+1}^{n} r_{ij} e_j \quad (i = 1, \dots, k).$$

*Proof.* We have

$$\begin{bmatrix} x_1, \dots, x_n \\ e_1, \dots, e_n \end{bmatrix} = \begin{bmatrix} x_1, & \dots & x_k, & 0, & \dots & 0 \\ e_1', & \dots & e_k', & e_{k+1}, & \dots & e_n \end{bmatrix},$$

by applying elementary operations. Since $H([x_1, \dots, x_k]) = R' \cdot (e_1' \wedge \cdots \wedge e_k')$ by the previous lemma, we have the claimed isomorphism. $\square$

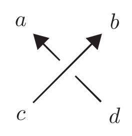
## 4 The lowest $q$-degree part

Let $D$ be a connected, oriented link diagram. By the definition, the lowest $q$-degree with a non-trivial chain group in $C(D)$ is $-2c_-(D)$, where $c_-(D)$ is the number of negative crossings in $D$. We wish to compute the homology $\overline{H}(D)$ on that part, as possible as we can.

*Remark* 9. The lowest $q$-degree in the homology $H(D)$ is $-c(D)+s(D)-1$, where $c(D)$ is the number of crossings in $D$ and $s(D)$ is the number of Seifert circles of $D$. The value $-c(D)+s(D)-1$ is known as the Morton bound [9] for the $q$-degree in the HOMFLY polynomial $P(D)$.

For our purposes, we may simplify the complexes $C_p(D)$. The simplified complexes are shown in figure 4, and they are denoted by $\widehat{C}_p(D)$. We define the complex $\widehat{C}(D)$ as the tensor product of $\widehat{C}_p(D)$ for all crossings $p$ in $D$ (over $R$).

$$\widehat{C}_p(D) = \begin{tikzcd} R \arrow[r, "a-c\ (=x_p)"] & \underline{R} \\ 0 \arrow[u, "0"] \arrow[r, "0"'] & R \arrow[u, "1"'] \end{tikzcd} \quad \text{for} \quad p =$$



$$\widehat{C}_p(D) = \begin{tikzcd} 0 \arrow[r, "0"] & R\{-2\} \\ 0 \arrow[u, "0"] \arrow[r, "0"'] & \underline{0} \arrow[u, "0"'] \end{tikzcd} \quad \text{for} \quad p =$$

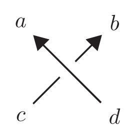


Figure 4: The simplified complex $\widehat{C}_p(D)$ for crossings

*Remark* 10. Because the horizontal differential of $C(D)$ has $q$-degree 2, we cannot work only on the lowest $q$-degree part for the moment. This is why we are still working over $R$, and why $\widehat{C}(D)$ still has elements with higher $q$-degree than $-2c_-(D)$.

As we can see, $\widehat{C}_p(D)$ has a unique $R$-summand for a negative crossing $p$. We can ignore them in computations of homologies, thus we will assume that $D$ has only positive crossings. Note that the lowest $q$-degree in $C(D)$ (or $\widehat{C}(D)$) is 0 for positive diagrams $D$.

Let $\Gamma(D)$ be the Seifert graph of $D$, and let $\varepsilon(D)$ be the set of edges in $\Gamma(D)$. For each crossing $p$ in $D$, we will denote by the same symbol $p$ its corresponding edge of $\Gamma(D)$.

Let us fix a crossing $p$ in $D$. We define an element $x_p \in R$ by $x_p = a - c$, where $a, b, c, d$ are edges in $G(D)$ near $p$ as shown in figure 4. Now we can consider that $\widehat{C}_p(D)$ is a complex of (horizontal) complexes $R \to [x_p]$, where $[x_p]$ is a Koszul complex for $x_p$.

The tensor product $\widehat{C}(D) = \otimes_p \widehat{C}_p(D)$ gives us a "cube complex" of horizontal complexes. A subset $A \subset \varepsilon(D)$ gives one vertex of the cube, and its corresponding horizontal complex is a Koszul complex $\otimes_{p\in A}[x_p] = [x_p]_{p\in A}$. In order to compute their homologies, we need the following lemma.

**Lemma 11.** *Let $A$ be a subset of $\varepsilon(D)$. The vectors $\{x_p\}_{p\in A}$ are linearly independent if and only if $A$ does not give a cut of the graph $\Gamma(D)$.*

*Proof.* Among all vectors $x_p$, we have one linear relation for each Seifert circle $S$ of $D$: the sum $\sum_p \pm x_p$ is zero if $p$ runs over crossings on $S$. (The sign of $x_p$ is determined by whether $p$ is on the right or left of $S$.)

If $A$ gives a minimum cut of $\Gamma(D)$, then we can deduce a linear relation on $\{x_p\}_{p\in A}$ as a linear sum of the above relations. Hence the only-if part is proven.

For a subset $A \subset \varepsilon(D)$, let $D_A$ be the diagram obtained from $D$ by smoothing all crossings in $A$.

In order to prove the if part of the lemma, let us assume that $A$ does not give a cut in $\Gamma(D)$. It is sufficient to prove that the set $\{x_p\}_{p\in A}$ can be extended to a basis of $V(D)$.

By the assumption on $A$, $D_A$ is still connected. Therefore, we can take a set $B$ of crossings in $D_A$ so that $D_{A\cup B}$ is a knot diagram. For a crossing $p \notin A \cup B$, we define a vector $y_p$ in $V(D)$ by $y_p = b - c$, where $a, b, c, d$ are as in the figure 4. Now the set $X = \{x_p\}_{p\in A\cup B} \cup \{y_p\}_{p\notin A\cup B}$ is spanning $V(D)$, because any difference of two edges in $G(D)$ can be written by a linear combination of vectors in $X$. Since $\dim V(D) = c(D)$ by the lemma 3, $X$ is a basis of $V(D)$. $\square$

Now we can compute the homologies of horizontal complexes $[x_p]_{p\in A}$, by using lemma 8 and lemma 11.

**Lemma 12.** *Let $U$ be a maximal uncut of $\varepsilon(D)$ contained in $A$. Then we have*

$$H([x_p]_{p\in A}) \cong \bigwedge\nolimits^{*}(R'\langle e_p\rangle_{p\in A-U}),$$

*where $R' = R/(x_p)_{p\in U}$. The lowest $q$-degree part of the homology is $\bigwedge^*(\mathbb{Q}\langle e_p\rangle_{p\in A-U})$.*

As the main result of this note, we compute the graded Euler characteristic of the lowest $q$-degree part of $\overline{H}(D)$.

**Theorem 13.** *Assume that $D$ is a connected, oriented, positive link diagram. Let $s$ be the number of Seifert circles of $D$. We have*

$$(-1)^{s-1}T_\Gamma(v^{-2}, 0) = \sum_{j,k}(-1)^{(k-j)/2}v^j \dim \overline{H}^{0,j,k}(D),$$

*where $T_\Gamma(x, y)$ is the Tutte polynomial of the graph $\Gamma = \Gamma(D)$.*

*Remark* 14. The Tutte polynomial $T_G(x, y)$ of a graph $G = (V, E)$ is

$$T_G(x, y) = \sum_{A\subset E}(x-1)^{k(A)-k(E)}(y-1)^{b_1(A)},$$

where $k(A)$ is the number of connected components in the subgraph $(V, A)$ and $b_1(A)$ is the first Betti number of $(V, A)$.

*Proof.* We can compute the graded Euler characteristic on the right-hand side before taking homology by vertical differentials, but after taking homology by horizontal differentials.

Let us fix a subset $A \subset \varepsilon(D)$. We denote the complement $\varepsilon(D) - A$ by $A^*$. Let $U$ be a maximal uncut of $\varepsilon(D)$ contained in $A$. If we contract $A^*$ in $\Gamma(D)$, then the edges in $A - U$ will give a spanning tree in the resulting graph. Therefore, we have $|A - U| = k(A^*) - 1$.

The lowest $q$-degree part of the homology $H([x_p]_{p \in A})$ gives

$$(-1)^{|A^*|}(1 - v^{-2})^{k(A^*)-1} = (-1)^{s-1} \cdot (-1)^{b_1(A^*)}(v^{-2} - 1)^{k(A^*)-1}$$

in the graded Euler characteristic of $\overline{H}(D)$. Therefore, we have the claimed equation. □

Kálmán and Murakami [5] showed that the lowest $q$-degree part of the HOMFLY polynomial for a positive diagram $D$ is written by the interior polynomial $I_{\mathcal{H}}(x)$ for the hypergraph $\mathcal{H}$ induced from $\Gamma(D)$. The interior polynomial is a generalization of the Tutte polynomial to hypergraphs. They are not equivalent as graph invariants, therefore we have the following corollary.

**Corollary 15.** *The homology $H(D)$ is not invariant under some Reidemeister moves.*

*Remark* 16. Abel [1] proved that $H(D)$ was not invariant under RIIb move, by using chain complexes for virtual link diagrams.

We shall end this note, by proposing the following question.

**Question 17.** *Can we "categorify" the Interior polynomial, in order to get well-defined HOMFLY homology for general diagrams?*

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