Chern-Simons perturbation theory and Reidemeister-Turaev torsion

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1 Introduction

Chern-Simons perturbation theory first established by Kontsevich in [6], Axelrod and Singer in [1] gives a sequence of invariants of a 3-manifold with an acyclic representation of the fundamental group. Here a representation is said to be acyclic if the local system given by the representation is acyclic.

There are several variations and related invariants of Chern-Simons perturbation theory. Bott and Cattaneo gave a refinement of Chern-Simons perturbation theoretical invariants by purely topological construction in [2],[3]. In the construction of Chern-Simons perturbation theory a propagator plays an important role. A (Poincaré dual of) propagator is a 4-chain in the two point configuration space of given 3-manifold. In Bott-Cattaneo's construction, a propagator is also used. Due to homological conditions for Bott-Cattaneo's propagator, the existence of propagator is not guaranteed in some case. In [4] and [8], Cattaneo and the author refined the conditions for Bott-Cattaneo's propagator and they showed that there exists refined propagators in any cases. In [8], the author introduced an invariant d of acyclic representation as a defect of a Bott-Cattaneo's propagator and a refined propagator.

The defect d was first introduced by Lescop in [7] for 3-manifold with $b_1 = 1$. Lescop defined an invariant of 3-manifolds with $b_1 = 1$. Her construction of the invariant was inspired by Chern-Simons perturbation theory and her invariant can be considered as a generalization of Chern-Simons perturbation theory for an abelian representation $H = \langle t \rangle \to \mathbb{Q}(t)^{\times}$. Here H is the degree-1 homology group of the manifold. Lescop showed that the defect d can be computed from the Alexander polynomial of the 3-manifold.

In this article, we give an invariant d' of an acyclic local system with a Euler structure. An Euler structure is a homotopy class of a non-vanishing vector field on the manifold. This d' is a refinement of d in [8] in the following sense: There is a special non-vanishing vector filed X such that $(-X)$ is homotopic to X. Then d' of the Euler structure represented by X coincides with d.

We next generalize Lescop's formula (for d') to any closed oriented 3-manifold. We show that d' can be computed from Reidemeister-Turaev torsion via logarithm derivative.

Remark 1.1. The invariant d' can be defined for any acyclic local system on any 3-manifold. Reidemeister torsion is also defined for any acyclic local system. Thus it is expected that our theorem can be extended for more general cases. Namely d' might be computed from Reidemeister torsion.

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2 Preliminaries

Let M be a closed oriented 3-manifold. Let we denote $H = H_1(M;\mathbb{Z})$. In this article, we consider H as a multiplicative group. For the simplicity, we assume that H is torsion free. Take a basis $\{t_1, \dots, t_k\} \in H$:

$$H = \{t_1^{n_1} \cdots t_k^{n_k} \mid n_i \in \mathbb{Z}\}.$$

Let

$$\mathbb{R}H = \{ \sum_{n_1,\dots,n_k} a_{n_1,\dots,n_k} t_1^{n_1} \cdots , t_k^{n_k} \}$$

be the group ring of H over $\mathbb{R}$ and let

$$Q(H) = \{f/g \mid f, g \in \mathbb{R}H, g \neq 0\}$$

be the quotient field of $\mathbb{R}H$. By using the basis, $Q(H)$ can be identified with

$$Q(H) = \mathbb{R}(t_1, \dots, t_k).$$

The natural representation

$$\rho_0 : H \to Q(H)^\times$$

given by $x \mapsto (y \mapsto yx)$ induces a homomorphism

$$\rho_0 : \mathbb{R}H \to Q(H)^\times.$$

(We use same symbol ρ_0 for the induced homomorphism).

We assume that ρ_0 is acyclic. Namely, the corresponding local system is acyclic:

$$H_*(M;\rho_0) = 0.$$

Let $\rho : H \to U(1) < \mathbb{C}^\times = Aut(\mathbb{C})$ be an acyclic representation: $H_*(M;\rho) = 0$. We use same symbol ρ for the extended map to $Q(H)$:

$$\rho : Q(H) \to \mathbb{C}.$$

An Euler structure on M is a homotopy class of non-vanishing vector field on M. Take an Euler structure e.

3 Reidemeister-Turaev torsion $\mathrm{Tor}(M,e)$

Reidemeister-Turaev torsion
$$\mathrm{Tor}(M,e) \in Q(H)^\times$$
is an invariant of (M,e) defined by Turaev as a refinement of Reidemeister torsion for abelian representations. In this section, we review the definition of $\mathrm{Tor}(M,e)$. For example, see [9] for more details.

Let $C_* = (C_*(M;\rho_0), \partial_*)$ be the chain complex of M with the local coefficient corresponding to ρ_0. Since ρ_0 is acyclic, there exist homomorphisms
$$g_i : C_i(M;\rho) \to C_{i+1}(M;\rho) \quad (i = 0,1,2,3)$$
satisfying
$$\partial_{i+1} \circ g_{i+1} + (-1)^{i+1} g_i \circ \partial_i = \mathrm{id}_{C_i}.$$
Then $\partial + g$ gives an isomorphism
$$\partial + g (= g_3 + \partial_2 + g_1) : C_{even} \to C_{odd}.$$
Here $C_{even} = C_2(M;\rho_0) \oplus C_0(M;\rho_0)$ and $C_{odd} = C_3(M;\rho) \oplus C_1(M;\rho)$. It is known that an Euler structure gives a basis of both C_{even} and C_{odd}. Therefore we can compute the determinant of $\partial + g$ with respect to the basis given by the Euler structure e:

Definition 3.1.
$$\mathrm{Tor}(M,e) = \det(\partial + g) \in Q(H)^\times.$$

4 An invariant $d'(\rho,e)$

For a pair of an acyclic representation $\rho : H \to U(1)$ and an Euler structure e, we define an invariant
$$d'(\rho,e) \in H_1(M;\mathbb{C}).$$
$d'(\rho,e)$ is a refinement of an invariant d defined in [8]. This d is deeply related to the Bott-Cattaneo's invariant. See Remark 4.2 for more details.

4.1 A construction of $d'(\rho,e)$

Recall that $\rho : H \to \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^\times$ is a representation such that its corresponding local system is acyclic. Then $\rho \boxtimes \rho^* : H \times H \to \mathbb{C}^\times \times \mathbb{C}^\times$ gives an acyclic local system on the 6-dimensional manifold $M \times M$, where $\rho^* : H \to \mathrm{Aut}(\mathrm{Hom}(\mathbb{C},\mathbb{C})) = \mathbb{C}^\times$ is the dual representation of ρ. The restriction $\rho \otimes \rho^* = (\rho \boxtimes \rho^*)|_\Delta$ of $\rho \boxtimes \rho^*$ to the diagonal $\Delta = \{(x,x) \mid x \in M\} \subset M \times M$ is trivial (A trivialization of $\rho \times \rho^*$ is given by
$$\mathbb{C}_x \times \mathbb{C}_x \to \mathbb{C}, (\alpha,\beta) \mapsto \alpha\overline{\beta}$$
for each $(x,x) \in \Delta$):
$$H_3(\Delta; \rho \boxtimes \rho^*|_\Delta) = H_3(\Delta; \rho \otimes \rho^*) = H_3(\Delta;\mathbb{C}).$$

Then the fundamental homology class $[\Delta] \in H_3(\Delta;\mathbb{C})$ gives a homology class of $H_3(M \times M; \rho \boxtimes \rho^*)$:

$$[\Delta] \in H_3(M \times M; \rho \boxtimes \rho^*).$$

Since $\rho \boxtimes \rho^*$ is acyclic, we have $[\Delta] = 0$. Therefore there exists a 4-chain $\Sigma \in C_4(M \times M; \rho \boxtimes \rho^*)$ satisfying

$$\partial\Sigma = \Delta.$$

Let v_e be a non-vanishing vector field on M representing the Euler structure e. Let

$$v_e(\Delta) = \{(x, v_e(x))\} \subset M \times M$$

be the 3-manifold given by pushing M along v_e.

There is an isotopy between the null-vecor filed and v_e. Thus $\rho \boxtimes \rho^*|_{v_e(\Delta)}$ can be identified with $\rho \boxtimes \rho^*|_{\Delta}$, and then $H_*(v_e(\Delta); \rho \boxtimes \rho^*|_{v_e(\Delta)})$ can be identified with $H_*(\Delta;\mathbb{C})$.

Definition 4.1.

$$d'(\rho, e) = [\Sigma \cap v_e(\Delta)] \in H_1(v_e(\Delta);\mathbb{C}) = H_1(\Delta;\mathbb{C}) = H_1(M;\mathbb{C}).$$

Remark 4.2. An invariant $d(\rho)$ defined in [8] can be computed from $d(\rho, e)$ as follows:

$$d(\rho) = \mathrm{Im}_*(d'(\rho, e)).$$

Here $\mathrm{Im}_* : H_1(M;\mathbb{C}) \to H_1(M;\mathbb{R})$ is the homomorphism induced by the projection to the imaginary part $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$. We note that a 4-chain Σ is a variation of propagators used in Chern-Simons perturbation theory.

5 Main theorem

We set a map

$$D = \sum_i \left(t_i \frac{\partial}{\partial t_i} \log \right) \otimes [t_i] : Q(H) \to Q(H) \otimes H$$

as

$$D(f) = \sum_i \frac{t_i}{f} \left(\frac{\partial}{\partial t_i}(f) \right) \otimes [t_i].$$

To compose D with the map $\rho \otimes \mathrm{id} : Q(H) \otimes H \to \mathbb{C} \otimes H = H_1(M;\mathbb{C})$, we have a map

$$\widetilde{D}_\rho = (\rho \otimes \mathrm{id}) \circ D : Q(H) \to H_1(M;\mathbb{C})$$

Example 5.1.

$$D(1 + t_1^3 + t_1^2 t_2^6) = \frac{1}{1 + t_1^3 + t_1^2 t_2^6} \left(3t_1^3 \otimes [t_1] + 2t_1^2 t_2^6 \otimes [t_1] + 6t_1^2 t_2^6 \otimes [t_2] \right),$$

$$\begin{aligned} & \widetilde{D}_\rho(1 + t_1^3 + t_1^2 t_2^6) \\ = \; & \rho\left(\frac{3t_1^3}{1 + t_1^3 + t_1^2 t_2^6} \right) \otimes [t_1] + \rho\left(\frac{2t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6} \right) \otimes [t_1] + \rho\left(\frac{6t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6} \right) \otimes [t_2] \\ = \; & \left(\frac{3\rho(t_1^3)}{1 + \rho(t_1^3) + \rho(t_1^2 t_2^6)} \right) [t_1] + \left(\frac{2\rho(t_1^2 t_2^6)}{1 + \rho(t_1^3) + \rho(t_1^2 t_2^6)} \right) [t_1] + \left(\frac{6\rho(t_1^2 t_2^6)}{1 + \rho(t_1^3) + \rho(t_1^2 t_2^6)} \right) [t_2] \\ & \in H_1(M;\mathbb{C}). \end{aligned}$$

Here $\{t_1, t_2\} \subset H$ is (a part of) a basis of H.

Theorem 5.2.

$$\widetilde{D}_\rho(\mathrm{Tor}(M, e)) = d'(\rho, e).$$

Remark 5.3. (1) We can check that the map $\widetilde{D}_\rho$ does not depend on the choice of basis.

(2) A similar formula holds, even if H has torsion elements. In the case we use the total quotient ring instead of $Q(H)$. See Example 5.6.

Example 5.4. Let $M = S^1 \times S^2$. In this case $H = H_1(M; \mathbb{Z}) = \langle t \rangle$. Here $t \in H$ is represented by $S^1 \times \{\mathrm{pt}\}$ for $\mathrm{pt} \in S^2$. It is known that, for a suitable Euler structure e, Reidemeister-Turaev torsion is given as

$$\mathrm{Tor}(S^1 \times S^2, e) = \frac{1}{(t-1)^2} \in Q(H).$$

Thus

$$D\left(\frac{1}{(t-1)^2}\right) = \frac{-2t}{t-1} \otimes [t],$$

$$\widetilde{D}_\rho(\mathrm{Tor}(S^2 \times S^2, e) = \frac{-2\rho(t)}{\rho(t)-1}[t] \in H_1(S^2 \times S^1; \mathbb{C}) = d'(\rho, e).$$

Here ρ is any non-trivial representation of H.

Example 5.5 (Lescop's formula)**.** Let M be a closed oriented 3-manifold with $b_1(M) = 1$, where $b_1(M)$ is the first Betti number. Take a generator $t \in H$ of H. We assume that the representation $\rho_0 : H \to Q(H)^\times = \mathbb{R}(t)^\times, t \mapsto (f \mapsto tf)$ is acyclic. To applying the main theorem, we get the Lescop's formula for $d(\rho)$ (Recall that $d(\rho) = d'(\rho, e)$ for a suitable Euler structure e):

$$d(\rho) = \left(\frac{1+t}{1-t} + t\frac{d}{dt}\log(\Delta(t))\right) t \in H_1(M; \mathbb{Q}(t)).$$

Here $\Delta(t)$ be the Alexander polynomial of M.

Example 5.6. Let $M = L(p, 1)$. In this case, $H = \langle t \mid t^p \rangle$ is torsion. For this H, the map $\frac{\partial}{\partial t}$ (and also D) does not make sense. However, we can prove that a formal computation of $\widetilde{D}_\rho(M, e)$ makes sense and gives $d'(\rho, e)$.

It is known that, for an appropriate Euler structure e, Reidemeister-Turaev torsion is given by

$$\mathrm{Tor}(M, e) = \frac{t^k}{(1-t)^2}.$$

Here k is an integer corresponding to the Euler structure.

Let $\rho : \pi_1 = \langle t \rangle \ni t \mapsto \zeta \in \mathbb{Z}/p\mathbb{Z} < U(1)$, where $\zeta^p = 1$.

$$\begin{aligned} D(\mathrm{Tor}(M, e)) &= t\frac{d}{dt}\log\left(\frac{t^k}{(1-t)^2}\right) \otimes [t] \\ &= t\frac{d}{dt}(k\log(t) - 2\log(1-t)) \otimes [t] \\ &= k \otimes [t] + \frac{2}{1-t} \otimes [t]. \end{aligned}$$

Therefore,

$$d'(\rho, e) = \left(k + \frac{2}{1-\zeta}\right)[t] \in H_1(M; \mathbb{Z}/p\mathbb{Z}).$$

6 Idea of proof

Take a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric on M satisfying the Morse-Smale conditions. The idea of the proof is as follows. We first give explicit descriptions of both $\mathrm{Tor}(M, e)$ and $d'(\rho, e)$ by using f (and the metric). Then we check the equation on the main theorem by concrete calculations.

6.1 Reidemeister-Turaev torsion from a Morse-Smale complex

Let C^f_* be the Morse-Smale complex of f with the local coefficient ρ. Take homomorphisms $g^f : C^f_* \to C^f_{*+1}$ satisfying $\partial^f_{i+1} \circ g^f_{i+1} + (-1)^{i+1} g^f_i \circ \partial^f_i = \mathrm{id}_{C^f_i}$ for $i = 0, 1, 2, 3$. These $g^f = \{g^f_i\}_i$ is called *combinatorial propagator* (see [5], [10]). We can compute Reidemeister-Turaev torsion $\mathrm{Tor}(M, e)$ from this Morse-Smale complex:

$$\mathrm{Tor}(M, e) = \det(\partial^f + g^f).$$

6.2 A description of d': Morse homotopy

The idea of Morse homotopy theory is to give explicit descriptions for some topological or algebraic objects as kind of moduli spaces related to Morse functions. To prove our main theorem, we used the Morse homotopy in Chern-Simons perturbation theory established by Fukaya [5] and Watanabe [10]. By using Morse functions, we can get an explicit description of a 4-chain Σ.Then we get an explicit description of a representative 1-cycle of $d'(\rho, e)$ as follows:

$$d'(\rho, e) = \sum_{\gamma:\text{ trajectory}} \rho(g^f \circ \gamma_*)[\gamma] + l_e$$

Here $[\gamma]$ is a 1-chain of M given by the image of a trajectory γ and l_e is also a 1-chain of M determined from the Euler structure e. Let γ be a trajectory from a critical point $p \in M$ to a critical point $q \in M$. Then γ_* is a homomorphism $\gamma_* : (\rho \boxtimes \rho)_{(p,p)} \to (\rho \boxtimes \rho)_{(q,q)}$. Therefore $g^f \circ \gamma_*$ is an automorphism of $(\rho \boxtimes \rho)_{(p,p)}$. Thus $\rho(g^f \circ \gamma_*) \in \mathbb{C}$.

6.3 Idea of the proof

Recall that the boundary homomorphism ∂^f of Morse-Smale complex is given by counting trajectories with the information of local coefficient ρ. Thus both $\det(\partial^f + g^f)$ and $d(\rho, e)$ can be computed from the information of trajectories of f. $\det(\partial^f + g^f)$ is given by taking determinant. $d'(\rho, e)$ is given by taking "trace". To investigate this point further, we can check the equation in the main theorem.

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