On quantum representation of knots via braided Hopf algebra

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1 Introduction

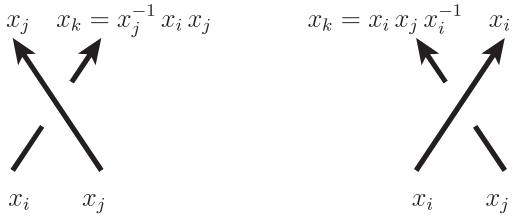
For a knot K and a linear algebraic group G, there is the space of G representations of K, which is the set of all homomorphisms from the fundamental group $\pi_1(S^3 \setminus K)$ to G. This space is reconstructed from the view point of the fundamental quandle and its representation associated with a Hopf algebra. Here we extend this construction to any braided Hopf algebra with braided commutativity. The typical example of a braided Hopf algebra is BSL(2), which is the braided quantum SL(2) introduced by S. Majid [3]. By applying the above construction to BSL(2), we get a quantized SL(2) representation of K. This is based on [4] which is a joint work with Roland van der Veen.

2 Wirtinger presentation for a closed braid

Let K be a knot in S^3 and D be its diagram. Then the fundamental group $\pi_1(S^3 \setminus K)$ of the complement of K has the following presentation.

$$\pi_1(S^3 \setminus K) = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_n \rangle$$

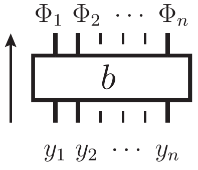
where n is the number of crossings of D, the generators $x_1, \cdots, x_n$ corresponds to the overpasses of D and r_i is the relation coming from the i-th crossing as follows.



Every knot can be expressed as a closed braid. For a knot K, let $b \in B_n$ be a braid whose closure is isotopic to K. Let $y_1, y_2, \cdots, y_n$ be elements of $\pi_1(S^3 \setminus K)$ corresponding to the overpasses at the bottom (and the top) of b. By applying the relations of the

Wirtinger presentation at every crossings from bottom to top, we get $\Phi_1(y_1, \ldots, y_n), \cdots,$ $\Phi_n(y_1, \cdots, y_n)$ at the top of b, and the Wirtinger presentation is equivalent to

$$\pi_1(S^3 \setminus K) = \langle y_1, \cdots, y_n \mid y_1 = \Phi_1(y_1, \cdots, y_n), \cdots, y_n = \Phi_n(y_1, \cdots, y_n) \rangle .$$



3 SL(2) representation space

An SL(2) representation ρ of $\pi_1(S^3 \setminus K)$ is determined by $\rho(y_1), \cdots, \rho(y_n) \in \mathrm{SL}(2)$ satisfying

$$\Phi_1(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_1),$$
$$\cdots,$$
$$\Phi_n(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_n).$$

Let I_b be the ideal in the tensor $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ of the coordinate ring of SL(2) generated by the above relations.

Theorem 1. *The quotient* $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/I_b$ *does not depend on the presentation of* $\pi_1(S^3 \setminus K)$ *and is called the* SL(2) *representation space of* $\pi_1(S^3 \setminus K)$.

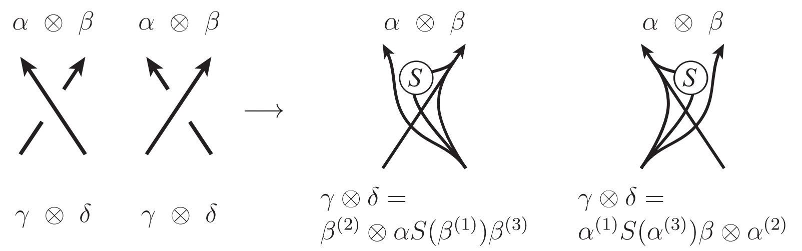
The ccordinate algebra $\mathbb{C}[\mathrm{SL}(2)]$ of SL(2) is generated by a, b, c, d corresponding to the matrix elements of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$. The algebra $\mathbb{C}[\mathrm{SL}(2)]$ has natural Hopf algebra structure coming from the group structure of SL(2).

$$\Delta : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)] \otimes \mathbb{C}[\mathrm{SL}(2)] \text{ with } \Delta(f)(x \otimes y) = f(xy),$$

$$S : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)] \text{ with } S(f)(x) = f(x^{-1}),$$

$$\varepsilon : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C} \text{ with } \varepsilon(f) = f(1).$$

Let $\Phi^* : \mathbb{C}[\mathrm{SL}(2)]^{\otimes n} \to \mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ be the dual map of $\Phi = (\Phi_1, \cdots, \Phi_n)$. At a crossing, Φ^* acts as follows.



Theorem 2. *Let J_b be the ideal generated by the image of $\Phi^* - id$, then J_b is equal to the previous ideal I_b and $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/J_b$ is the* SL(2) *representation space of $\pi_1(S^3 \setminus K)$.*

Remark. This construction can be generalized to any commutative Hopf algebra.

4 Braided Hopf algebra

Definition 1. An algebra A is called a braided Hopf algebra if it is equipped with following linear maps satisfying the relations given in the next picture.

multiplication $\mu : A \otimes A \to A$, **counit** $\varepsilon : A \to k$,

comultiplication $\Delta : A \to A \otimes A$, **antipode** $S : A \to A$,

unit $1 : k \to A$, **braiding** $\Psi : A \otimes A \to A \otimes A$.

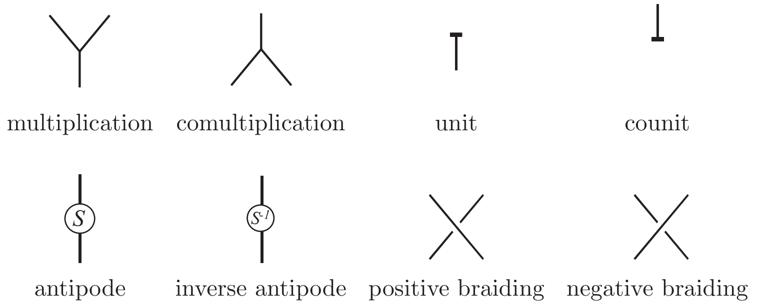
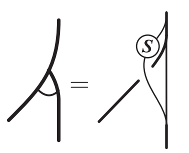


Figure 1: Operations of a braided Hopf algebra

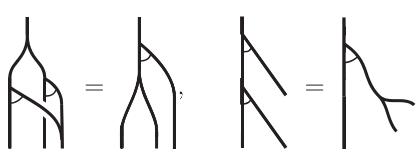
Definition 2. The adjoint coaction ad : $A \to A \otimes A$ is defined by

$$\mathrm{ad}(x) = (id \otimes \mu)(\Psi \otimes id)(S \otimes \Delta)\Delta(x).$$

The adjoint coaction is explained graphically as follows.



The adjoint coaction ad satisfies the following relations.



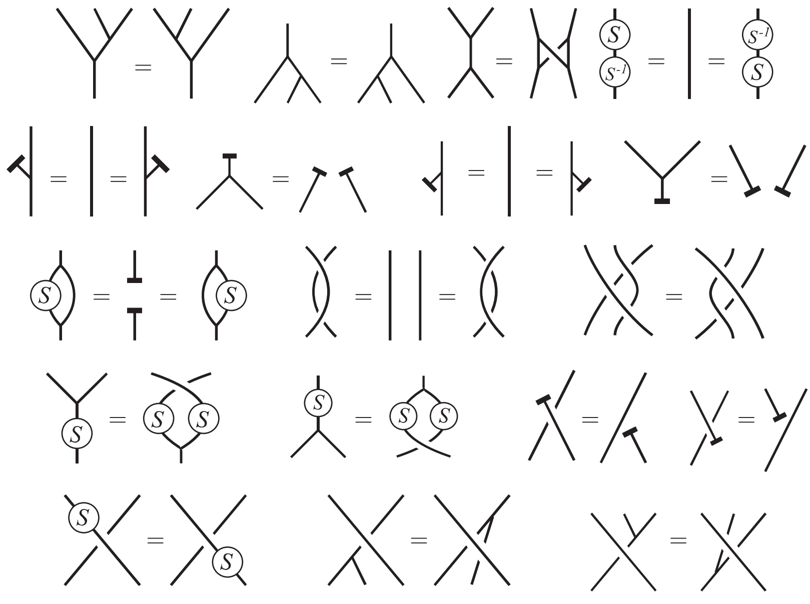


Figure 2: Relations of a braided Hopf algebra

The first relation means

$$(id \otimes id \otimes \mu)(id \otimes \Psi \otimes id)(\mathrm{ad} \otimes \mathrm{ad})\,\Delta = (\Delta \otimes id)\mathrm{ad}.$$

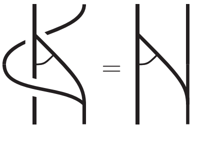
The second relation means

$$(\mathrm{ad} \otimes id)\mathrm{ad} = (id \otimes \Delta)\mathrm{ad}.$$

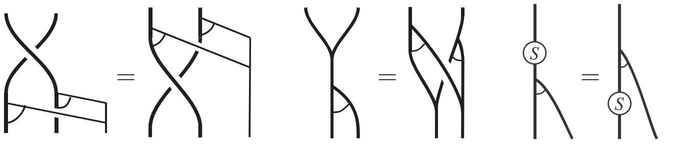
Now we introduce braided commutativity, which is a weakened version of the commutativity.

Definition 3. A braided Hopf algebra A is braided commutative if it satisfies

$$(id \otimes \mu)(\Psi \otimes id)(id \otimes \mathrm{ad})\Psi = (id \otimes \mu)(\mathrm{ad} \otimes id).$$

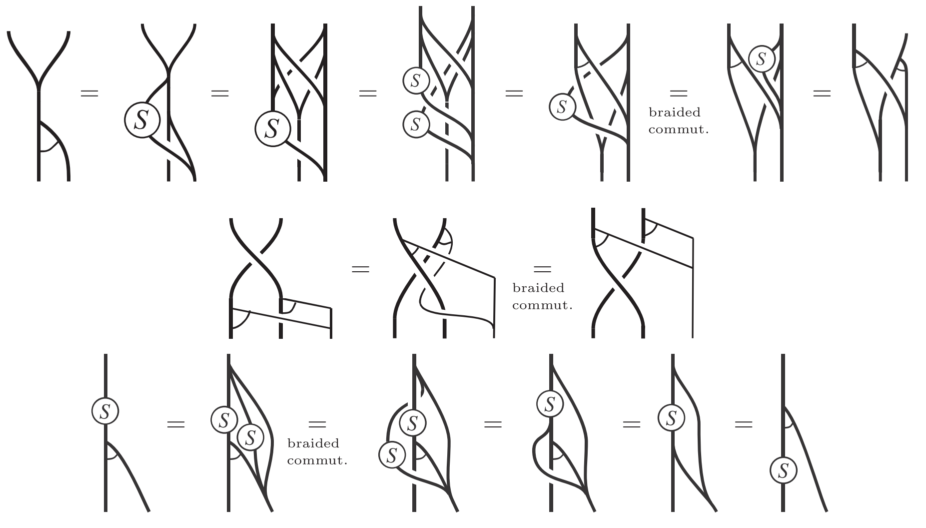


If A is braided commutative, the following relations hold.



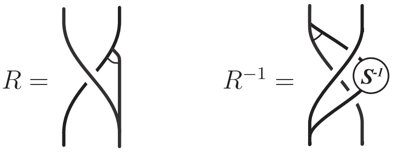
ad $\otimes$ ad and Ψ ad and μ ad and S

As an example, we prove the last relation. These relations are proved graphically as follows.



5 Representation space from a braided Hopf algebra

We first construct a representation of the braid group by using a braided Hopf algebra. Let A be a braided Hopf algebra which may not be braided commutative. Let R and R^{-1} be elements of $\mathrm{End}(A^{\otimes 2})$ given by the following.



For $\sigma_i^{\pm 1} \in B_n$, let $\rho(\sigma_i) = id^{\otimes(i-1)} \otimes R \otimes id^{\otimes(n-i-1)}$ and $\rho(\sigma_i^{-1}) = id^{\otimes(i-1)} \otimes R^{-1} \otimes id^{\otimes(n-i-1)}$.

Theorem 3. *The above ρ defined for generators of B_n extends to a representation of B_n in* $\mathrm{End}(A^{\otimes n})$.

If A is a usual Hopf algebra, such representation of the braid group was constructed in [5]. The proof for this theorem was given in [2] for a usual Hopf algebra, and their proof is easily generalized for a braided Hopf algebra.

From now on, we assume that the braided Hopf algebra A is braided commutative. For $b \in B_n$, let $\rho(b) \in \mathrm{End}(A^{\otimes n})$ be the representation of b defined as above. Let I_b be the left ideal of $A^{\otimes n}$ generated by the image of the map $\rho(b) - id^{\otimes n}$.

Proposition 1. *The left ideal I_b is a two-sided ideal.*

This proposition comes from the following lemma.

Lemma 1. *For $\boldsymbol{x}, \boldsymbol{y} \in A$, we have*

$$\rho(b)\mu(\boldsymbol{x} \otimes \boldsymbol{y}) = \mu(\rho(b)\boldsymbol{x} \otimes \rho(b)\boldsymbol{y}).$$

To prove this lemma, we need the braided commutativity.

Theorem 4. *Let X be a set of generators of A and $\boldsymbol{x}_i = 1^{\otimes(i-1)} \otimes \boldsymbol{x} \otimes 1^{\otimes(n-i)}$ for $\boldsymbol{x} \in X$. Then the ideal I_b in $A^{\otimes n}$ is generated by*

$$\{\rho(b)\boldsymbol{x}_i - \boldsymbol{x}_i \mid \boldsymbol{x} \in X,\ i = 1, \cdots, n-1\}.$$

Proof. Since

$$\begin{aligned} &d(b)\,\mu_n(\boldsymbol{x} \otimes \boldsymbol{y}) - \mu_n(\boldsymbol{x} \otimes \boldsymbol{y}) \\ &= \mu_n\Big(d(b)\,\boldsymbol{x} \otimes \big(d(b)\,\boldsymbol{y} - \boldsymbol{y}\big)\Big) + \mu_n\Big(\big(d(b)\,\boldsymbol{x} - \boldsymbol{x}\big) \otimes \boldsymbol{y}\Big). \end{aligned}$$

and $d(b)\,\boldsymbol{x} - \boldsymbol{x}$, $d(b)\,\boldsymbol{y} - \boldsymbol{y}$ are both contained in I_b. □

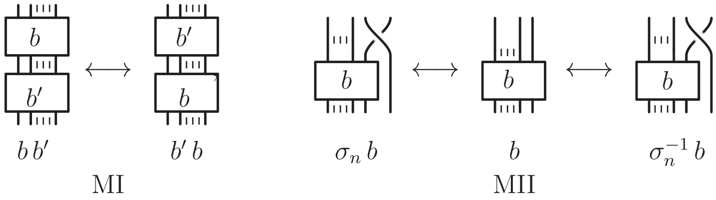
For an n braid b, let $A_b = A^{\otimes n}/I_b$.

Theorem 5 (Main theorem)**.** *If the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic, then A_{b_1} and A_{b_2} are isomorphic algebras. Moreover, A_{b_1} and A_{b_2} are isomorphic A-comodules with adjoint coaction. In other words, A_b is an invariant of the knot (or link) $\widehat{b}$, which is the closure of b.*

Definition 4. The quotient algebra $A_b = A^{\otimes n}/I_b$ is called the A representation space of the closure $\widehat{b}$.

To prove the above theorem, we show that the quotient algebra A_b is invariant under the Markov moves.

Definition 5. These moves are called the Markov moves and such b_1 and b_2 are called Markov equivalent.



Theorem 6. *The closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic in S^3 if and only if there is a sequence of the following two types of moves connecting b_1 to b_2.*

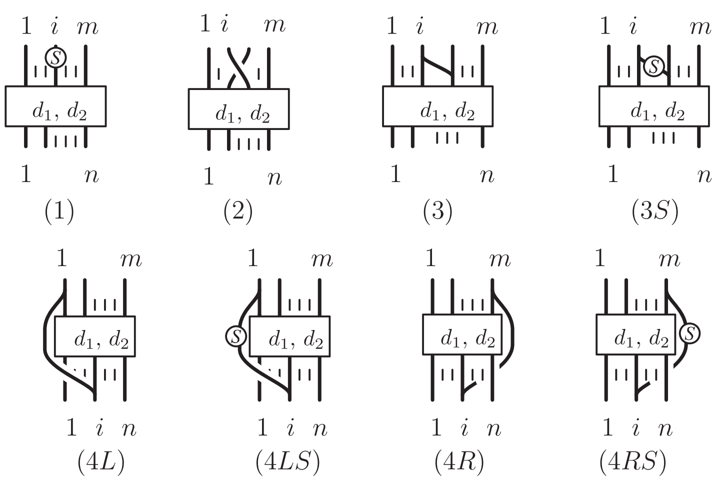
The main theorem is proved by showing the invariancde under MI and MII. To prove the invariance under MI is not difficult, but the invariance under MII is not so easy. To show this, we need to introduce some moves of diagrams which are duals of Tietze transformations which are moves to change the presentation of a combinatorial group defined by generators and relations.

Definition 6. For $b \in B_n$ we present $I_{\rho(b)}$ by $\rho(b) \sim \rho(1)$. Similarly, for two diagrams d_1, d_2 representing elements of $\mathrm{Hom}(A^{\otimes m}, A^{\otimes n})$, $d_1 \sim d_2$ present a two-sided ideal I_{d_1,d_2} in $A^{\times n}$ generated by

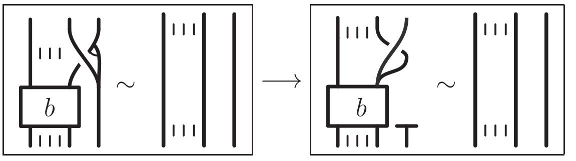
$$d_1(x_1 \otimes \cdots \otimes x_m) - d_2(x_1 \otimes \cdots \otimes x_m)$$

for $x_1, \cdots, x_m \in A$. Such d_1 and d_2 are called *the equivalent pair* of diagrams corresponding to the two-sided ideal I_{d_1,d_2} and the quotient algebra $A^{\otimes n}/I_{d_1,d_2}$.

Proposition 2 (dual of Tietze tranformation)**.** *Let $d_1 \sim d_2$ be an equivalent pair and let $d_1' \sim d_2'$ be the equivalent pair where d_1' and d_2' are obtained from d_1 and d_2 respectively by one of the following operations (1), (2), (3), (3S), (4L), (4LS), (4R), (4RS) illustrated in the following. Then the corresponding ideals I_{d_1,d_2} and $I_{d_1',d_2'}$ are equal.*



The invariance under MII is proved by transforming the equivalent pair $b\sigma_n \sim e$ to the diagram in the next figure by using Proposition 2.



Let p_n be the surjection from $A^{\otimes(n+1)}$ to $A^{\otimes n}$ defined by

$$p_n(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = x_1 \otimes \cdots \otimes x_{n-1} \otimes \mu\left(\Psi^{-1}(x_n, x_{n+1})\right).$$

Then the above picture means that $I_{d(b\sigma_n)}$ is generated by $(d(b) \circ p_n)(\boldsymbol{x}) \otimes 1 - \boldsymbol{x}$ for $\boldsymbol{x} \in A^{\otimes(n+1)}$. For $\boldsymbol{y} \in I_{d(b)}$, $\boldsymbol{y} \otimes 1 \in I_{d(b\sigma_n)}$ and $p_n(\boldsymbol{y} \otimes 1) = \boldsymbol{y}$, so $p_n(I_{d(b\sigma_n)}) = I_{d(b)}$. For $\boldsymbol{x} \in \operatorname{Ker} p_n$,

$$(d(b) \circ p_n)(\boldsymbol{x}) \otimes 1 - \boldsymbol{x} = -\boldsymbol{x},$$

and so $\boldsymbol{x} \in I_{d(b\sigma_n)}$. This means that $\operatorname{Ker} p_n \subset I_{d(b\sigma_n)}$, which implies $p_n^{-1}(I_{d(b)}) = I_{d(b\sigma_n)}$ since $p_n(I_{d(b\sigma_n)}) = I_{d(b)}$. Therefore p_n gives an isomorphism $A^{\otimes(n+1)}/I_{d(b\sigma_n)} \cong A^{\otimes n}/I_{d(b)}$.

Example. The figure eight knot 4_1 is isomorphic to the closure of the braid $b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$. The graphical expression of $d(b)$ is given in Figure 3. So the space of A representation of 4_1 is $A \otimes A \otimes A / I_{d(b)}$ where $I_{d(b)}$ is generated by $d(b)(x \otimes y \otimes z) - x \otimes y \otimes z$ for $x, y, z \in A$. We will see the relation for $x_1 = x \otimes 1 \otimes 1$ and $x_2 = 1 \otimes x \otimes 1$. Let $d(b_1)_i$

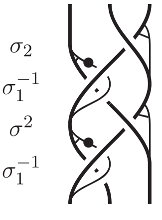


Figure 3: A graphical expression of $d(b)$

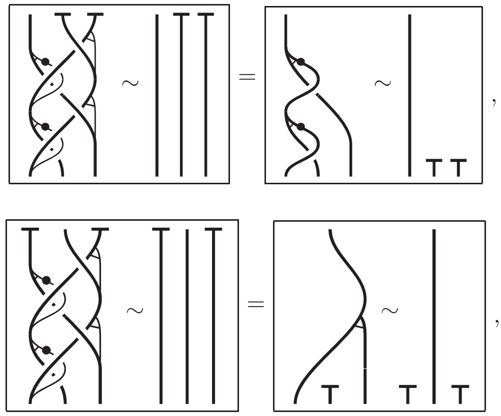
is a mapping from A to $A^{\otimes 3}$ sending $x \in A$ to $d(b)(x_i) \in A^{\otimes 3}$ for $i = 1, 2$. Then the ideal $I_{d(b)}$ is generated by $\{d(b)_1(x) - x \otimes 1 \otimes 1,\ d(b)_2(x) - 1 \otimes x \otimes 1 \mid x \in A\}$ where

$$d(b)_1(x) = (\Psi^{-1} \otimes id)(id \otimes S^{-1} \otimes id)(\mathrm{ad} \otimes id)\Psi^{-1}(id \otimes S^{-1})\,\mathrm{ad}(x),$$
$$d(b)_2(x) = (id \otimes \Psi)(\mathrm{ad}(x) \otimes 1).$$

These elements are explained graphically in Figure 4. Let p be a mapping from $A^{\otimes 3}$ to $A^{\otimes 2}$ defined by $p(x \otimes y \otimes z) = (\mu \otimes \mu)(x \otimes ad(y) \otimes z)$. Then p is surjective, $\ker p$ is

$d(b)_1(x) \sim x \otimes 1 \otimes 1$

$d(b)_2(x) \sim 1 \otimes x \otimes 1$



where $\bullet = (S^{-1})$.

Figure 4: Relations for the figure eight knot.

generated by $d(b)_2(x) - 1 \otimes x \otimes 1$ for $x \in A$, and $A^{\otimes 3}/\ker p \cong A^{\otimes 2}$. Let $I' = p(I_{d(b)})$, then I' is generated by

$$p\,(d(b)_1(x) - x \otimes 1 \otimes 1) = (\mu \otimes \mu)\, ad_2(\Psi^{-1} \otimes id)\, S_2^{-1}\, \mathrm{ad}_1\, \Psi^{-1}\, S_2^{-1}\, \mathrm{ad}(x) - x \otimes 1$$

where ad_i, S_i^{-1} act to the i-th component of the tensor product. The ideal I' is graphically explained in Figure 5. Moreover, the mapping p gives the isomorphism between $A \otimes A \otimes A/I_{d(b)}$ and $A \otimes A/I'$, so $A \otimes A/I'$ is isomorphic to the A representation space of 4_1. It corresponds to a presentation of $\pi_1(S^3 \setminus 4_1)$ with two generators and one relator.

$p\,(d(b)_1(x)) \sim x \otimes 1$

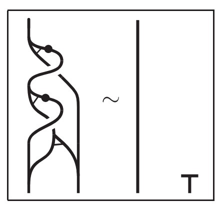


Figure 5: Equivalent diagram for I'.

6 Braided SL(2)

Definition 7. A braided SL(2) is a one-parameter deformation of $\mathbb{C}[\mathrm{SL}(2)]$ defined in [3] by the following. It is denoted by BSL(2).

$$b\,a = t\,a\,b, \quad c\,a = t^{-1}\,a\,c, \quad d\,a = a\,d, \quad d\,b = b\,d + (1 - t^{-1})\,a\,b,$$
$$c\,d = d\,c + (1 - t^{-1})\,c\,a, \quad b\,c = c\,b + (1 - t^{-1})\,a\,(d - a), \quad a\,d - t\,c\,b = 1,$$

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(c) = c \otimes a + d \otimes c,$$
$$\Delta(d) = c \otimes b + d \otimes d, \quad S(a) = (1-t)\,a + t\,d \quad S(b) = -t\,b, \quad S(c) = -t\,c,$$
$$S(d) = a, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1,$$
$$\Psi(x \otimes 1) = 1 \otimes x, \Psi(1 \otimes x) = x \otimes 1, \Psi(a \otimes a) = a \otimes a + (1-t)\,b \otimes c, \Psi(a \otimes b) = b \otimes a,$$
$$\Psi(a \otimes c) = c \otimes a + (1-t)(d-a) \otimes c,\ \Psi(a \otimes d) = d \otimes a + (1-t^{-1})\,b \otimes c,$$
$$\Psi(b \otimes a) = a \otimes b + (1-t)\,b \otimes (d-a),\ \Psi(b \otimes b) = t\,b \otimes b,\ \Psi(d \otimes b) = b \otimes d,$$
$$\Psi(b \otimes c) = t^{-1}\,c \otimes b + (1+t)(1-t^{-1})^2\,b \otimes c - (1-t^{-1})(d-a) \otimes (d-a),$$
$$\Psi(b \otimes d) = d \otimes b + (1-t^{-1})\,b \otimes (d-a),\ \Psi(c \otimes a) = a \otimes c,\ \Psi(c \otimes b) = t^{-1}\,b \otimes c,$$
$$\Psi(c \otimes c) = t\,c \otimes c,\ \Psi(c \otimes d) = d \otimes c,\ \Psi(d \otimes a) = a \otimes d + (1-t^{-1})\,b \otimes c,$$
$$\Psi(d \otimes c) = c \otimes d + (1-t^{-1})(d-a) \otimes c,\ \Psi(d \otimes d) = d \otimes d - t^{-1}\,(1-t^{-1})\,b \otimes c.$$

Theorem 7. *The braided Hopf algebra* BSL(2) *is braided commutative.*

Since BSL(2) is an example of a braided commutative braided Hopf algebra, we have BSL(2) representations of K, which is $\mathrm{BSL}(2)^{\otimes n}/I_{d(b)}$. We also call it the space of quantized $\mathrm{SL}(2,\mathbb{C})$ representations of K. It follows directly from the relations and Proposition I.8.17 of [1] that BSL(2) is Noetherian, so the ideal $I_{d(b)}$ is finitely generated.

References

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