CLOCK THEOREM AND DISTANCE FORMULA FOR STATES OF TRINITIES

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Abstract. The purpose of this note is twofold. First, we recall a recent extension, by the author and C. Hine, of Kauffman's Clock Theorem to certain combinatorial objects called trinities. The essential part of our theorem is that the state transition graph of a trinity is always connected.
Our second purpose is to announce a formula for the distance, in the state transition graph, between two arbitrary states. The proof of this formula will appear elsewhere.

1. Trinities and their states

A *trinity* is a properly three-colored triangulation of the two-sphere. This notion can be viewed as an extension of (isotopy classes of) generically immersed spherical curves, see [3, Section 2.2]. Our main observation is that Kauffman's Clock Theorem [8] generalizes to trinities. This result also relates to work of Tutte who, in the proof of his Tree Trinity Theorem [10, 11], introduced certain bijections which become the objects that generalize Kauffman's states.

Thus our work has an obvious connection to the 'formal knot theory' treatment of the Alexander polynomial. There are some other ways in which trinities supply case studies in topology. They do exhibit interesting phenomena in relation to Floer homology [4], contact geometry [6], and the Homfly polynomial [7].

We will call the colors of the vertices of the trinity red, green, and blue. Triangles are colored white or black according to the cyclic order of the three colors around them. It is also natural to color each edge with the 'third color' not occurring among its endpoints. It is then easy to see that in each trinity red edges, green edges, blue edges, white triangles, and black triangles all have the same number n. By Euler's formula, the number of vertices exceeds n by 2.

A *Tutte matching* or *state* of a trinity is a bijection between vertices and incident white triangles. To adjust the sizes of the two sets, we exclude an 'outer' white triangle and its three vertices, called the 'roots.' Following Tutte, we will indicate matches between triangles and vertices by drawing an arrow across the triangle and terminating at the vertex. At the tail end, the arrows are extended beyond the white triangle and into the adjacent black one, all the way to the opposite vertex. That way the two ends of the arrow, and the edge that it crosses, are all of the same color. We let the arrow inherit this color as well. In other words, arrows of color X are chosen from among the edges of the dual graph G_X^* of the graph G_X formed by the edges of the trinity of color X.

Note that as G_X is bipartite, G_X^* is naturally oriented, exactly by the rule (the same for all three colors X) that the tail end of each arrow is in a black triangle. Tutte showed that in a Tutte matching, the set of arrows of color X is a *spanning* *arborescence* of G^*_X: this means that they form a spanning tree of G^*_X in which each arrow points away from the root of color X. (In what follows we will keep calling the edges of our directed graphs *arrows*.)

The arrows that constitute a Tutte matching may not cross each other over white triangles but may cross over black ones. On the other hand, black triangles may also be *empty* (void of arrows), something that does not happen to the white ones, save the outer triangle. For an empty black triangle with vertices a, b, c to exist in a state, see Figure 1, it means that none of the three adjacent white triangles is matched to its 'third' vertex that is not in $\{a, b, c\}$. So the three triangles and $\{a, b, c\}$ are matched to each other, which can be achieved in exactly two ways; we refer to the empty black triangle as *clockwise* or *counterclockwise* accordingly.

Now, as explained in [3, Section 2.2], the role of Kauffman's state transpositions (clock moves) is played by the following *state transitions*, in which a *clockwise move* turns a clockwise triangle into a counter-clockwise one, see Figure 1. Its inverse is a *counter-clockwise move.*

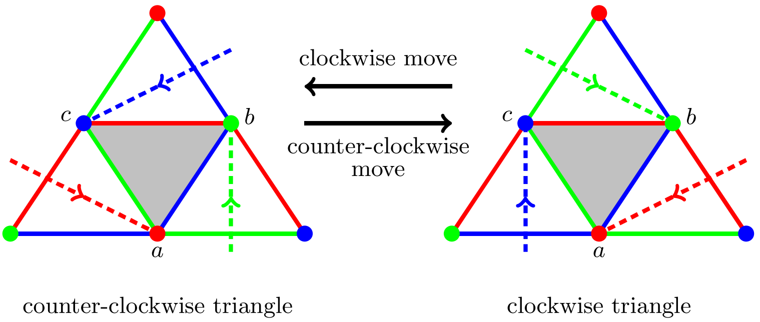

Figure 1. State transition about an empty black triangle.

We define a *state transition graph*, where the vertices correspond to the states (matchings), and two vertices are connected if it is possible to go from one to the other by performing a clockwise or counter-clockwise move. We also record the directions of our moves, wherein moving 'up' in the state transition graph will correspond to a clockwise move, and moving 'down' will correspond to a counter-clockwise move.

Please refer to [3, Section 2.3] for the definition and role of irreducibility. Roughly speaking, it means that the trinity can not be obtained by splicing together two smaller trinities along a triangle. This is a rather simple matter that we omit here.

Theorem 1.1. *The state transition graph of an irreducible trinity is connected. In fact, it has the structure of a distributive lattice.*

This extends theorems of Kauffman [8] and of Gilmer and Litherland [2], which make the same claims for the particular kind of trinity that arises from Kauffman's 'universes.' Let us note that in [8] the connectedness result is an essential step in proving the state expansion formula for the Alexander polynomial — a theorem that became even more significant when it was used to show [9] that knot Floer homology categorifies the Alexander polynomial.

An immediate consequence is that global maxima and minima exist in our state transition graphs. After Kauffman, we call these the *clocked* and *counterclocked* *states*, respectively. Their construction is crucial for the definition of the lattice structure and the proof of Theorem 1.1.

2. A distance formula

Given the connectedness of the state transition graph, it is natural to wonder how far apart its vertices may be. Cohen and Teicher [1] answered this question in Kauffman's original setup. We extend their results as follows.

We will use the standard notion of distance, in which each edge of the graph has length 1. Thus states that differ by a single clock move have distance 1, and in general the distance is the minimum number of clock moves that are required to transform one state to another.

Let σ and τ be two states of a given trinity. This time, let us represent them by the the 'last one-third' of each of their constituent arrows. More precisely, let us place points $t_1, t_2, \ldots, t_{n-1}$ in our non-outer white triangles and if the triangle containing t_i is matched with its vertex x, then let us connect, within the triangle, t_i to x with a directed arc. From now on, σ and τ will refer to the singular 1-chain, with integer coefficients, obtained as the union of all appropriate arcs.

Since states are bijections between the same fixed sets, in the above sense $\sigma - \tau$ is an oriented 1-cycle. For example, by examining Figure 1, we see that if σ and τ differ by a single clock move then $\sigma - \tau$ surrounds the corresponding black triangle once, and it does not surround any other black triangles.

To be more precise, let us use stereographic projection, with respect to some interior point of the outer white triangle, to transform our spherical pictures into planar ones. That way we may also speak of the winding number of a directed 1-chain with respect to a point that is not along the chain. For our 1-chains $\sigma - \tau$, the winding number is constant over black triangles. In the example of the previous paragraph, if a clockwise move around the black triangle b takes σ to τ, then the winding number is -1 for b and it is 0 for all other black triangles. For the inverse move, represented by the 1-chain $\tau - \sigma$, the winding number is 1 for b and 0 elsewhere.

Theorem 2.1. *Let σ and τ be Tutte matchings of some trinity and let us associate the* 1*-chain $\sigma - \tau$ to them. Furthermore, let us consider the winding number $w(b)$ of $\sigma - \tau$ with respect to each black triangle b of the trinity. Then, the distance between σ and τ in the state transition graph is the sum of the absolute values $|w(b)|$ over all black triangles.*

In fact, a sequence of state transitions can be found in which each black triangle b is changed exactly $|w(b)|$ times (and always in the same direction, in accordance with the sign of w). We do not go into the details of the proof, but it is not hard based on the fact that for any two states, the obvious path through their join (or wedge) is shortest.

As a special case, let us fix τ to be the unique counterclocked state of the trinity. By associating the function $w(b)$ to the arbitrary state σ, we obtain a one-to-one morphism from the state lattice into the distributive lattice of integer-valued functions over the set of black triangles. This is in fact the way (although not with the same words) in which Gilmer and Litherland [2] established their distributive lattice. In [3], Hine and the author generalized their approach.

The *height* of the clock lattice is the distance between its extrema, that is, the clocked and counterclocked states. Theorem 2.1 provides a formula for this quantity, too. We remark that the height (and thus the isomorphism class) of the clock lattice depends not only on the trinity, but also on the choice of the outer white triangle. In fact, we are not aware of any results that relate the lattice structure to other (either combinatorial or topological) invariant quantities that are associated to trinities, such as interior polynomials [5].

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