The Strong Slope Conjecture for Whitehead doubles

Kenneth L. Baker, Kimihiko Motegi and Toshie Takata

1 Introduction

Let K be a knot in the 3–sphere S^3. The Slope Conjecture of Garoufalidis [8] and the Strong Slope Conjecture of Kalfagianni and Tran [18] propose relationships between a quantum knot invariant, the sequence of the degrees of the colored Jones function of K, and classical invariants, the boundary slope and the topology of essential surfaces in the exterior of K.

The *colored Jones function* of K is a sequence of Laurent polynomials $J_{K,n}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ for $n \in \mathbb{N}$, where $J_{\bigcirc,n}(q) = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$ for the unknot $\bigcirc$ and $\frac{J_{K,2}(q)}{J_{\bigcirc,2}(q)}$ is the ordinary Jones polynomial of K. Since the colored Jones function is q–holonomic [10, Theorem 1], the degrees of its terms are given by *quadratic quasi-polynomials* for suitably large n [9, Theorem 1.1 & Remark 1.1]. For the maximum degree $d_+[J_{K,n}(q)]$, we set the quadratic quasi-polynomials to be

$$\delta_K(n) = a(n)n^2 + b(n)n + c(n)$$

for rational valued periodic functions $a(n), b(n), c(n)$ with integral period. We call $4a(n)$ a *Jones slope* of K and define

$$js(K) = \{4a(n) \mid n \in \mathbb{N}\}.$$

In the present article we allow surfaces to be disconnected, and we say a properly embedded surface in a 3–manifold is *essential* if each component is orientable, incompressible, and boundary-incompressible. A number $p/q \in \mathbb{Q} \cup \{\infty\}$ is a *boundary slope* of a knot K if there exists an essential surface in the knot exterior $E(K) = S^3 - \mathrm{int}N(K)$ with a boundary component representing $p[\mu] + q[\lambda] \in H_1(\partial E(K))$ with respect to the standard meridian μ and longitude λ. Now define the set of boundary slopes of K:

$$bs(K) = \{r \in \mathbb{Q} \cup \{\infty\} \mid r \text{ is a boundary slope of } K\}.$$

Since a Seifert surface of minimal genus is an essential surface, $0 \in bs(K)$ for any knot. Let us also remark that $bs(K)$ is always a finite set [12, Corollary].

Garoufalidis conjectures that Jones slopes are boundary slopes.

Conjecture 1.1 (**Slope Conjecture** [8]). *For any knot K in S^3, every Jones slope is a boundary slope. That is $js(K) \subset bs(K)$.*

Example 1.2 (Garoufalidis [8]).

$$\delta_{T_{2,3}}(n) = \frac{3}{2}n^2 - \frac{3}{2}, \quad js(T_{2,3}) = \{4 \cdot \frac{3}{2}\} = \{6\} \subset \{0, 6\} = bs(T_{2,3})$$

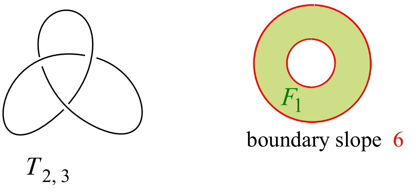

Figure 1.1: Slope Conjecture for the trefoil knot, $T_{2,3}$

Example 1.3 (Futer-Kalfagianni-Purcell [6]).

$$\delta_{4_1}(n) = 1n^2 - \frac{1}{2}n - 1, \quad js(4_1) = \{4 \cdot 1\} = \{4\} \subset \{0, \pm 4\} = bs(4_1)$$

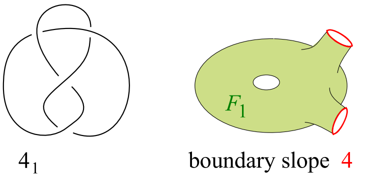

Figure 1.2: Slope Conjecture for the figure-eight knot, 4_1

Example 1.4 (Garoufalidis [8]).

$$\delta_{P(-2,3,7)}(n) = \frac{37}{8}n^2 - \frac{1}{4}n + \varepsilon(n) \quad (\varepsilon(n) \text{ has period } 4),$$

$$js(P(-2,3,7)) = \{4 \cdot \frac{37}{8}\} = \{\frac{37}{2}\} \subset \{0, 16, \frac{37}{2}, 20\} = bs(P(-2,3,7))$$

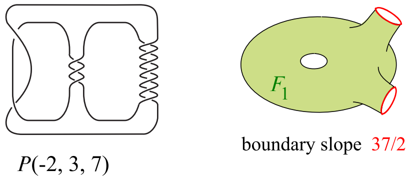

Figure 1.3: Slope Conjecture for $P(-2,3,7)$

Garoufalidis' Slope Conjecture concerns only the quadratic terms of $\delta_K(n)$. Kalfagianni and Tran propose the Strong Slope Conjecture which subsumes the Slope Conjecture and asserts that the topology of the surfaces whose boundary slopes are Jones slopes may be predicted by the linear terms of $\delta_K(n)$. Define

$$jx(K) = \{2b(n) \mid n \in \mathbb{N}\}.$$

Conjecture 1.5 (**The Strong Slope Conjecture** [18, 16]). [1] *Let K be a knot in S^3. For any Jones slope p/q there exists an essential surface $F_n \subset E(K)$ such that F_n has boundary slope $p/q = 4a(n)$ and $\dfrac{\chi(F_n)}{|\partial F_n|q} = 2b(n)$ for some $n \in \mathbb{N}$. We call F_n a Jones surface.*

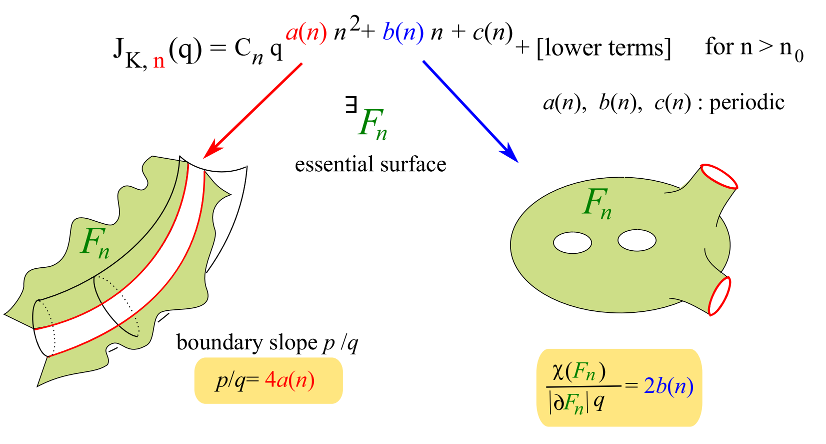

Figure 1.4: Strong Slope Conjecture

[1] In the present article we restrict our attention to simple situations, we adopt the simplified form instead of the definition in [2].

Let us examine six potential situations where the Strong Slope Conjecture holds or does not hold, schematically illustrated in Figure 1.5(i)–(vi).

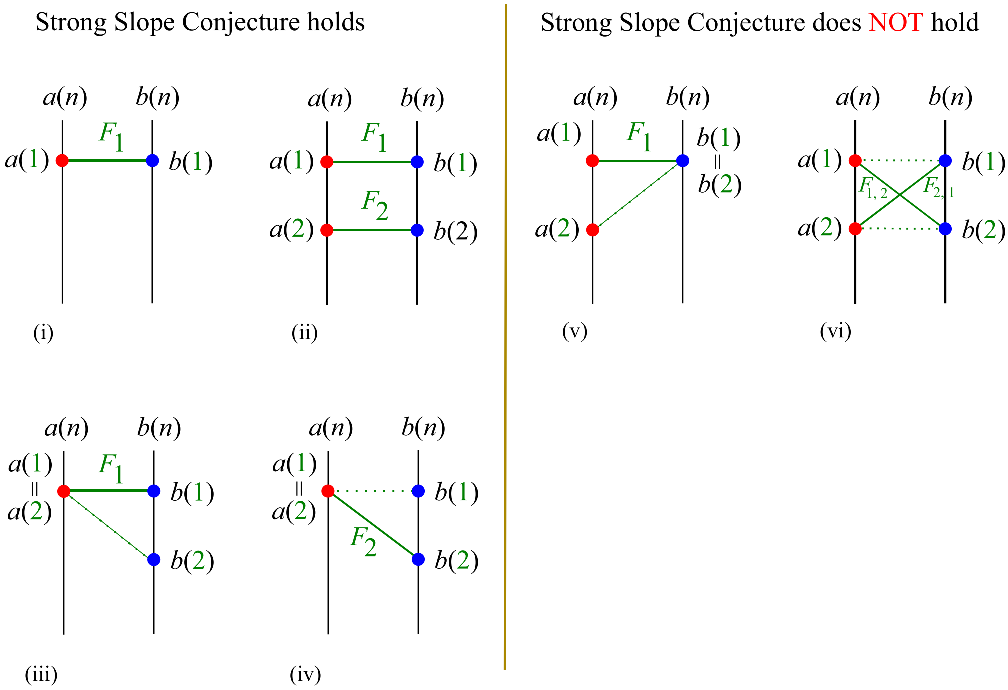

Figure 1.5: Six potential situations for relationships between $\delta_K(n)$, properly embedded essential surfaces, and the Strong Slope Conjecture

In most computed examples we have situation (i): both $a(n)$ and $b(n)$ are constant and F_1 is a Jones surface. The horizontal segment represents a Jones surface F_1 which has boundary slope $\frac{p}{q} = 4a(1)$ and satisfies $\frac{\chi(F_1)}{|\partial F_1|q} = 2b(1)$. In situation (ii), $a(n)$ and $b(n)$ have period 2 and there are Jones surfaces F_1 and F_2 for which F_i has boundary slope $\frac{p_i}{q_i} = 4a(i)$ and $\frac{\chi(F_i)}{|\partial F_i|q_i} = 2b(i)$ for each $i = 1, 2$. In both of these situations the Strong Slope Conjecture holds as each Jones slope has a Jones surface.

Both (iii) and (iv) describe situations where $a(n)$ is constant, while $b(n)$ has period 2. In situation (iii) the horizontal segment corresponds to a Jones surface F_1 which has boundary slope $\frac{p}{q} = 4a(1)$ and satisfies $\frac{\chi(F_1)}{|\partial F_1|q} = 2b(1)$. However, as the slanting dotted segment suggests, we have no essential surface whose boundary slope is $\frac{p}{q} = 4a(2)$ and $\frac{\chi(F_2)}{|\partial F_2|q} = 2b(2)$. Similarly in situation (iv) the slanting segment corresponds to a Jones surface F_2 which has boundary slope $\frac{p}{q} = 4a(2)(= 4a(1))$ and satisfies $\frac{\chi(F_2)}{|\partial F_2|q} = 2b(2)$, but, as the horizontal dotted segment suggests, we have no essential surface whose boundary slope is $\frac{p}{q} = 4a(1)$ and $\frac{\chi(F_1)}{|\partial F_1|q} = 2b(1)$. In either situation, for the unique Jones slope $\frac{p}{q}$ we have a Jones surface with boundary slope $\frac{p}{q}$. Thus by definition, the Strong Slope Conjecture holds.

Situations (v) and (vi) show what may go wrong when $a(n)$ has period 2 (or greater). In situation (v), $a(1) \neq a(2)$ while $b(1) = b(2)$. Here, although F_1 is a Jones surface with Jones slope $4a(1)$, we have no Jones surface with Jones slope $4a(2)(\neq 4a(1))$. Hence the Strong Slope Conjecture does not hold. In situation (vi), we have surfaces $F_{1,2}$ with boundary slope $4a(1)$ and $\frac{\chi(F_{1,2})}{|\partial F_{1,2}|q} = 2b(2)$, and $F_{2,1}$ with $4a(2)$ and $\frac{\chi(F_{2,1})}{|\partial F_{2,1}|q} = 2b(1)$, However, we have no Jones surfaces F_1, F_2, so the Slope Conjecture does not hold. The dotted horizontal lines in Figure 1.5(vi) suggests the absence of F_1 and F_2.

Example 1.6. *Let K be a knot which appears among the families in the following list.*

(1) *Torus knots [8], [18, Theorem 3.9].*

(2) *B–adequate knots [6], [18, Lemma 3.6, 3.8], hence adequate knots, and in particular alternating knots.*

(3) *Non-alternating knots with up to* 9 *crossings except for* 8_{20}, 9_{43}, 9_{44} *[8], [18, 15].*

Then, writing $\delta_K(n) = a(n)n^2 + b(n)n + c(n)$, we have that $a(n), b(n)$ are constant, and $c(n)$ has period at most two. Moreover, K satisfies the Slope Conjecture and the Strong Slope Conjecture. Note also that if K is nontrivial, then $b(n) = b \leq 0$. See [20, 11, 21] for further examples.

Remark 1.7. *The non-alternating knots* 8_{20}, 9_{43}, 9_{44} *satisfy the Strong Slope Conjecture [18], but for these knots the coefficient $b(n)$ has period* 3.

2 Extending a class of knots with the Strong Slope Conjecture

In this note we discuss the extension of a class of knots which satisfy the Slope Conjecture and the Strong Slope Conjecture.

Let V be a standardly embedded solid torus in S^3 with a preferred meridian-longitude (μ_V, λ_V), and take a pattern (V, k) where k is a knot in the interior of V illustrated by Figure 2.1. Given a knot K in S^3 with a preferred meridian-longitude (μ_K, λ_K), let $f \colon V \to S^3$ an orientation preserving embedding which sends the core of V to the knot $K \subset S^3$ such that $f(\mu_V) = \mu_K$ and $f(\lambda_V) = \lambda_K$. Then the image $f(k)$ is called a *satellite knot* with a companion knot K and pattern (V, k). In particular, if k is a (p, q)–torus knot in V such that it winds p times meridionally and q times longitudinally, then $f(K)$ is called a (p, q)*–cable of* K and denoted by $C_{p,q}(K)$ as in Figure 2.1(1); if the pattern

(V,k) is given as in Figure 2.1(2), then $f(K)$ is called a *Whitehead double of* K and denoted by $W(K)$.

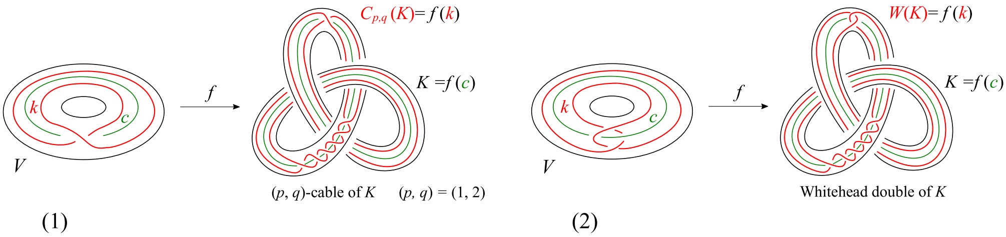

Figure 2.1: $f\colon V \to S^3$ is a faithful embedding and it maps the core of V to K; The left is a (p,q)–cable of K with $(p,q) = (1,2)$, and the right is a Whitehead double of K.

We state a main result in the following weaker form. See [2] for the more general setting.

Theorem 2.1. *Any knot obtained by a finite sequence of cabling, Whitehead doubling, and connected sums of torus knots or B–adequate knots satisfies the Slope Conjecture and the Strong Slope Conjecture.*

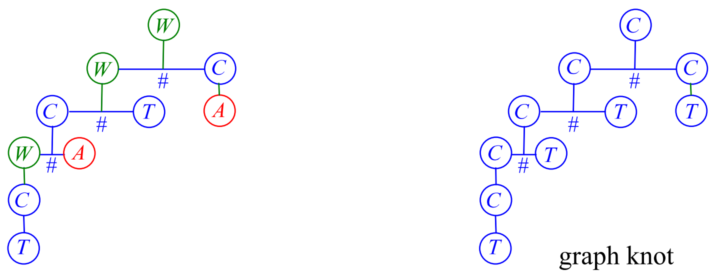

Figure 2.2: Strong Slope Conjecture and cablings, Whitehead doublings and connected sums; Ⓣ and Ⓐ denote a torus knot and B–adequate knot, respectively. Ⓒ, Ⓦ and $\sharp$ represent a cable, the Whitehead double and a connected sum, respectively.

In Theorem 2.1, if we start with the family of torus knots and apply sequences of cablings and connected sums to create new knots, we obtain graph knots. Thus as a particular case, we obtain:

Corollary 2.2. *Every graph knot satisfies the Slope Conjecture and the Strong Slope Conjecture.*

The aim of this expository paper is to give an idea of how to prove Theorem 2.1. For details and more generality, see [2, Theorem 1.6], [3] and [18].

For our investigations into the Strong Slope Conjecture for knots obtained from torus knots, B–adequate knots by a finite sequence of operations cablings, Whitehead doubles and connected sums, technical reasons lead us to introduce two conditions which are preserved under these operations in this setting.

Definition 2.3 (Sign Condition)**.** *Let $\varepsilon_n(K)$ be the sign of the coefficient of the term of the maximum degree of $J_{K,n}(q)$. A knot K satisfies the Sign Condition if $\varepsilon_m(K) = \varepsilon_n(K)$ for $m \equiv n \bmod 2$.*

♠ In general $d_+[J_{K,n}(q)]$ forms a quadratic quasi-polynomial when $n \geq n_0$ for some large integer n_0. We call $\{n \mid n \geq n_0\}$ the *stable range* of $d_+[J_{K,n}(q)]$, and $\{n \mid 1 \leq n < n_0\}$ the *unstable range* of $d_+[J_{K,n}(q)]$. When we consider cables or Whitehead doubles of a knot K, if there is no unstable range of $d_+[J_{K,n}(q)]$, we do not need the Sign Condition for K. When the unstable range of K exists, the Sign Condition precludes potential unexpected and problematic cancellations and enables us to determine the maximum degree of the colored Jones polynomials of a cable or Whitehead double of K.

Remark 2.4. *In [2] we asked if every knot satisfies the Sign Condition. However, a computer experiments shows that some knots do not satisfy the Sign Condition. See [3, §4.2].*

Definition 2.5 (Condition δ)**.** *We say that a knot K satisfies Condition δ if*

(1) $\delta_K(n) = an^2 + bn + c(n)$ *has period at most* 2*,*

(2) $b \leq 0$ [2]*,*

(3) $4a \in \mathbb{Z}$ [3]*, and*

(4) $b = 0 \implies a \neq 0$ [4]*.*

Definition 2.6. *Let $\mathcal{K}$ be the maximal set of knots in S^3 of which each is either the trivial knot or satisfies the Sign Condition, Condition δ, and the Strong Slope Conjecture.*

Example 2.7. *Torus knots and B–adequate knots belong to $\mathcal{K}$.*

- *The trivial knot is in $\mathcal{K}$ by definition.*

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2) For the trivial knot $b = 1/2 > 0$. It is conjectured that $b \leq 0$ for any nontrivial knot [18, Conjecture 5.1].

3) This condition is used when cabling.

4) If $b = 0$, then it is conjectured that K is composite or cabled [18, Conjecture 5.1]; for a cabled knot it is known that $a \neq 0$.

- *Any nontrivial torus knot satisfies the Strong Slope Conjecture and Condition δ via [18, Theorem 3.9]. More precisely, if K is a positive torus knot, then $0 < 4a \in \mathbb{Z}$ and $b = 0$. If K is a negative torus knot, then $a = 0$ and $b < 0$. Furthermore, it follows from [3, Proposition 4.3] that any torus knot satisfies the Sign Condition. Hence torus knots are in $\mathcal{K}$.*

- *Any nontrivial B–adequate knot satisfies the Strong Slope Conjecture and Condition δ via [6, 7] and [18, Lemma 3.6, 3.8]. More precisely, $0 \leq 4a \in \mathbb{Z}, b \leq 0$. See [18, Lemma 3.6]. If $b = 0$, then K is a torus knot and $a > 0$ ([18, Lemma 3.8]). Furthermore, it follows from [3, Proposition 4.4] that any B–adequate knot satisfies the Sign Condition. Hence B–adequate knots are in $\mathcal{K}$.*

Theorem 2.8. *The set $\mathcal{K}$ is closed under connected sum, cabling and Whitehead doubling.*

For simplicity we call a connected sum, cabling and Whitehead double a *knot operation.* Very roughly, for each knot operation, we proceed in the following steps.

Plan of the proof

For each knot operation:

- Compute of the maximum degree of the colored Jones polynomial of a knot obtained by the knot operation.
- Find an essential surface in each decomposing piece which satisfies suitable boundary conditions, and glue them to construct an essential surface in the exterior of the resulting knot with desired Jones slope.
- Check the Euler characteristic condition in the Strong Slope Conjecture.
- Check that the Sign Condition and Condition δ are preserved under the knot operation, which allows us to apply the knot operation repeatedly.

3 Connected sums and cablings

3.1 The Strong Slope Conjecture for connected sums

Theorem 3.1. *Let K_1 and K_2 be knots in $\mathcal{K}$. Then a connected sum $K_1 \sharp K_2$ also belongs to $\mathcal{K}$.*

Our proof of the Strong Slope Conjecture for connected sums serves as a nice example to illustrate the general method of proof for these knot operations. We now sketch a proof of the above theorem. For details, see [25, 3].

Proof. By the assumption that K_1 and K_2 satisfy Condition δ, we may write $\delta_{K_i}(n) = a_i n^2 + b_i n + c_i(n)$ $(i = 1, 2)$.

Let S_i be an essential surface in $E(K_i)$ that has boundary slope $p_i/q_i = 4a_i$ and satisfies $\dfrac{\chi(S_i)}{|\partial S_i|q_i} = 2b_i$.

One observes that

$$\delta_{K_1\sharp K_2}(n) = \delta_{K_1}(n) + \delta_{K_2}(n) - \frac{1}{2}n + \frac{1}{2} = (a_1 + a_2)n^2 + (b_1 + b_2 - \frac{1}{2})n + c_1(n) + c_2(n) + \frac{1}{2}.$$

Note that $E(K_1\sharp K_2)$ is decomposed into $E(K_1)$ and $E(K_2)$ along an essential annulus A whose core is meridian of each K_1 and K_2; see Figure 3.1.

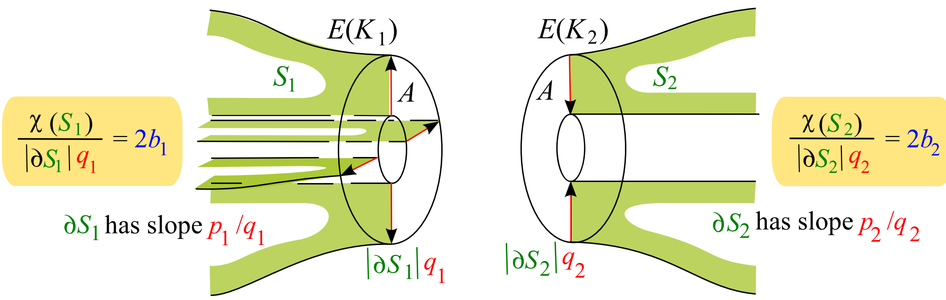


Figure 3.1: Gluing essential surfaces in $E(K_1)$ and $E(K_2)$

Gluing m_1 copies of S_1 and m_2 copies of S_2 along A, we obtain $F = m_1S_1 \cup m_2S_2$ in $E(K_1\sharp K_2)$. For F to be a properly embedded surface, the gluing condition requires

- $m_1|\partial S_1|q_1 = m_2|\partial S_2|q_2 = |F \cap A| = |\partial F|q$.

Then following [25, Claim 2.3] each component of ∂F on $\partial E(K_1\sharp K_2)$ has slope $p/q = p_1/q_1 + p_2/q_2$ $(q > 0)$, which equals $4(a_1 + a_2)$. The surface F may be non-orientable, so we take the frontier $\widetilde{F}$ of the tubular neighborhood of F in $E(K_1\sharp K_2)$. Then the orientable surface $\widetilde{F}$ also has boundary slope $p/q = p_1/q_1 + p_2/q_2$ $(q > 0)$ and satisfies $\dfrac{\chi(\widetilde{F})}{|\partial\widetilde{F}|q} = \dfrac{\chi(F)}{|\partial F|q}$ [2, Lemma 5.1]. As shown in [25, Claim 2.4] $\widetilde{F}$ is essential in $E(K_1\sharp K_2)$.

One calculates

- $\chi(F) = \chi(m_1S_1 \cup m_2S_2) = m_1\chi(S_1) + m_2\chi(S_2) - |F \cap A|$.

Hence we have:

$$\begin{aligned}\frac{\chi(\widetilde{F})}{|\partial \widetilde{F}|q} = \frac{\chi(F)}{|\partial F|q} &= \frac{m_1\chi(S_1) + m_2\chi(S_2) - |F \cap A|}{|\partial F|q} \\ &= \frac{m_1\chi(S_1)}{m_1|\partial S_1|q_1} + \frac{m_2\chi(S_2)}{m_2|\partial S_2|q_2} - \frac{|\partial F|q}{|\partial F|q} = \frac{\chi(S_1)}{|\partial S_1|q_1} + \frac{\chi(S_2)}{|\partial S_2|q_2} - 1 \\ &= 2b_1 + 2b_2 - 1 = 2(b_1 + b_2 - 1/2) \\ &= 2b.\end{aligned}$$

This shows that $K_1 \sharp K_2$ satisfies the Strong Slope Conjecture.

Finally, see [3] for $K_1 \sharp K_2$ satisfying the Sign Condition and Condition δ so that it belongs to $\mathcal{K}$. □

3.2 The Strong Slope Conjecture for cablings

A version of the following proposition is essentially given in [18, Theorem 3.9].

Proposition 3.2. *Let K be a knot which belongs to $\mathcal{K}$, i.e. it satisfies Condition δ, the Sign Condition, and the Strong Slope Conjecture. Then a non-trivial cable $K_{p,q}$ satisfies Condition δ, the Sign Condition, and the Strong Slope Conjecture, hence $K_{p,q} \in \mathcal{K}$.*

Proof. Since we assume that $K \in \mathcal{K}$, the maximum degree of the colored Jones polynomial of $K_{p,q}$ is explicitly given in the following form.

Proposition 3.3 (More generally, see [18] and [3])**.** *Let K be a knot which belongs to $\mathcal{K}$ with $\delta_K(n) = an^2 + bn + c(n)$. Then*

$$\delta_{K_{p,q}}(n) = \begin{cases} q^2an^2 + \Big(qb + \frac{(q-1)(p-4qa)}{2}\Big)n & \\ \qquad + \big(a(q-1)^2 - (b + \frac{p}{2})(q-1) + c(i)\big) & \text{for } \frac{p}{q} < 4a, \\ \frac{pq}{4}n^2 - \frac{pq}{4} + C_\sigma(K_{p,q}) & \text{for } \frac{p}{q} \geq 4a, \end{cases}$$

where $i \equiv_{(2)} q(n-1)+1$, $\sigma \equiv_{(2)} n$, and $C_\sigma(K_{p,q})$ is a number that only depends on the knot K, the numbers p and q, and the parity σ of n. Furthermore, $K_{p,q}$ also satisfies the Sign Condition. □

It is easy to see that $K_{p,q}$ also satisfies the Condition δ, [2, Lemma 6.8].

Using Proposition 3.3 instead of [18, Proposition 3.2], one applies the construction in [25] to obtain an essential surface satisfying the Strong Slope Conjecture. □

4 Whitehead doubles

Theorem 4.1. *Let K be a knot in $\mathcal{K}$. Then the Whitehead double of K also belongs to $\mathcal{K}$.*

To prove this theorem we need to know the behavior of the maximum degree of the colored Jones polynomials of the Whitehead double $W(K)$ of K.

Proposition 4.2 (More generally, see [2, Propositions 2.3 and 2.4])**.** *Let K be a knot which belongs to $\mathcal{K}$ with $\delta_K(n) = an^2 + bn + c(n)$. Assume K is non-trivial. Then*

$$\delta_{W(K)}(n) = \begin{cases} 4an^2 + (-4a + 2b - \frac{1}{2})n + a - b + c(1) + \frac{1}{2} & (a > 0), \\ 0n^2 - \frac{1}{2}n + C_+(K, 0) + \frac{1}{2} & (a \le 0). \end{cases}$$

Proof. Apply the graphical calculus of Masbaum and Vogel [23, 22] following the method of Tanaka [26]. □

Outline of the proof of Theorem 4.1. It follows from [2, Lemma 6.9] that a Whitehead double $W(K)$ satisfies the Sign Condition and Condition δ whenever $K \in \mathcal{K}$. For details, see [2, Lemma 6.9].

In the remaining we prove that the Whitehead double $W(K)$ of K also satisfies the Strong Slope Conjecture when K belongs to $\mathcal{K}$. We divide into two cases depending upon $a > 0$ or $a \le 0$; see Proposition 4.2.

Case 1. $a \le 0$. In this case, since

$$\delta_{W(K)}(n) = 0n^2 - \frac{1}{2}n + C_+(K, 0) + \frac{1}{2},$$

the Jones slope is 0. So let F be a minimal genus Seifert surface of $W(K)$.

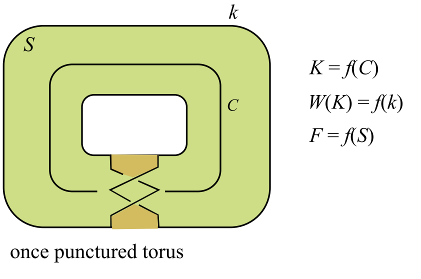


Figure 4.1: A minimal genus Seifert surface of $W(K)$

Then

- F is a once punctured torus with boundary slope 0.
- $\dfrac{\chi(F)}{|\partial F|q} = \dfrac{\chi(F)}{|\partial F|} = \dfrac{-1}{1} = -1 = -2(\dfrac{1}{2})$.

Thus F is the desired essential surface.

Case 2. $a > 0$. Then since $\delta_K(n) = an^2 + bn + c(n)$ implies

$$\delta_{W(K)}(n) = 4an^2 + (-4a + 2b - \frac{1}{2})n + a - b + c(1) + \frac{1}{2},$$

the Jones slope is $4 \cdot 4a = 16a$.

Let us consider the decomposition of $E(W(K)) = E(K) \cup X$ as shown in Figure 4.2, where $X = f(V - \mathrm{int}N(k)) \cong V - \mathrm{int}N(k)$, the Whitehead link exterior; see Figure 2.1(2). Since K satisfies the Strong Slope Conjecture, we have an essential surface $S_K \subset E(K)$ with boundary slope $4a$.

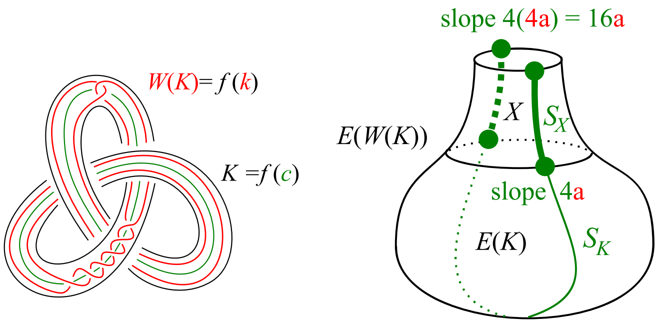


Figure 4.2: A decomposition of $E(W(K))$; in general $|\partial S_K| \neq |\partial S_X \cap \partial E(K)|$!

As the next step, we find an essential surface $S_X \subset X$ such that $\partial S_X \cap \partial E(K)$ has slope $4a$ and $\partial S_X \cap \partial E(W(K))$ has slope $16a$; see Figure 4.2.

Let k_1 be a core of the solid torus $S^3 - \mathrm{int}V$ and put $k_2 \subset V$ as in Figure 4.3. Then $k_1 \cup k_2$ is a two-bridge link $L_{\frac{3}{8}}$.

Hatcher and Thurston [13] show how a certain collection of "minimal edge paths" in the Farey diagram from $1/0$ to p/q are in correspondence with the properly embedded incompressible and ∂–incompressible surfaces with boundary in the exterior of the two bridge knot $\mathcal{L}_{p/q}$. Floyd and Hatcher [5] extend this to two-bridge links of two components from which Hoste and Shanahan [14] discern the boundary slopes of such surfaces, building upon work of Lash [19].

Here, for use with satellite constructions, we use the works of Floyd-Hatcher [5] and Hoste-Shanahan[14] to catalog all the properly embedded essential surfaces in the exterior of the Whitehead link $\mathcal{L}_{3/8}$, their Euler characteristics, their boundary slopes, and number of boundary components.

Figure 4.4 shows three diagrams. The diagram D_1 is the common Farey diagram. Pair adjacent triangles into quadrilaterals containing a diagonal so that a vertex is an endpoint

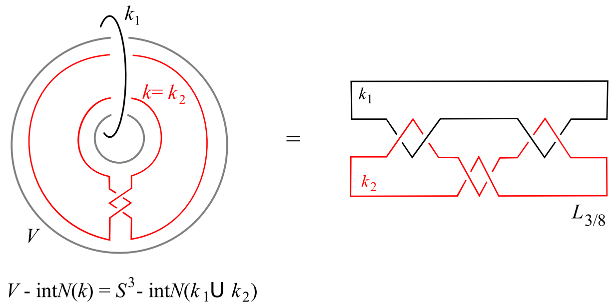


Figure 4.3: $k_1 \cup k_2$ forms a two-bridge link $L_{3/8}$.

of either all or none of the diagonals of the incident quadrilaterals. The diagram D_0 is obtained by switching the diagonal in each of the quadrilaterals. The diagram D_t is obtained by replacing these diagonals with inscribed quadrilaterals. Actually, D_t represents a parameterized family of diagrams for $t \in [0, \infty]$: with appropriate parameterizations of the edges of the quadrilaterals by $[0, 1]$ the vertices of the inscribed quadrilaterals in D_t are located at either t or $1/t$. The diagrams $D_0 = D_\infty$ and D_1 arise as limits where the inscribed quadrilaterals degenerate to diagonals. The edges of D_1 are labeled A and C, the edges of $D_0 = D_\infty$ are labeled B and D, and these induce labels on D_t.

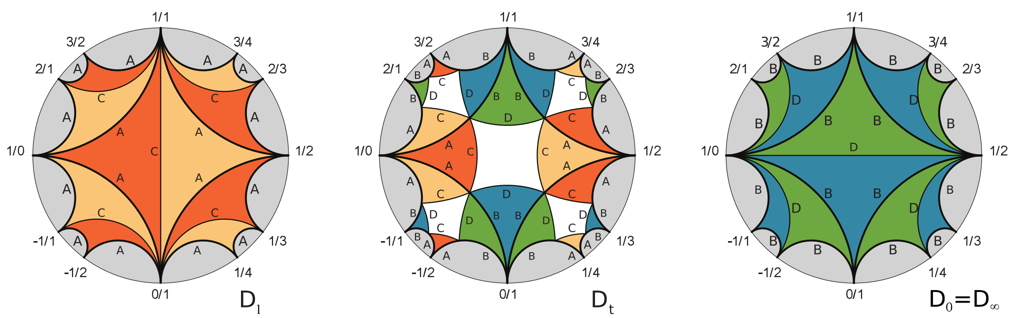


Figure 4.4: The diagrams D_1, D_t, and $D_0 = D_\infty$, cf. [14, Figures 3 and 4].

For a two bridge link $\mathcal{L}_{p/q}$ (where q is even), Floyd and Hatcher show that a properly embedded essential surface in the exterior of the link is carried by one of finitely many branched surfaces associated to "minimal edge paths" in D_t from $1/0$ to p/q. A *minimal edge path* in D_t is a consecutive sequence of edges of D_t such that the boundary of any face of D_t contains at most one edge of the path. Then for each minimal edge path, a branched surface is assembled from the sequence of edges by stacking four blocks of basic branched surface $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_D$ corresponding to the labels A, B, C, D that are positioned according to the endpoints and orientation of its edge and whether $t < 1$ or $t > 1$. Let us take a closer look at portions of the diagrams D_1, D_t, and $D_0 = D_\infty$ that carry the minimal edge paths from $\frac{1}{0}$ to $\frac{3}{8}$.

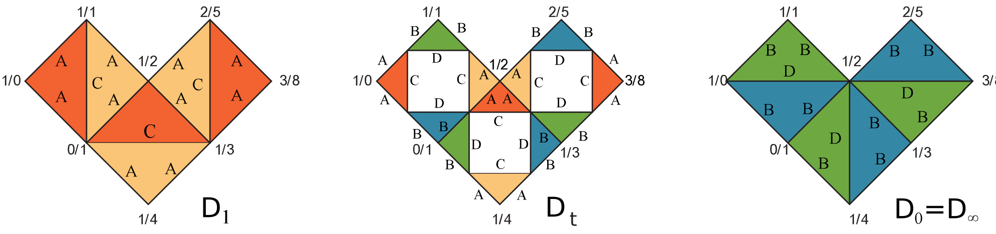
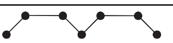
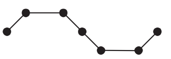
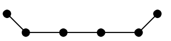
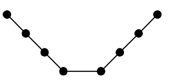


Figure 4.5: Portions of the diagrams D_1, D_t, and $D_0 = D_\infty$ that carry the minimal edge paths from $\frac{1}{0}$ to $\frac{3}{8}$.

Table 4.1[5] below provides minimal edge paths in D_t from 1/0 to 3/8 with $t = \alpha/\beta > 1$.

Table 4.1: The minimal edge paths in D_t from 1/0 to 3/8, the branch patterns of their supporting branched surfaces, and the Euler characteristics for the surfaces when $t = \alpha/\beta > 1$ are shown.

HS path	path picture	branch pattern	χ
γ_1		$ADAADA$	$-\alpha - \beta$
γ_2		$ADAADA$	$-\alpha - \beta$
γ_3		$ADAADA$	$-\alpha - \beta$
γ_5		$ADCDA$	$-\alpha$
γ_6		$ABBCBBA$	$-\alpha$

In this manner, every minimal edge path in D_t for $t \in (0,1) \cup (1,\infty)$ produces a weighted branched surface spanning the two-bridge link, with weights in terms of the parameters α and β such that $t = \alpha/\beta$.

A surface carried by one of the branched surfaces is determined by α and β, the number of sheets of the surface running along each component of $L_{p/q}$, and by how the surface branches in each segment $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_D$. For details see [5].

These minimal edge paths γ in D_t with their parameters α, β describe specific surfaces $F_{\gamma,\alpha,\beta} \subset S^3 - \mathrm{int}N(k_1 \cup k_2) = V - \mathrm{int}N(k)$ which may have multiple components and may

[5] The HS path name is established in [14, Table 2].

be non-orientable. If it is non-orientable, then we may replace $F_{\gamma,\alpha,\beta}$ by the boundary of a tubular neighborhood (a twisted I–bundle over $F_{\gamma,\alpha,\beta}$), which is orientable and associated with parameters $2\alpha, 2\beta$; so the resulting orientable essential surface is associated with $F_{\gamma,2\alpha,2\beta}$. In the following we omit parameters α, β and assume that $F_\gamma \subset V - \mathrm{int}N(k)$ is orientable, but it may have multiple components.

Let us choose a minimal edge path γ_1 and take an essential surface F_{γ_1} with $t = \alpha/\beta > 1$. Then put $S_X = f(F_{\gamma_1}) \subset X$. See Figures 2.1 and 4.6

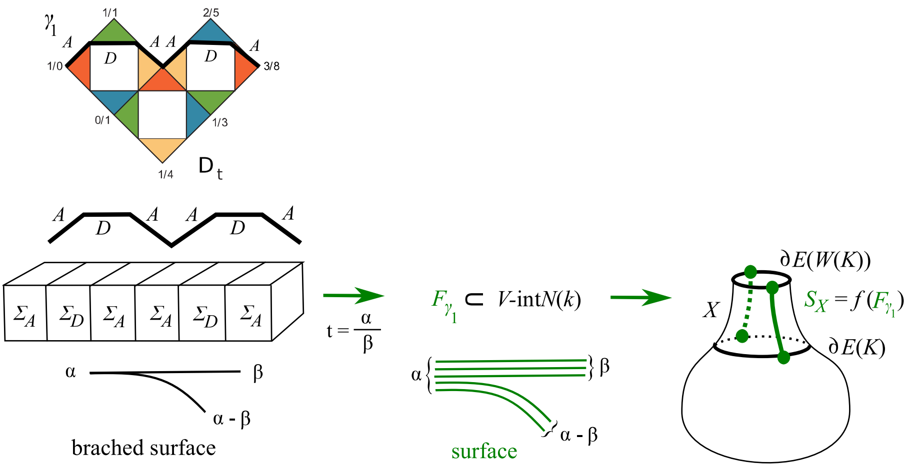


Figure 4.6: The essential surface $F_{\gamma_1} \subset X$ associated with the minimal edge path γ_1 with $t = \alpha/\beta$ and $S_X = f(F_{\gamma_1})$

The surface S_X enjoys the following property.

Property of S_X with weight $\alpha/\beta > 1$

- S_X is essential (Floyd-Hatcher [5]).
- The pair of boundary slopes of S_X is (slope on $\partial N(K)$, slope on $\partial N(W(K))) = (\frac{\alpha}{2\beta}, \frac{2\alpha}{\beta})$ (Hoste-Shanahan [14]).

  (Note [14] describes the boundary slopes in terms of the meridian-longitude pair μ_i, λ_i of k_i (Figure 4.3). We have changed (μ_1, λ_1) to (λ_K, μ_K).)
- $(|\partial S_X \cap \partial N(K)|, |\partial S_X \cap \partial N(W(K))| = (\gcd(\alpha, 2\beta), \gcd(2\alpha, \beta))$ (Hoste-Shanahan [14]).
- $\chi(S_X) = -\alpha - \beta$ ([2]).

Recall that

$$\delta_K(n) = an^2 + bn + c(n) \quad (a > 0)$$

$$\delta_{W(K)}(n) = a_W n^2 + b_W n + c_W = 4an^2 + (-4a + 2b - \frac{1}{2})n + a - b + c(1) + \frac{1}{2}$$

Let us write $a = \frac{r}{s}$ and take $\alpha = 16r$, $\beta = 2s$ [6]. Then we have

- $(\frac{\alpha}{2\beta}, \frac{2\alpha}{\beta}) = (\frac{16r}{4s}, \frac{32r}{2s}) = (\frac{4r}{s}, \frac{16r}{s}) = (4a, 16a) = (4a, 4a_W)$ [7], i.e. S_X has boundary slope $4a$ on $\partial E(K)$ and boundary slope $16a$ on $\partial E(W(K))$, which satisfy our desired condition; see Figure 4.2.
- $(|\partial S_X \cap \partial N(K)|, |\partial S_X \cap \partial N(W(K))|) = (\gcd(\alpha, 2\beta), \gcd(2\alpha, \beta)) = (4\gcd(4, s), 2\gcd(16, s))$.
- $\chi(S_X) = -\alpha - \beta = -16r - 2s$.

Since K satisfies the Strong Slope Conjecture,

$$\frac{\chi(S_K)}{|\partial S_K| \cdot \frac{s}{\gcd(4,s)}} = 2b.$$

Gluing m copies of S_K and n copies of S_X along $\partial E(K)$, we obtain $F = mS_K \cup nS_X$ in $E(W(K))$. For F to be a properly embedded surface, the gluing condition requires

- $m|\partial S_K| = n|\partial S_X \cap \partial N(K)|$.

Then $|\partial S| = n|\partial S_X \cap \partial N(W(K))| = 2n\gcd(16, s)$.

The surface F may be non-orientable. So we take the frontier $\widetilde{F}$ of the tubular neighborhood of F in $E(W(K))$. Then $\widetilde{F}$ is an essential surface with boundary slope $4a_W = \frac{16r}{s}$ and satisfies $\dfrac{\chi(\widetilde{F})}{|\partial \widetilde{F}| \cdot \frac{s}{\gcd(16,s)}} = \dfrac{\chi(F)}{|\partial F| \cdot \frac{s}{\gcd(16,s)}}$.

---

[6] To ensure that S_X is orientable, as we mentioned before, we take $\alpha = 16r$, $\beta = 2s$ rather than $\alpha = 8r$, $\beta = s$

[7] In fact, Condition δ implies $4a \in \mathbb{Z}$, so the denominators of $4a$ and $16a$ are 1, i.e. $\frac{s}{\gcd(4,s)} = \frac{s}{\gcd(16,s)} = 1$.

Finally, we observe

$$\begin{aligned}
\frac{\chi(\widetilde{F})}{|\partial\widetilde{F}|\cdot\frac{s}{\gcd(16,s)}} = \frac{\chi(F)}{|\partial F|\cdot\frac{s}{\gcd(16,s)}} &= \frac{m\chi(S_K)+n\chi(F_{\gamma_1})}{2n\gcd(16,s)\cdot\frac{s}{\gcd(16,s)}}\\
&= \frac{2bm|\partial S_K|\cdot\frac{s}{\gcd(4,s)}+n(-16r-2s)}{2ns}\\
&= \frac{8bn\gcd(4,s)\cdot\frac{s}{\gcd(4,s)}-2n(8r+s)}{2ns}\\
&= \frac{8bns-2n(8r+s)}{2ns}\\
&= 4b-8r/s-1 = 2(-4a+2b-\frac{1}{2}) = 2b_W
\end{aligned}$$

as desired. □

5 Non-adequate Whitehead doubles

Recall that B–adequate knots are known to satisfy the Strong Slope Conjecture [6, 7, 18]. On the other hand, we may expect that most Whitehead doubles are not B–adequate. However, to the best our knowledge, there are no explicit such examples. So for completeness we give explicit family of Whitehead doubles which are not B–adequate.

Theorem 5.1. *Let K be the torus knot $T_{2,-(2m+1)}$ for $m \geq 1$. Then $W_-(K)$ is not B–adequate.*

Since $K = T_{2,-(2m+1)}$ is alternating, it is adequate. So this result shows that even when K is adequate, its Whitehead double $W_-(K)$ may not be adequate.

Proof. Let us denote the Turaev genus of K by $g_T(K)$, which was introduced by Turaev [27]. We first observe that if $W_-(T_{p,-q})$ is adequate for $p, q > 1$, then $g_T(W_-(T_{p,-q})) = 1$. This follows from the computation of $\delta_{W_-(T_{p,-q})}(n)$ and $\delta^*_{W_-(T_{p,-q})}(n)$ (the minimal degree) and [17, Theorem 1.1].

Next we show that $g_T(W_-(T_{p,-q})) \neq 1$. Actually if $g_T(W_-(T_{p,-q})) = 1$, then $W_-(T_{p,-q})$ admits an alternating diagram on a standardly embedded torus in S^3. Recall that since any Whitehead double has (Seifert) genus one, it is prime. Then following [1] $W_-(T_{p,-q})$ is either a torus knot or a hyperbolic knot, a contradiction. Thus the $g_T(W_-(T_{p,-q})) \neq 1$.

Hence $W_-(T_{p,-q})$ is not adequate. In particular, if we put $p = 2,\ q = 2m+1 \geq 3$, $W_-(T_{2,-(2m+1)})$ is not adequate. On the other hand, we can see that $W_-(T_{2,-(2m+1)})$ is A–adequate. Thus we conclude that $W_-(T_{2,-(2m+1)})$ is not B–adequate. □

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Department of Mathematics
University of Miami
Coral Gables
FL 33146
USA
E-mail address: *k.baker@math.miami.edu*

University of Miami・Kenneth L. Baker

Department of Mathematics
Nihon University
Tokyo 156–8550
JAPAN
E-mail address: *motegi.kimihiko@nihon-u.ac.jp*

日本大学・文理学部　茂手木公彦

Graduate School of Mathematics
Kyushu University
Fukuoka 819–0395
JAPAN

九州大学・数理学府　高田敏恵