“Monodromy groups” of Heegaard surfaces of 3-manifolds –Research announcement–

Yuya Koda and Makoto Sakuma
Department of Mathematics, Hiroshima University
Advanced Mathematical Institute, Osaka City University

1 Introduction

This is an announcement of an ongoing research project concerning a certain natural subgroup, which we call the “monodromy group”, of the mapping class group of a closed orientable surface embedded in a closed orientable 3-manifold. The group is particularly interesting in the case where the surface is a Heegaard surface, and there are a lot of natural questions which we believe worth being studied.

Let Σ be a closed orientable surface. The *extended mapping class* $\mathrm{MCG}(\Sigma)$ of Σ is the group of isotopy classes of (possibly orientation-reversing) self-homeomorphisms of Σ. When there is no fear of confusion, we do not distinguish between a self-homeomorphism f of Σ and the element $[f]$ of $\mathrm{MCG}(\Sigma)$ represented by f. The *mapping class group*, $\mathrm{MCG}^+(\Sigma)$, of Σ is the index 2 subgroup of $\mathrm{MCG}(\Sigma)$, consisting of those elements which are orientation-preserving.

Now, assume that Σ is embedded in a closed orientable 3-manifold M, and let $j : \Sigma \to M$ be the inclusion map. Following the terminology introduced by Jaco-Shalen [26, Section 3] (see also [25, Chapter 5]), we define a *spatial deformation* of Σ in M to be a homotopy $F = \{f_t\}_{t\in I} : \Sigma \times I \to M$ which satisfies the following two conditions.

1. f_0 is the inclusion $j : \Sigma \to M$, i.e., $f_0(x) = x$ for every $x \in \Sigma$.
2. f_1 is an embedding with image Σ, and so it induces a self-homeomorphism of Σ.

Here $f_t : \Sigma \to M$ $(t \in I = [0, 1])$ is the continuous map from Σ to M defined by $f_t(x) = F(x, t)$. We regard $f_0 = j$ and $f_1 = f$ as elements of $\mathrm{MCG}(\Sigma)$, and call them the *initial end* and the *terminal end*, respectively, of the spatial deformation. (We note our setting is different from that in [25, 26], where M is an n-manifold with nonempty boundary and Σ is a subset of ∂M, and it is only required that f_1 is an embedding of Σ into ∂M.)

We are interested in the subgroup of $\mathrm{MCG}(\Sigma)$ consisting of the terminal ends f_1 of spatial deformations $\{f_t\}_{t\in I}$ of Σ in M.

Definition 1.1 Let Σ be a closed orientable surface embedded in a closed orientable 3-manifold M. The the *monodromy group* or the *spatial deformation group* of Σ in M, denoted by $\Gamma(M,\Sigma)$ (or $\Gamma(\Sigma)$ in brief), is the subgroup of the extended mapping class group $\mathrm{MCG}(\Sigma)$ defined by

$$\begin{aligned}\Gamma(M,\Sigma) &= \{[f]\in \mathrm{MCG}(\Sigma) \mid \text{There is a spatial deformation } \{f_t\}_{t\in I} \text{ with } f=f_1.\}\\ &= \{[f]\in \mathrm{MCG}(\Sigma) \mid j\circ f:\Sigma\to M \text{ is homotopic to the inclusion } j.\}.\end{aligned}$$

There are two reasons why we call $\Gamma(M,\Sigma)$ the monodromy group. One reason is that if M is a Σ-bundle over S^1 with monodromy φ, then $\Gamma(M,\Sigma)$ is the cyclic group $\langle\varphi\rangle$ generated by the monodromy φ (see Theorem 2.3). Another reason is that in the virtual branched fibration theorem, the monodromies of the surface bundles, which appear as double branched coverings of a closed orientable 3-manifold M with a Heegaard surface Σ, are special elements of $\Gamma(M,\Sigma)$ (see Theorem 8.1).

This announcement is organized as follows. In Section 2, we treat the case where Σ is an incompressible surface in a Haken manifold M, and give a complete description of the group $\Gamma(M,\Sigma)$ (Theorem 2.3). The remaining sections are devoted to the case where Σ is a Heegaard surface. In Section 3, we recall various natural subgroups of $\mathrm{MCG}(\Sigma)$ associated with a Heegaard surface and describe their relations with the group $\Gamma(M,\Sigma)$. We then state the main question which we treat in this work. After studying the special case of a genus 1 Heegaard splitting of a lens space in Section 4, we give a partial answer to the question in Sections 5, 6 and 7 (Theorem 5.1). In the final section, we state the branched fibration theorem (Theorem 8.1) which gives another motivation for defining and studying the monodromy group $\Gamma(M,\Sigma)$.

2 Monodromy groups of incompressible surfaces in Haken manifolds

Example 2.1 Let φ be an element of $\mathrm{MCG}(\Sigma)$, and let $M := \Sigma\times\mathbb{R}/(x,t)\sim(\varphi(x),t+1)$ be the Σ-bundle over S^1 with monodromy φ. We denote the image of $\Sigma\times 0$ in M by the same symbol Σ and call it a *fiber surface*. Then we have a natural spatial deformation $\{f_t\}$ of Σ in M defined by $f_t(x)=[x,t]$, where $[x,t]$ is the element of M represented by (x,t). Its terminal end is equal to φ^{-1}, because $f_1(x)=[x,1]=[\varphi^{-1}(x),0]=\varphi^{-1}(x)$. Thus φ belongs to the monodromy group $\Gamma(M,\Sigma)$. We show in Theorem 2.3(1) that $\Gamma(M,\Sigma)$ is equal to the cyclic group $\langle\varphi\rangle$ generated by φ.

Example 2.2 Let h be an orientation-reversing free involution of a closed orientable surface Σ, and let $N := \Sigma\times[-1,1]/(x,t)\sim(h(x),-t)$ be the twisted I-bundle over the closed non-orientable surface Σ/h. The boundary ∂N is identified with Σ by the homeomorphism $\Sigma\to\partial N$ mapping x to $[x,1]$, where $[x,t]$ denotes the element of N represented by (x,t). Then we have a natural spatial deformation of $\Sigma=\partial N$ in N, in the sense of [26], defined by $f_t([x,t])=[x,1-2t]$. Its terminal end is equal to h, because $f_1(x)=f_1([x,1])=[x,-1]=[h(x),1]=h(x)$ for every $x\in\Sigma=\partial N$. Let N' be any compact orientable 3-manifold whose boundary is identified with Σ, i.e., a homeomorphism $\partial N'\cong\Sigma$ is fixed, and let $M=N\cup N'$ be the closed orientable 3-manifold obtained by gluing N and N' along the common boundary Σ. Then the preceding argument shows that h is an element of $\Gamma(M,\Sigma)$. Theorem 2.3(2) shows that if N' is a Haken manifold which is not a twisted I-bundle then $\Gamma(M,\Sigma)$ is the order 2 cyclic group generated by h. If N' is a twisted bundle associated with an orientation-reversing involution h' of Σ, then h' also belongs to $\Gamma(M,\Sigma)$. Theorem 2.3(3) shows that if Σ has positive genus (i.e., $\Sigma \not\cong S^2$), then $\Gamma(M,\Sigma)$ is the (possibly cyclic) dihedral group generated by the two involutions h and h'.

The following theorem is proved by using the positive solution of Simon's conjecture [52] concerning manifold compactifications of covering spaces, with finitely generated fundamental groups, of compact 3-manifolds. A proof of Simon's conjecture can be found in Canary's expository article [8, Theorem 9.2], where he attributes it to Long and Reid.

Theorem 2.3 *Let M be a closed orientable Haken manifold M, and let Σ be an incompressible surface in M. Then the following hold.*

1. *Suppose that M is a Σ-bundle over S^1 with monodromy φ and Σ is a fiber surface. Then $\Gamma(M,\Sigma)$ is the cyclic group $\langle\varphi\rangle$.*
2. *Suppose that Σ separates M into two submanifolds, M_1 and M_2, precisely one of which is a twisted I-bundle. Then $\Gamma(M,\Sigma)$ is the order 2 cyclic group generated by the orientation-reversing involution of Σ associated with the twisted I-bundle structure.*
3. *Suppose that Σ separates M into two submanifolds, M_1 and M_2, both of which are twisted I-bundles. Then $\Gamma(M,\Sigma)$ is the (finite or infinite, and possibly cyclic) dihedral group generated by the two orientation-reversing involutions of Σ associated with the twisted I-bundle structures.*
4. *Suppose none of the above conditions hold. Then $\Gamma(M,\Sigma)$ is the trivial group.*

3 Natural mapping class groups associated with Heegaard surfaces and bridge spheres

For an orientable manifold X and its subspaces, $Y_1,\dots,Y_n$, let $\mathrm{MCG}(X,Y_1,\dots,Y_n)$ be the extended mapping class group of $(X,Y_1,\dots,Y_n)$, i.e., the group of self-homeomorphisms of X which preserve each Y_i $(1\le i\le n)$, modulo isotopy preserving the subsets $Y_1,\dots,Y_n$. We do not distinguish between a self-homeomorphism f of $(X,Y_1,\dots,Y_n)$ and the element $[f]$ of $\mathrm{MCG}(X,Y_1,\dots,Y_n)$ represented by f. $\mathrm{MCG}^+(X,Y_1,\dots,Y_n)$ denotes the subgroup, of index 1 or 2, consisting of the elements represented by orientation-preserving homeomorphism of X.

Let $M=V_1\cup_\Sigma V_2$ be a Heegaard splitting of a closed orientable 3-manifold M, namely Σ is a closed orientable surface in M which separates M into handlebodies V_1 and V_2. Then $\mathrm{MCG}(\Sigma)$ contains the following subgroups which are naturally associated with the Heegaard splitting $M=V_1\cup_\Sigma V_2$.

1. The *handlebody group* $\mathrm{MCG}(V_i)$ of the handlebody V_i, which is identified with a subgroups of $\mathrm{MCG}(\Sigma)$, by restricting a self-homeomorphism of V_i to its boundary $\partial V_i=\Sigma$. This has been a target of various works (see a survey by Hensel [22] and references therein).

2. The intersection $\mathrm{MCG}(V_1) \cap \mathrm{MCG}(V_2)$, which is identified with $\mathrm{MCG}(M, V_1, V_2)$. This group or its orientation-subgroup $\mathrm{MCG}^+(M, V_1, V_2)$ is called the *Goeritz group* of the Heegaard splitting $M = V_1 \cup_\Sigma V_2$ and it has been extensively studied. In particular, the problem of when this group is finite, finitely generated, or finitely presented attracts attention of various researchers (cf. Minsky [20, Question 5.1]). The work on this problem goes back to Goeritz [19], which gave a finite generating set of the Goeritz group of the genus 2 Heegaard splitting of S^3. In these two decades, great progress was achieved by many authors [50, 43, 1, 11, 27, 28, 12, 13, 14, 15, 17, 16, 24], however, it still remains open whether the Goeritz group a Heegaard splitting of S^3 is finitely generated when the genus is at least 4.
3. The group $\langle \mathrm{MCG}(V_1), \mathrm{MCG}(V_2) \rangle$ generated by $\mathrm{MCG}(V_1)$ and $\mathrm{MCG}(V_2)$. Minsky [20, Question 5.2] asked when this subgroup is the free product with amalgamation amalgamated over $\mathrm{MCG}(V_1) \cap \mathrm{MCG}(V_2)$. A partial answer to this question was given by Bestvina-Fujiwara [4].
4. The mapping class group $\mathrm{MCG}(M, \Sigma)$ of the pair (M, Σ). This contains $\mathrm{MCG}(M, V_1, V_2)$ as a subgroup of index ≤ 2. The result of Scharlemann-Tomova [51] says that the natural map $\mathrm{MCG}(M, \Sigma)$ to $\mathrm{MCG}(M)$ is surjective if the Hempel distance $d(\Sigma)$ (see [21]) is greater than $2g(\Sigma)$. On the other hand, it is proved by Johnson [27], improving the result of Namazi [43], that the natural map $\mathrm{MCG}(M, \Sigma)$ to $\mathrm{MCG}(M)$ is injective if the Hempel distance $d(\Sigma)$ is greater than 3. Hence, the natural map gives an isomorphism $\mathrm{MCG}(M, \Sigma) \cong \mathrm{MCG}(M)$ if $g(\Sigma) \geq 2$ and $d(\Sigma) > 2g(\Sigma)$. Building on the work of McCullough-Miller-Zimmermann [39] on finite group actions on handlebodies, finite group actions on the pair (M, Σ) are extensively studied (see Zimmermann [53, 54] and references therein.)
5. The subgroup $\ker(\mathrm{MCG}(M, \Sigma) \to \mathrm{MCG}(M))$, which forms a subgroup of the monodromy group $\Gamma(M, \Sigma)$. Johnson-Rubinstein [31] gave systematic constructions of periodic, reducible, pseudo-Anosov elements in this group. Johnson-McCullough [30] called this group the Goeritz group, and they used the group to study the homotopy type of the space of Heegaard surfaces.
6. The group $\Gamma(V_i) := \ker(\mathrm{MCG}(V_i) \to \mathrm{Out}(\pi_1(V_i)))$. (We employ the symbol $\Gamma(V_i)$, for it is contained in the monodromy group $\Gamma(M, \Sigma)$.) It was shown by Luft [33] that its index 2 subgroup $\Gamma^+(V_i) := \ker(\mathrm{MCG}^+(V_i) \to \mathrm{Out}(\pi_1(V_i)))$ is the *twist group*, that is, the subgroup of $\mathrm{MCG}^+(V_i)$ generated by the Dehn twists about meridian disks. McCullough [38] proved that $\Gamma(V_i)$ is not finitely generated by showing that it admits a surjection onto a free abelian group of infinite rank. A typical orientation-reversing element of $\Gamma(V_i)$ is a *vertical I-bundle involution*, namely an involution h on V_i for which there is an I-bundle structure such that h preserves each fiber setwise and acts on it as a reflection.
7. The group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ generated by $\Gamma(V_1)$ and $\Gamma(V_2)$, which is contained in $\Gamma(M, \Sigma)$. This group arises in the question raised by Minsky [20, Question 5.4], and it was proved by Bowditch-Ohshika-Sakuma [44] (see also Bestvina-Fujiwara [4]) that its orientation-preserving subgroup $\langle \Gamma^+(V_1), \Gamma^+(V_2) \rangle$ is the free product $\Gamma^+(V_1) * \Gamma^+(V_2)$

if the Hempel distance $d(\Sigma)$ is high enough. (The question of whether the same conclusion holds for $\langle\Gamma(V_1), \Gamma(V_2)\rangle$ is still an open question.)

There are natural analogues of the above groups for bridge spheres. Let K be a link in S^3, and let S be a bridge sphere of K. (We regard a knot as a one-component link.) Then (S^3, K) is a union of two trivial tangles (B_1^3, t_1) and (B_2^3, t_2) such that $(S, S \cap K) = \partial(B_1^3, t_1) = \partial(B_2^3, t_2)$. We denote the surface $(S, S \cap K)$ with marked points by the same symbol S, and consider its extended mapping class group $\mathrm{MCG}(S)$. Then the extended mapping class group $\mathrm{MCG}(B_j^3, t_j)$ of the pair (B_j^3, t_j) can be identified with a subgroup of $\mathrm{MCG}(S)$ by restricting a self-homeomorphism of (B_j^3, t_j) to S. Let $\Gamma(t_j)$ be the subgroup of $\mathrm{MCG}(B_j^3, t_j)$ consisting of mapping classes represented by self-homeomorphisms pairwise-homotopic to the identity. Let $\Gamma(K, S)$ be the subgroup of $\mathrm{MCG}(S)$ consisting of the elements represented by self-homeomorphisms f such that the composition $j \circ f : (S, S \cap K) \to (S^3, K)$ is pairwise homotopic to the inclusion $j : (S, S \cap K) \to (S^3, K)$. Then the subgroup $\langle\Gamma(t_1), \Gamma(t_2)\rangle$ is contained in the group $\Gamma(K, S)$.

The group $\langle\Gamma(t_1), \Gamma(t_2)\rangle$ for the 2-bridge spheres S of 2-bridge links K play key roles in the work of Ohtsuki-Riley-Sakuma [45] on systematic construction of epimorphisms between 2-bridge link groups and the series of joint works of Lee-Sakuma [34, 35, 36, 37] which (i) solves a problem for 2-bridge spheres corresponding to [20, Question 5.4], which in turn implies a characterization of epimorphism between 2-bridge link groups (see [2, Theorem 8.1]), (ii) solves the conjugacy problem of essential simple loops on the 2-bridge sphere in a 2-bridge link complement in terms of the group $\langle\Gamma(t_1), \Gamma(t_2)\rangle$, and (iii) applies these results to establish a variation of McShane's identity for hyperbolic 2-bridge links. (See [34] and [44, Theorem 1] for summary.) Moreover, the second author recognized through discussion with Ken Baker [3] and through a comment of the referee of [37] (see [37, p.5] and Proposition 3.2 below)) that it is more natural to work with $\Gamma(K, S)$ than to work with its subgroup $\langle\Gamma(t_1), \Gamma(t_2)\rangle$. This paper, as well as [44], is motivated by the following natural question.

Question 3.1 To what extent do the results for the 2-bridge spheres described above hold in general setting?

We now describe the question that we study in this announcement. Recall the following inclusion:

$$\langle\Gamma(V_1), \Gamma(V_2)\rangle < \Gamma(M, \Sigma) \quad \text{and} \quad \langle\Gamma(t_1), \Gamma(t_2)\rangle < \Gamma(K, S)$$

For 2-bridge sphere of 2-bridge links, the following result can be proved by using [35, Main-Theorem 2.3].

Proposition 3.2 *For the* 2*-bridge sphere* S *of a* 2*-bridge link* (S^3, K) *which is not a* 2*-component trivial link, the following hold.*

1. *If* K *is hyperbolic (i.e., the Hempel distance of the* 2*-bridge sphere is greater than* 2*), then* $\langle\Gamma(t_1), \Gamma(t_2)\rangle = \Gamma(K, S)$.
2. *If* K *is not hyperbolic (i.e., the Hempel distance of the* 2*-bridge sphere is* 1 *or* 2*), then* $\langle\Gamma(t_1), \Gamma(t_2)\rangle$ *has index* 2 *in* $\Gamma(K, S)$*. The gap comes from a "book rotation" (see [37, p.5, Figure 3]).*

Moreover, except for the case where K is a trivial knot, the image of $\langle \Gamma(t_1), \Gamma(t_2)\rangle$ in the automorphism group of the curve complex of the 4-times punctured sphere (namely the Farey tessellation) is the free product of the images of $\Gamma(t_1)$ and $\Gamma(t_2)$.

The only gaps between $\Gamma(M, \Sigma)$ and its subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$, which we know, are book rotations (see the next section for the precise definition of a book rotation) and those associated with genus 1 Heegaard splitting of lens spaces (see Proposition 4.2 in the next section). So we would like to pose the following question.

Question 3.3 What is the difference between the two groups $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ and $\Gamma(M, \Sigma)$? Is it true that if the Heegaard surface is complicated enough, e.g., a high Hempel distance, then these groups are identical?

After treating in Section 4 the special case of genus 1 Heegaard splittings of lens spaces, we give a partial answer to this question in Sections 5, 6 and 7. In fact, we prove that a gap can actually exist, by showing that, in the case Σ is induced from an open book decomposition, book rotations are not contained in the subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ provided that the 3-manifold M is aspherical or has positive Gromov norm (see Theorem 5.1).

4 Monodromy groups of genus 1 Heegaard surfaces of lens spaces

We first note the following characterization of the monodromy group $\Gamma(M, \Sigma)$ in terms of the induced homomorphisms between the fundamental groups.

Lemma 4.1 *Let Σ be a closed orientable surface embedded in a closed orientable 3-manifold M which is irreducible. Then a mapping class $[f] \in \mathrm{MCG}(\Sigma)$ belongs to $\Gamma(M, \Sigma)$ if and only if the homomorphism $(j \circ f)_* : \pi_1(\Sigma) \to \pi_1(M)$ is equal to the homomorphism j_* modulo post composition of an inner-automorphism of $\pi_1(M)$.*

This lemma follows from the fact that $\pi_2(M) = 0$ which in turn is a consequence of the irreducibility of M and the sphere theorem.

Let $L(p, q) = V_1 \cup_\Sigma V_2$ be a genus 1 Heegaard splitting of the Lens space $L(p, q)$. Thus the meridian of V_1 is identified with p(longitude)$+\,q$(meridian) of V_2. We identify V_1 with $S^1 \times D^2$ and $\Sigma = \partial V_1$ with $S^1 \times \partial D^2 = S^1 \times S^1$. We identify $\mathrm{MCG}(\Sigma)$ with $\mathrm{GL}(2, \mathbb{Z})$ by identifying $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the self-homeomorphism of $\Sigma = S^1 \times S^1$ that maps (z_1, z_2) to $(z_1^a z_2^b, z_1^c z_2^d)$. Let λ and μ be the elements of $\pi_1(\Sigma) = H_1(\Sigma)$ represented by $S^1 \times 1$ and $1 \times S^1$, respectively. Then the automorphism A_* of $H_1(\Sigma)$ induced by the self-homeomorphism A is given by

$$(A_*(\lambda), A_*(\mu)) = (\lambda, \mu)A.$$

Then the following proposition can be easily proved by using Lemma 4.1.

Proposition 4.2 *For the genus 1 Heegaard splitting $L(p, q) = V_1 \cup_\Sigma V_2$ of the (p, q)-lens space, the following holds.*

$$\Gamma(L(p, q), \Sigma) = \left\{ A \in \mathrm{GL}(2, \mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ * & \pm 1 \end{pmatrix} \pmod{p} \right\}.$$

In particular, $\Gamma^+(L(p,q),\Sigma)$ *is a principal congruence subgroup of* $\mathrm{SL}(2,\mathbb{Z})$ *of level* p.

This proposition says that $\Gamma(L(p,q),\Sigma)$ is solely determined by p and independent of q which together with p determines a homeomorphism type of the lens space. It also implies that $\Gamma(L(p,q),\Sigma)$ is much bigger than the subgroup $\langle\Gamma(V_1),\Gamma(V_2)\rangle$ in general, because $\Gamma(L(p,q),\Sigma)$ has finite index in the full mapping class group $\mathrm{MCG}(\Sigma)=\mathrm{GL}(2,\mathbb{Z})$, whereas $\langle\Gamma(V_1),\Gamma(V_2)\rangle$ has infinite index except when $p\le 4$ or $L(p,q)\cong L(5,2)$. We think these are rather exceptional phenomena caused by the specificity of lens spaces that their fundamental groups are cyclic.

5 Monodromy groups of Heegaard surfaces induced from open book decompositions

In the following, we always identify S^1 with $\mathbb{R}/\mathbb{Z}$. In our notation, we will not distinguish between an element of S^1 and its representative in $\mathbb{R}$.

Let M be a closed, orientable 3-manifold. Recall that an *open book decomposition* of M is defined to be the pair (L,π), where

1. L is a (fibered) link in M; and
2. $\pi: M-L\to S^1$ is a fibration such that $\pi^{-1}(\theta)$ is the interior of a Seifert surface Σ_θ of L for each $\theta\in S^1$.

We call L the *binding* and Σ_θ a *page* of the open book decomposition (L,π). The monodromy of the fibration π is called the *monodromy* of (L,π). We think of the monodromy φ of (L,π) as an element of $\mathrm{MCG}^+(\Sigma_0\,\mathrm{rel}\,\partial\Sigma_0)$, the mapping class group of Σ_0 relative to $\partial\Sigma_0$, i.e., the group of self-homeomorphisms of Σ_0 which fix $\partial\Sigma_0$, modulo isotopy fixing $\partial\Sigma_0$. The pair (M,L), as well as the projection π, is then recovered from Σ_0 and φ. Indeed, we can identify (M,L) with

$$(\Sigma_0\times\mathbb{R},\partial\Sigma_0\times\mathbb{R})/\sim,$$

where $\sim$ is defined by $(x,s)\sim(\varphi(x),s+1)$ for $x\in\Sigma_0$ and any $t\in\mathbb{R}$, and $(y,0)\sim(y,s)$ for $y\in\partial\Sigma_0$ and any $t\in\mathbb{R}$. By this identification, we see that (M,L) admits an $\mathbb{R}$-action $\{h_t\}_{t\in\mathbb{R}}$, called a *book rotation*, defined by $h_t([x,s)])=[x,s+t]$, where $[x,s]$ denotes the point of M represented by (x,s).

Given an open book decomposition (L,π) of M, we obtain a Heegaard splitting $M=V_1\cup_\Sigma V_2$, where $V_1=\pi^{-1}([0,1/2])\cup L$, $V_2=\pi^{-1}([1/2,1])\cup L$, and $\Sigma=\Sigma_0\cup\Sigma_{1/2}$. We call this the Heegaard splitting of M *induced from* the open book decomposition (L,π). For this Heegaard splitting, we define two particular elements of the extended mapping class group $\mathrm{MCG}(\Sigma)$ as follows. The first one, denoted by $\rho=\rho_{(L,\pi)}$, is defined by $\rho(x)=h_{1/2}(x)$ for every $x\in\Sigma$. The second one, denoted by $\sigma=\sigma_{(L,\pi)}$, is defined by

1. $\sigma(x)=h_1(x)$ for every $x\in\Sigma_0$; and
2. $\sigma(x)=x$ for every $x\in\Sigma_{1/2}$.

Clearly, both ρ and σ are elements of $\Gamma(M,\Sigma)$. Indeed, $\{f_t\}_{t\in I}$ with $f_t=h_{t/2}$ $(t\in[0,1])$, see Figure 1, gives a homotopy from the inclusion $j:\Sigma\to M$ to ρ, while $\{g_t\}$ with

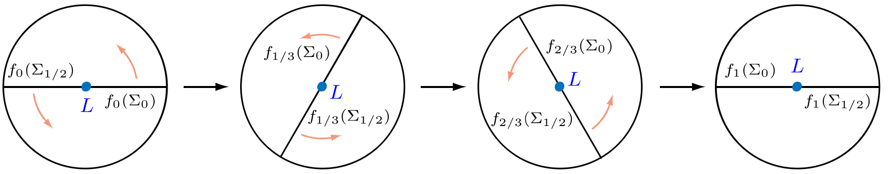

Figure 1: The homotopy $\{f_t\}_{t\in I}$.

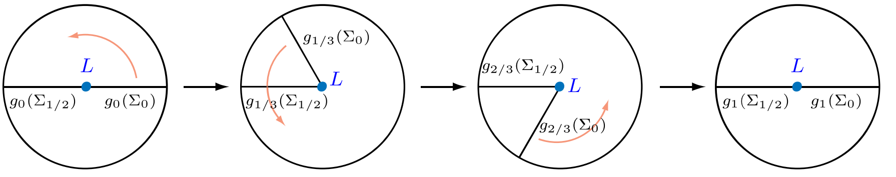

Figure 2: The homotopy $\{g_t\}_{t\in I}$.

1. $g_t(x) = h_t(x)$ for $x \in \Sigma_0$, $t \in [0,1]$; and
2. $g_t(x) = x$ for $x \in \Sigma_{1/2}$, $t \in [0,1]$, see Figure 2,

gives a homotopy from j to σ. We note that ρ is orientation-reversing and σ is orientation-preserving. These elements are related by $\sigma^2 = \rho^2 \circ \iota_2 \circ \iota_1$, where ι_1 (ι_2, respectively) is the vertical I-bundle involution on V_1 (V_2, respectively) with respect to the natural I-bundle structure given by (L, π). The map σ restricted to the page Σ_0 is nothing but the monodromy of the open book decomposition (L, π).

We prove the following theorem in Sections 6 and 7.

Theorem 5.1 *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M induced from an open book decomposition. If M is aspherical or M has a positive Gromov norm, then we have $\Gamma(M,\Sigma) \gneq \langle \Gamma(V_1), \Gamma(V_2)\rangle$.*

In fact, we will see that neither ρ nor σ, defined above, is not contained in $\langle \Gamma(V_1), \Gamma(V_2)\rangle$. To show this, we define a $\mathbb{Z}^2$-valued invariant, called the *homological degree*, for elements of $\Gamma(M,\Sigma)$.

6 Definition of the homological degree and non-zero degree maps for Heegaard splittings

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M. We will adopt the following convention. Given an orientation of M, or equivalently, a fundamental class $[M] \in H_3(M)$, we always choose the fundamental classes $[V_i] \in H_3(V_i, \partial V_i)$ $(i = 1, 2)$ and $[\Sigma] \in H_2(\Sigma)$ so as to satisfy the following:

- $[M] = [V_2] - [V_1]$; and
- $[\Sigma] = [\partial V_1]$, where $[\partial V_1]$ is the one induced from $[V_1]$.

By $[I] \in H_1(I; \partial I)$ we always mean the fundamental class corresponding to the canonical orientation of I. The inclusion map $\Sigma \hookrightarrow M$ is always denoted by j. We define a map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ as follows. First, we fix an orientation of M. Let $f \in \Gamma(M, \Sigma)$. By definition, there exists a homotopy $F = \{f_t\}_{t \in I} : (\Sigma \times I, \Sigma \times \partial I) \to (M, \Sigma)$ with $f_0 = j$ and $f_1 = j \circ f$. The induced map

$$F_* : H_3(\Sigma \times I, \Sigma \times \partial I) \to H_3(M, \Sigma) \cong H_3(V_1, \partial V_1) \oplus H_3(V_1, \partial V_1)$$

takes $[\Sigma \times I]$ to $d_1[V_1] + d_2[V_2]$ for some $d_1, d_2 \in \mathbb{Z}$, where $[\Sigma \times I]$ is the cross product of $[\Sigma]$ and $[I]$. The image $\mathrm{Deg}(f)$, called the *homological degree* of f, is then define to be the pair (d_1, d_2). We note that this element in $\mathbb{Z}^2$ does not depend on the choice of the orientation of M. However, this does depend on the choice of the above homotopy F in general, as shown by the following lemma.

Lemma 6.1 *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M. Then the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined if and only if there exits no non-zero degree map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ).*

There are various examples of non-zero degree maps from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ), as shown below. We note that open book decompositions with trivial monodromies play key roles in the construction.

Example 6.2

1. Let $M = \#_g(S^2 \times S^1)$, and Σ a Heegaard surface of M. Then there exists a degree-d map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ) for any integer d.
2. Let M be a closed, orientable *spherical* 3-manifold (i.e., a 3-manifold admitting the S^3-geometry) with $|\pi_1(M)| = n$, and Σ a Heegaard surface for M. Then there exists degree-d map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ) for any integer d with $n|d$.
3. Let $M = \mathbb{RP}^3 \# \mathbb{RP}^3$, and Σ a Heegaard surface for M. Then there exists a degree-d map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ) for any even integer d.
4. Let M be a closed, orientable 3-manifold, Σ a Heegaard surface for M, and Σ' the Heegaard surface obtained by a stabilization of Σ. If there exists a degree-d map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ), then there exists a degree-d map from $(\Sigma' \times S^1, \Sigma' \times \{0\})$ to (M, Σ') as well.

By using the Seifert fiber space Conjecture established by Gabai [18] and Casson-Jungreis [10] and the Geometrization Theorem of Perelman [46, 47, 48] (see [5, 9, 32, 41, 42] for exposition), we can prove the following complete chracterization of closed, orientable, prime 3-manifolds M which admit a non-zero degree map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ) for some Heegaard surface Σ.

Theorem 6.3 *Let M be a closed, orientable, prime 3-manifold, and Σ a Heegaard surface for M. Then there exists a non-zero degree map from $(\Sigma \times S^1, \Sigma \times \{0\})$ to (M, Σ) if and only if M is non-aspherical.*

In addition to the above theorem, there is a well-known obstruction, in terms of the Gromov norm, for the existence of a nonzero degree map from $\Sigma \times S^1$, which is applicable to all (not necessarily prime) closed orientable 3-manifolds. So the above theorem together with Lemma 6.1 implies the following corollary.

Corollary 6.4 *Let M be a closed, orientable 3-manifold, and Σ a Heegaard surface for M. Then if M is aspherical or has a positive Gromov norm $||M|| > 0$, then the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined. Furthermore, when M is geometric or prime, the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined if and only if M is aspherical.*

7 Proof of Theorem 5.1

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M, where the homological degree is well-defined. We begin with easy examples of homological degrees of elements of $\langle \Gamma(V_1), \Gamma(V_2) \rangle$.

Example 7.1
1. If f is a Dehn twist about a meridian of V_i ($i \in \{1, 2\}$), then $\mathrm{Deg}(f) = (0, 0)$.
2. If f is a vertical I-bundle involution on V_1 (V_2, respectively), then $\mathrm{Deg}(f)$ is $(-2, 0)$ ($(0, -2)$, respectively).
3. Let ρ be an element of $\Gamma(M, \Sigma)$ defined in Section 5. Then $\mathrm{Deg}(\rho) = (-1, -1)$.

Next, we present basic properties of the homological degree.

Lemma 7.2 *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M, where the homological degree is well-defined. Then the following hold.*

1. *If $\mathrm{Deg}(f) = (d_1, d_2)$ for $f \in \Gamma(M, \Sigma)$, then $d_1 + d_2 = -1 + \deg f$.*
2. *For any $f, g \in \Gamma(M, \Sigma)$, we have $\mathrm{Deg}(g \circ f) = \deg f \cdot \mathrm{Deg}(g) + \mathrm{Deg}(f)$.*
3. *For any $f \in \Gamma(V_1)$, $\mathrm{Deg}(f) = (-1 + \deg f, 0)$; and*
4. *For any $f \in \Gamma(V_2)$, $\mathrm{Deg}(f) = (0, -1 + \deg f)$.*

By using the above lemma, we can show the following proposition by induction.

Proposition 7.3 *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M, where the homological degree is well-defined. If $f \in \langle \Gamma(V_1), \Gamma(V_2) \rangle$, then $\mathrm{Deg}(f)$ is equal to $(2n, -2n)$ or $(2n - 2, -2n)$ for some $n \in \mathbb{Z}$. In particular, the mod 2 reduction of $\mathrm{Deg}(f)$ is $(0, 0) \in (\mathbb{Z}/2\mathbb{Z})^2$.*

We can now prove Theorem 5.1 as follows. Suppose that M is aspherical or M has a positive Gromov norm. Then by Corollary 6.4, the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined. Let (L, π) be an open book decomposition (L, π) of M with $V_1 = \pi^{-1}([0, 1/2]) \cup L$, $V_2 = \pi^{-1}([1/2, 1]) \cup L$, and $\Sigma = \Sigma_0 \cup \Sigma_{1/2}$. Let ρ be an element of $\Gamma(M, \Sigma)$ defined in Section 5. Then by Example 7.1 we have $\mathrm{Deg}(\rho) = (-1, -1)$. Therefore, ρ does not belong to $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ by Proposition 7.3, which implies the assertion. Note that we also have $\mathrm{Deg}(\sigma) = (-1, 1)$ for $\sigma \in \Gamma(M, \Sigma)$ defined in Section 5, and thus, we can prove the assertion using this element as well instead of ρ.

8 Monodromy groups of Heegaard surfaces and the virtual branched fibration theorem

In the final section of this paper, we give yet another motivation for the study of the monodromy group $\Gamma(M, \Sigma)$ and its subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ associated with a Heegaard splitting $M = V_1 \cup_\Sigma V_2$. To describe this, let $\mathcal{I}(V_i)$ be the set of torsion elements of $\Gamma(V_i)$. (In fact, this set is equal to the set of vertical I-bundle involutions of V_i.) Then we have the following theorem, refining the observation [49, Addendum 1] that every closed orientable 3-manifold M admits a surface bundle as a double branched covering.

Theorem 8.1 *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed orientable 3-manifold M. Then there is a double branched covering $p : \tilde{M} \to M$ that satisfies the following conditions.*

(i) $\tilde{M}$ is a surface bundle over S^1 whose fiber is homeomorphic to Σ.

(ii) The inverse image $p^{-1}(\Sigma)$ of the Heegaard surface Σ is a union of two (disjoint) fiber surfaces.

Moreover, the set $D(M, \Sigma)$ of monodromies of such bundles is equal to the set $\{h_1 h_2 \mid h_i \in \mathcal{I}(V_i)\}$, up to conjugation and inversion.

Brooks [7] and Montesinos [40] independently proved that $D(M, \Sigma)$ contains a pseudo-Anosov element whenever $g(\Sigma) \geq 2$. Hirose and Kin [23] studied the asymptotic behavior of the minimum of the dilatations of pseudo-Anosov elements contained in $D(S^3, \Sigma_g)$ as $g \to \infty$, where Σ_g is a genus g Heegaard surface of S^3.

For the special case where $g(\Sigma) = 1$ and M is a lens space, we can easily see that $D(M, \Sigma)$ contains an Anosov element. Moreover, we can observe that the minimum translation length of the action on the curve complex of any element of $D(M, \Sigma)$ with $g(\Sigma) = 1$ is "comparable" with $2d(\Sigma)$ where $d(\Sigma)$ is the Hempel distance of Σ. We hope this toy example leads us to a refinement of the results of Brooks and Montesinos.

Problem 8.2 Compare the complexity of the Heegaard surface Σ of M, e.g. the Hempel distance, and the complexities of the elements in $D(M, \Sigma)$, e.g. the minimum translation lengths of the actions on the curve complex.

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References

[1] E. Akbas, *A presentation for the automorphisms of the* 3*-sphere that preserve a genus two Heegaard splitting*, Pacific J. Math. **236** (2008), no. 2, 201–222.

[2] S. Aimi, D. Lee, S. Sakai, and M. Sakuma, *Classification of parabolic generating pairs of Kleinian groups with two parabolic generators*, arXiv:2001.11662 [math.GT].

[3] K. Baker, *Private discussion*, Hiroshima, June, 2013.

[4] M. Bestvina and K. Fujiwara, *Handlebody subgroups in a mapping class group*, In the tradition of Ahlfors-Bers. VII, 29–50, Contemp. Math., 696, Amer. Math. Soc., Providence, RI, 2017.

[5] L. Bessiéres, G. Besson, M. Boileau, S, Maillot, J. Porti, *Geometrisation of 3-manifolds*, EMS Tracts in Mathematics, **13**. European Math. Soc. (EMS), Zürich, 2010. x+237 pp.

[6] M. Boileau, *Private discussion*, Oberwolfach, Feburary, 2020.

[7] R. Brooks, *On branched coverings of 3-manifolds which fiber over the circle*, J. Reine Angew. Math. **362** (1985), 87–101.

[8] R. D. Canary, *Marden's tameness conjecture: history and applications*, Geometry, analysis and topology of discrete groups, 137–162, Adv. Lect. Math. (ALM), **6**, Int. Press, Somerville, MA, 2008.

[9] H.-D. Cao and X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), 165–492: *Erratum*, ibid, 663.

[10] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), no. 3, 441–456.

[11] S. Cho, *Homeomorphisms of the* 3*-sphere that preserve a Heegaard splitting of genus two*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 1113–1123.

[12] S. Cho, *Genus-two Goeritz groups of lens spaces*, Pacific J. Math. **265** (2013), no. 1, 1–16.

[13] S. Cho and Y. Koda, *The genus two Goeritz group of* $S^2 \times S^1$, Math. Res. Lett. **21** (2014), no. 3, 449–460.

[14] S. Cho and Y. Koda, *Disk complexes and genus two Heegaard splittings for nonprime* 3*-manifolds*, Int. Math. Res. Not. IMRN **2015**, no. 12, 4344–4371.

[15] S. Cho and Y. Koda, *Connected primitive disk complexes and genus two Goeritz groups of lens spaces*, Int. Math. Res. Not. IMRN **2016**, no. 23, 7302–7340.

[16] S. Cho and Y. Koda, *The mapping class groups of reducible Heegaard splittings of genus two*, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2473–2502.

[17] M. Freedman and M. Scharlemann, M, *Powell moves and the Goeritz group*, arXiv:1804.05909.

[18] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) **136** (1992), no. 3, 447–510.

[19] L. Goeritz, *Die abbildungen der brezelfläche und der vollbrezel vom geschlecht* 2, Abh. Math. Sem. Univ. Hamburg **9** (1933), no. 1, 244–259.

[20] C. Gordon, *Problems*, Workshop on Heegaard Splittings, 401–411, Geom. Topol. Monogr. **12**, Geom. Topol. Publ., Coventry, 2007.

[21] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology **40** (2001), no. 3, 631–657.

[22] S. Hensel, *A primer on handlebody groups*, preprint.

[23] S. Hirose and E. Kin, *On hyperbolic surface bundles over the circle as branched double covers of the* 3*-sphere*, Proc. Amer. Math. Soc. **148** (2020), no. 4, 1805–1814.

[24] D. Iguchi, Y. Koda, *Twisted book decompositions and the Goeritz groups*, Topology Appl. **272** (2020), 107064, 15 pp.

[25] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, **43**. American Mathematical Society, Providence, R.I., 1980. xii+251 pp.

[26] W. Jaco and P. B. Shalen, *Peripheral structure of 3-manifolds*, Invent. Math. **38** (1976), no. 1, 55–87.

[27] J. Johnson, *Mapping class groups of medium distance Heegaard splittings*, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4529–4535.

[28] J. Johnson, *Automorphisms of the three-torus preserving a genus-three Heegaard splitting*, Pacific J. Math. **253** (2011), no. 1, 75–94.

[29] J. Johnson, *Heegaard splittings and open books*, arXiv:1110.2142 [math.GT].

[30] J. Johnson and D. McCullough, *The space of Heegaard splittings*, J. Reine Angew. Math. **679** (2013), 155–179.

[31] J. Johnson and H. Rubinstein, *Mapping class groups of Heegaard splittings*, J. Knot Theory Ramifications **22** (2013), no. 5, 1350018, 20 pp.

[32] B. Kleiner and J. Lott, *Notes on Perelman's papers*, Geom. Topol. **12** (2008), 2587–2855.

[33] R. Luft, *Actions of the homeotopy group of an orientable 3-dimensional handlebody*, Math. Ann. **234** (1978), no. 3, 279–292.

[34] D. Lee and M. Sakuma, *Simple loops on* 2*-bridge spheres in* 2*-bridge link complements*, Electron. Res. Announc. Math. Sci. **18** (2011), 97–111.

[35] D. Lee and M. Sakuma, *Epimorphisms between* 2*-bridge link groups: Homotopically trivial simple loops on* 2*-bridge spheres*, Proc. London Math. Soc. **104** (2012), 359–386.

[36] D. Lee and M. Sakuma, *A variation of McShane's identity for* 2*-bridge links*, Geometry & Topology **17** (2013), 2061–2101.

[37] D. Lee and M. Sakuma, *Homotopically equivalent simple loops on* 2*-bridge spheres in* 2*-bridge link complements (I), (II) and (III)*, Geom. Dedicata **171** (2014), 1–28, ibid. 29–56, ibid. 57–91.

[38] D. McCullough, *Twist groups of compact 3-manifolds*, Topology **24** (1985), no. 4, 461–474.

[39] D. McCullough, A. Miller, B. Zimmermann, *Group actions on handlebodies*, Proc. London Math. Soc. (3) **59** (1989), no. 2, 373–416.

[40] J. M. Montesinos, *On 3-manifolds having surface bundles as branched coverings*, Proc. Amer. Math. Soc. **101** (1987), no. 3, 555–558.

[41] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs, **3**. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. xlii+521 pp.

[42] J. Morgan and G. Tian, *The geometrization conjecture*, Clay Mathematics Monographs, **5**. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014. x+291 pp.

[43] H. Namazi, *Big Heegaard distance implies finite mapping class group*, Topology Appl. **154** (2007), no. 16, 2939–2949.

[44] K. Ohshika and M. Sakuma, *Subgroups of mapping class groups related to Heegaard splittings and bridge decompositions*, Geom. Dedicata **180** (2016), 117–134.

[45] T. Ohtsuki, R. Riley and M. Sakuma, *Epimorphisms between* 2*-bridge link groups*, Geom. Topol. Monogr. **14** (2008), 417–450.

[46] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 [math.DG].

[47] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109 [math.DG].

[48] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245 [math.DG].

[49] M. Sakuma, *Surface bundles over* S^1 *which are 2-fold branched cyclic coverings of* S^3, Math. Sem. Notes Kobe Univ. **9** (1981), no. 1, 159–180.

[50] M. Scharlemann, *Automorphisms of the 3-sphere that preserve a genus two Heegaard splitting*, Bol. Soc. Mat. Mexicana (3) **10** (2004), Special Issue, 503–514.

[51] M. Scharlemann and M. Tomova, *Alternate Heegaard genus bounds distance*, Geom. Topol. **10** (2006), 593–617.

[52] J. Simon, *Compactification of covering spaces of compact 3-manifolds*, Michigan Math. J. **23** (1976), 245–256 (1977).

[53] B. Zimmermann, *Finite group actions on handlebodies and equivariant Heegaard genus for 3-manifolds*, Topology Appl. **43** (1992), no. 3, 263–274.

[54] B. Zimmermann, *On large orientation-reversing finite group-actions on 3-manifolds and equivariant Heegaard decompositions*, Monatsh. Math. **191** (2020), no. 2, 437–444.

Department of Mathematics, Hiroshima University
Higashi-Hiroshima, 739-8526, Japan
E-mail address: `ykoda@hiroshima-u.ac.jp`

広島大学先進理工系科学研究科数学プログラム　古宇田 悠哉

Advanced Mathematical Institute, Osaka City University
3-3-138, Sugimoto, Sumiyoshi, Osaka City 558-8585, Japan
E-mail address: `sakuma@hiroshima-u.ac.jp`

大阪市立大学数学研究所，広島大学　作間 誠