The symplectic derivation Lie algebra of the free commutative algebra

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1 Introduction

There are three Lie algebras $\mathfrak{l}_g, \mathfrak{a}_g, \mathfrak{c}_g$ defined by Kontsevich [7]. They are related to various geometric objects, e.g. moduli spaces of graphs and Riemann surfaces. In particular, $\mathfrak{c}_g$, the main topic in this paper, is used in perturbative Chern-Simons theory, which provides the extension of Vassiliev invariants [1, 6].

Each of three Lie algebras, denoted by $\mathfrak{h}_g$ here, has a certain ideal $\mathfrak{h}_g^+$. By an argument of a spectral sequence,

$$H_\bullet(\mathfrak{h}_g) \cong H_\bullet(\mathfrak{sp}(2g;\mathbb{Q})) \otimes H_\bullet(\mathfrak{h}_g^+)^{\mathrm{Sp}}$$

holds in the stable range. Here $H_\bullet(\mathfrak{h}_g^+)^{\mathrm{Sp}}$ is the symplectic invariant part of $H_\bullet(\mathfrak{h}_g^+)$. $\mathfrak{h}_g^+$ is relatively easy to compute, and enable us to construct cohomology classes of higher degree by taking duals or cup products. This method is applied to $\mathfrak{l}_g$ and $\mathfrak{a}_g$ to study them by Morita [8].

Kontsevich's theorem shows each of three corresponds to a kind of graph complex. In the case of $\mathfrak{c}_g$,

$$PH_\bullet(\mathfrak{c}_\infty) \cong PH_\bullet(\mathfrak{sp}(2\infty;\mathbb{Q})) \oplus (\text{commutative graph homology})\,.$$

In fact, both homology groups of $\mathfrak{c}_\infty := \lim_{g\to\infty} \mathfrak{c}_g$ and $\mathfrak{sp}(2\infty;\mathbb{Q}) := \lim_{g\to\infty} \mathfrak{sp}(2g;\mathbb{Q})$ have natural Hopf algebra structures. We denote by $PH_\bullet(\mathfrak{c}_\infty)$ and $PH_\bullet(\mathfrak{sp}(2\infty;\mathbb{Q}))$ the primitive parts of $H_\bullet(\mathfrak{c}_\infty)$ and $H_\bullet(\mathfrak{sp}(2\infty;\mathbb{Q}))$ respectively. There are some computational results from the viewpoint of graph homology theory (e.g. [2]). Conant-Gerlits-Vogtmann [3] computed the part up to degree 12. Willwacher-Živković [9] determined the generating function of Euler characteristic and displayed it up to weight 60.

The homology group $H_\bullet(\mathfrak{c}_g^+)$ has a $\mathbb{Z}_{\geq 0}$-grading called weight. It decomposes $H_\bullet(\mathfrak{c}_g^+)$ into direct summands $H_\bullet(\mathfrak{c}_g^+)_w$, which is generated by homogeneous elements of weight w. It is easy to see that $H_1(\mathfrak{c}_g^+) = S^3\mathbb{Q}^{2g}$, however, the higher degree of $H_\bullet(\mathfrak{c}_g^+)$ is still unknown. We proved $H_2(\mathfrak{c}_g^+)_w = 0$ for $g, w \geq 4$. Moreover, we determined $H_2(\mathfrak{c}_g^+)$ in terms of Sp-modules as a corollary.

This paper is a summary of [5], in which more details of the proof are.

2 The Lie algebra $\mathfrak{c}_g$

Let $g \geq 4$ be an integer. We write $H \coloneqq \mathbb{Q}^{2g}$ and consider the canonical $\mathrm{Sp}(2g;\mathbb{Q})$-action. Let $\mu\colon H \otimes H \to \mathbb{Q}$ be the canonical symplectic form, and $a_1, \dots, a_g, b_1, \dots, b_g$ be a symplectic basis with respect to μ.

Definition 2.1. For $w \geq 0$, let $\mathfrak{c}_g(w) \coloneqq S^{w+2}H$, which is the $(w+2)$-nd symmetric power, and set

$$\mathfrak{c}_g \coloneqq \bigoplus_{w\geq 0} \mathfrak{c}_g(w) \supset \bigoplus_{w\geq 1} \mathfrak{c}_g(w) \eqqcolon \mathfrak{c}_g^+.$$

We regard $\mathfrak{c}_g$ or $\mathfrak{c}_g^+$ as sets of polynomial functions on H of degree higher than 2 or 3 respectively. Let $[,]$ be the classical Poisson bracket on H, i.e.

$$[f,h] = \sum_{i=1}^{g} \left(\frac{\partial f}{\partial a_i}\frac{\partial h}{\partial b_i} - \frac{\partial f}{\partial b_i}\frac{\partial h}{\partial a_i} \right) \quad (f, h \in \mathfrak{c}_g).$$

Then $\mathfrak{c}_g^+ \subset \mathfrak{c}_g$ becomes a Lie subalgebra. We consider the Chevalley-Eilenberg chain complex $(\wedge^\bullet \mathfrak{c}_g, \partial)$. Then $\wedge^\bullet \mathfrak{c}_g^+ \subset \wedge^\bullet \mathfrak{c}_g$ becomes a chain subcomplex.

We introduce a $\mathbb{Z}_{\geq 0}$-grading on $\wedge^\bullet \mathfrak{c}_g$.

Definition 2.2.
- For $f_1 \in \mathfrak{c}_g(w_1), \dots, f_k \in \mathfrak{c}_g(w_k)$, we say that $f_1 \wedge \cdots \wedge f_k \in \wedge^k \mathfrak{c}_g$ is of *weight* $w_1 + \cdots + w_k$.
- $\left(\wedge^k \mathfrak{c}_g^+\right)_w \coloneqq \mathrm{Span}\left\{\omega \in \wedge^k \mathfrak{c}_g^+ \;\middle|\; \omega \text{ is of weight } w\right\}$

If $f_1 \in \mathfrak{c}_g(w_1) = S^{w_1+2}H$ and $f_2 \in \mathfrak{c}_g(w_2) = S^{w_2+2}H$, then

$$[f_1, f_2] \in S^{(w_1+2)-1+(w_2+2)-1}H = \mathfrak{c}_g(w_1 + w_2).$$

In other words, the bracket $[,]$ preserves weights. We see that the symplectic action on $\wedge^\bullet \mathfrak{c}_g^+$ preserves weights and that so does the differential ∂, hence we have a decomposition $\bigoplus_{w\geq 1} (\wedge^\bullet \mathfrak{c}_g^+)_w = \wedge^\bullet \mathfrak{c}_g^+$ as a chain complex.

Definition 2.3. $H_\bullet(\mathfrak{c}_g^+)_w \coloneqq H_\bullet(((\wedge^\bullet \mathfrak{c}_g^+)_w, \partial))$

Hence $H_n(\mathfrak{c}_g^+) = \bigoplus_{w\geq 1} H_n(\mathfrak{c}_g^+)_w$. Now we state the main theorem.

Theorem 2.1 (H., 2020). $H_2(\mathfrak{c}_g^+)_w = 0$ if $g, w \geq 4$.

The proof is done by showing all the cycles are boundaries.

If $g, w \geq 2$ then $H_1(\mathfrak{c}_g^+) = S^3H = \mathfrak{c}_g(1)$ because the differential map

$$\partial = [,]\colon \wedge^2 \mathfrak{c}_g^+ \to \bigoplus_{w\geq 2} \mathfrak{c}_g(w).$$

is surjective. This follows from the equation

$$\partial_2(a_1^w a_g \wedge a_1^2 b_g) = [a_1^w a_g, a_1^2 b_g] = a_1^{w+2} \in \mathfrak{c}_g(w)$$

and the fact that each $\mathfrak{c}_g(w) = S^{w+2}H$ is Sp-irreducible. We want to adopt the similar method, however, the chain space $(\wedge^2\mathfrak{c}_g^+)_w$ is not Sp-irreducible for general w. Therefore, we must find its Sp-irreducible decomposition and their generators.

3 Representation theory of $\mathrm{Sp}(2g;\mathbb{Q})$

Let us review the classical representation theory (see e.g. [4]).

The following is an important fact for the proof of the main theorem.

Theorem 3.1.

$$\left\{ \begin{array}{c} \text{Finite dimensional polynomial} \\ \text{irreducible } \mathrm{Sp}(2g;\mathbb{Q}) \text{ representation} \end{array} \right\} \Big/ \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{Young diagram with} \\ \text{at most } g \text{ rows} \end{array} \right\}$$

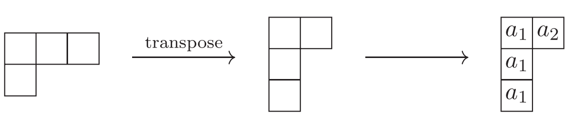
$$[V_\lambda] \longleftarrow\!\!\!\shortmid \; \lambda$$

Here V_λ is the submodule of $(\wedge^{\lambda'_1}H) \otimes \cdots \otimes (\wedge^{\lambda'_d}H)$ generated by

$$a_\lambda := (a_1 \wedge \cdots \wedge a_{\lambda'_1}) \otimes \cdots \otimes (a_1 \wedge \cdots \wedge a_{\lambda'_d}) \ \in (\wedge^{\lambda'_1}H) \otimes \cdots \otimes (\wedge^{\lambda'_d}H)$$

as an $\mathrm{Sp}(2g;\mathbb{Q})$-module and ${}^t\lambda = [\lambda'_1 \cdots \lambda'_d]$ $\quad (g \geq \lambda'_1 \geq \cdots \geq \lambda'_d \geq 1)$ is the transpose of λ.

Example 3.1.
- If $\lambda = [4] \cong S^4H$, then $a_\lambda = a_1^{\otimes 4}$.
- If $\lambda = [1111]$, then $a_\lambda = a_1 \wedge a_2 \wedge a_3 \wedge a_4$.
- Let $\lambda = [31]$, then ${}^t\lambda = [211]$. Thus $a_\lambda = (a_1 \wedge a_2) \otimes a_1 \otimes a_1$.



We easily see that the chain space $\wedge^2\mathfrak{c}_g^+$ decomposes into

$$\wedge^2\mathfrak{c}_g^+ \cong \bigoplus_{w\geq 2} \left(\bigoplus_{\substack{k>l\geq 1 \\ k+l=w}} \mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l) \oplus \bigoplus_{\substack{k\geq 1 \\ 2k=w}} \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(k) \right).$$

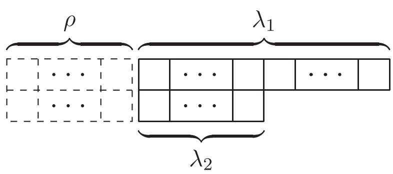
It is enough to discuss for each of the components $\mathfrak{c}_g(k)\otimes\mathfrak{c}_g(l)$ and $\mathfrak{c}_g(k)\wedge\mathfrak{c}_g(k)$ because it is finite dimensional so that its Sp-irreducible decomposition always exists. We identify each of its irreducible components with a corresponding Young diagram through Theorem 3.1 for fixed k and l.

Lemma 3.1.

(i) For $k > l \geq 1$, $\mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l) \cong \displaystyle\bigoplus_{0\leq\lambda_2\leq l+2} \bigoplus_{0\leq\rho\leq l+2-\lambda_2} [(k+l+4-\lambda_2-2\rho)\quad \lambda_2]$

(ii) For $k \geq 1$, $\mathfrak{c}_g(k) \wedge \mathfrak{c}_g(k) \cong \displaystyle\bigoplus_{0\leq\lambda_2\leq k+2} \bigoplus_{\substack{0\leq\rho\leq k+2-\lambda_2 \\ \rho+\lambda_2 \text{ is odd}}} [(2k+4-\lambda_2-2\rho)\quad \lambda_2]$

This lemma follows from the Littlewood-Richardson rule and branching rules. In particular, the multiplicity of Sp-irreducible components of $\mathfrak{c}_g(k)\otimes\mathfrak{c}_g(l)$ or $\mathfrak{c}_g(k)\wedge\mathfrak{c}_g(k)$ is always 1. We regard each irreducible component of $\mathfrak{c}_g(k)\otimes\mathfrak{c}_g(l)$ or $\mathfrak{c}_g(k)\wedge\mathfrak{c}_g(k)$ as a Young diagram



satisfying the same conditions as ones in the lemma. The part described by dashed lines means the part "chopped off" by the branching rules.

4 Sketch of the proof

Note that the differential ∂ is Sp-equivariant so that it maps an Sp-irreducible component to another Sp-irreducible component isomorphically, otherwise to 0.

We show the main theorem by the following steps:

1. Fix $w \geq 4$ and $k \geq l \geq 1$ such that $k+l=w$.

2-1. Take an irreducible component $\lambda = [\lambda_1\lambda_2] \neq [w+2]$ of $\mathfrak{c}_g(k)\otimes\mathfrak{c}_g(l)$ or $\mathfrak{c}_g(k)\wedge\mathfrak{c}_g(k)$.

2-2. Find $\omega_3 \in (\wedge^3\mathfrak{c}_g^+)_w$ such that $(\partial\omega_3)|_\lambda$ generates λ as an $\mathrm{Sp}(2g;\mathbb{Q})$-module.

2′. Find the kernel of $\partial : (\wedge^2\mathfrak{c}_g^+)_w \to (\wedge^1\mathfrak{c}_g^+)_w = \mathfrak{c}_g(w)$ restricted to the isotypical component corresponding to $\lambda = [w+2]$.

The way to find ω_3 varies depending on the conditions which $k, l, \rho, \lambda_1, \lambda_2$ satisfies. We do not discuss here the details of the construction of ω_3 but how to determine if $\partial(\omega_3)$ generates λ as an Sp-module.

We define two homomorphisms.

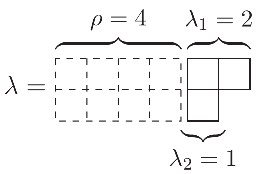
$$\begin{aligned} \mu_{\mathrm{end}}\colon \quad & H^{\otimes(w+2)} \longrightarrow H^{\otimes w}, \\ & x_1 \otimes \cdots \otimes x_{w+2} \longmapsto \mu(x_1, x_{w+2}) x_2 \otimes \cdots \otimes x_{w+1} \end{aligned}$$

$$\begin{aligned} \Lambda_{\mathrm{end}}\colon \quad & H^{\otimes(w+2)} \longrightarrow (\wedge^2 H) \otimes H^{\otimes w}. \\ & x_1 \otimes \cdots \otimes x_{w+2} \longmapsto (x_1 \wedge x_{w+2}) \otimes x_2 \otimes \cdots \otimes x_{w+1} \end{aligned}$$

We consider $\mathfrak{c}_g(k) = S^{k+2}H \subset H^{\otimes(k+2)}$ and $\mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l) \subset H^{\otimes(k+l+4)}$. Similarly we consider $\mathfrak{c}_g(k) \wedge \mathfrak{c}_g(k) \subset H^{\otimes(2k+4)}$, like $a_1^3 \wedge a_2^4 = a_1^3 \otimes a_2^4 - a_2^4 \otimes a_1^3 \in H^{\otimes 7}$ for example. Hence μ_{end} and Λ_{end} can be applied to an element of $\wedge^2 \mathfrak{c}_g^+$.

Let $\eta \in \mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l)$ and let $\lambda \subset \mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l)$ be an Sp-irreducible component. Let us consider the situation that η is mapped to a_λ (in Theorem 3.1) by some compositions of μ_{end}s and Λ_{end}s. Then the isotypical component of η corresponding to λ, which is denoted by $\eta|_\lambda$, generates λ because both μ_{end} and Λ_{end} are Sp-equivariant. We use this technique for the proof.

Example 4.1. Consider the case $w = 7$, $k = 4$, $l = 3$, and $\lambda = [2\ 1]$.



Since $\lambda \neq [9]$, we have to find $\omega_3 \in (\wedge^3 \mathfrak{c}_g^+)_7$ with $(\partial\omega_3)|_\lambda$ generating λ. In fact, it is enough to define $\omega_3 := a_1^2 a_4 \wedge a_3^4 b_4 \wedge a_2 b_3^4$. Then $\partial\omega_3 = a_1^2 a_3^4 \wedge a_2 b_3^4 - 16 a_1^2 a_4 \wedge a_2 a_3^3 b_3^3 b_4$.

Let us check $(\partial\omega_3)|_\lambda$ generates λ. For the first term of $\partial\omega_3$, we have

$$a_1^2 a_3^4 \xmapsto{(\mu_{\mathrm{end}})^{\circ 4}} 24^2 a_1^2 \otimes a_2 \xmapsto{\Lambda_{\mathrm{end}}} 2 \cdot 24^2 (a_1 \wedge a_2) \otimes a_1 = 24^2 a_{[2\ 1]} \in (\wedge^2 H) \otimes H.$$

Here $(\mu_{\mathrm{end}})^{\circ 4}$ is the 4-time compositions of μ_{end}. Hence $(a_1^2 a_3^4)|_{[2\ 1]}$ generates $[2\ 1]$ as an Sp-module. For the second term, we have $(a_1^2 a_4 \wedge a_2 a_3^3 b_3^3 b_4)|_{[2\ 1]} = 0$ because $\mathfrak{c}_g(1) \otimes \mathfrak{c}_g(6)$ does not contain Sp-irreducible components isomorphic to $[2\ 1]$ by Lemma 3.1.

Therefore, $(\partial\omega_3)|_\lambda$ generates $[2\ 1] \subset \mathfrak{c}_g(4) \otimes \mathfrak{c}_g(3)$. This shows that $[2\ 1] \subset (\mathfrak{c}_g(4) \otimes \mathfrak{c}_g(3)) \cap \mathrm{Im}\,(\partial\colon \wedge^3 \mathfrak{c}_g^+ \to \wedge^2 \mathfrak{c}_g^+)$.

5 Lower weight cases

By Theorem 2.1, we have

$$H_2(\mathfrak{c}_g^+) = \bigoplus_{w\geq 1} H_2(\mathfrak{c}_g^+)_w = \bigoplus_{w=1}^{3} H_2(\mathfrak{c}_g^+)_w$$

In order to determine $H_2(\mathfrak{c}_g^+)$, it is enough to discuss the case $w = 1, 2, 3$.

Lemma 5.1. If $g \geq 4$, then $H_2(\mathfrak{c}_g^+)_1 = 0$, $H_2(\mathfrak{c}_g^+)_2 = [51] + [33] + [22] + [11] + [0]$, and $H_2(\mathfrak{c}_g^+)_3 = [1]$.

Proof. $H_2(\mathfrak{c}_g^+)_1 = 0$ is obvious because no $k \geq l \geq 1$ satisfy $k + l = 1$.

Since the weight 2 part of $\wedge^3\mathfrak{c}_g^+$ is zero and since $\partial_2 = [,]\colon \wedge^2\mathfrak{c}_g(1) \to \mathfrak{c}_g(2) = S^4H = [4]$ is surjective, we have $H_2(\mathfrak{c}_g^+)_2 = \wedge^2\mathfrak{c}_g(1)/\mathfrak{c}_g(2)$. The Sp-irreducible decomposition of $\wedge^2\mathfrak{c}_g(1)$ is $[51] + [33] + [4] + [22] + [11] + [0]$, therefore the statement follows.

The Sp-irreducible decomposition of $\mathfrak{c}_g(2) \otimes \mathfrak{c}_g(1)$ is

$$\mathfrak{c}_g(2) \otimes \mathfrak{c}_g(1) = [7] + [61] + [52] + [43] + [5] + [41] + [32] + [3] + [21] + [1].$$

The space $\wedge^3\mathfrak{c}_g(1)$ does not have [5] and [1] as its Sp-irreducible components. We can use the same method as in the case $w \geq 4$ for all the other Sp-irreducible components and obtain appropriate ω_3s for each of them. Again, since $\partial_2 = [,]\colon \mathfrak{c}_g(2) \wedge \mathfrak{c}_g(1) \to \mathfrak{c}_g(3) = S^5H = [5]$ is surjective, we have $H_2(\mathfrak{c}_g^+)_3 = (\mathfrak{c}_g(2) \otimes \mathfrak{c}_g(1))/\left(\mathfrak{c}_g(3) \oplus \mathrm{Im}\,(\partial_3\colon \wedge^3 \mathfrak{c}_g(1) \to \mathfrak{c}_g(2) \wedge \mathfrak{c}_g(1))\right) = [1]$. ■

Corollary 5.1. If $g \geq 4$, then $H_2(\mathfrak{c}_g^+) = [51]+[33]+[22]+[11]+[1]+[0]$ as an Sp-module.

From the proof of Lemma 5.1, we also have the following.

Corollary 5.2. If $g \geq 4$, then $H_3(\mathfrak{c}_g^+)_3 = [711]+[63]+[531]+[333]+[52]+[421]+[322]+[41]+2[311]+2[3]$ as an Sp-module.

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