Problems on Low-dimensional Topology, 2020

Edited by T. Ohtsuki[1]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the online conference "Intelligence of Low-dimensional Topology" whose live streaming is distributed from Research Institute for Mathematical Sciences, Kyoto University in May 13–15, 2020.

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[1]Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, JAPAN
Email: tomotada@kurims.kyoto-u.ac.jp
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1 Cyclotomic expansions of twist knots

(Wataru Yuasa)

The colored Jones polynomial $J_n(K) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ is a quantum invariant of knot K obtained from the $(n+1)$-dimensional irreducible representation of the quantum group of $\mathfrak{sl}_2$. $J_n(K)$ is normalized such that $J_n(U) = \{n+1\}/\{1\}$ for 0-framed unknot U and $\{m\} = q^{\frac{m}{2}} - q^{-\frac{m}{2}}$.

It is known, as Habiro's *cyclotomic expansion* for the colored Jones polynomials [12], that for any 0-framed knot K, $J_n(K)$ can be presented in the form

$$\frac{J_n(K)}{J_n(U)} = \sum_{k=0}^{n-1} h_{K,k}(q) C(n,k),$$

for some $h_{K,k}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, where

$$C(n,k) = \frac{\{n-k\}\{n-k+1\}\cdots\{n+k-1\}\{n+k\}}{\{n+1\}}$$

for $0 < k < n$ and $C(n,0) = 1$.

Chen, Liu and Zhu [4] gave conjectural formula of the cyclotomic expansion for the quantum $\mathfrak{sl}_N$ invariant with one-row colorings. Let $J_n^{\mathfrak{sl}_N}(K)$ be the quantum $\mathfrak{sl}_N$ invariant obtained from the irreducible representation corresponding to the one-row Young diagram of n boxes.

Conjecture 1.1 (cyclotomic expansion for the $J_n^{\mathfrak{sl}_N}(K)$ invariant [4])**.** *For any 0-framed knot K,*

$$\frac{J_n^{\mathfrak{sl}_N}(K)}{J_n^{\mathfrak{sl}_N}(U)} = \sum_{k=0}^{n-1} h_{K,k}^{(N)}(q) C^{(N)}(n,k),$$

for some $h_{K,k}^{(N)}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, where

$$C^{(N)}(n,k) = \frac{\{n-k+1\}\{n-k+2\}\cdots\{n+k+N-1\}}{\{n+1\}\{n+2\}\cdots\{n+N-1\}}$$

for $0 < k < n$ and $C^{(N)}(n,0) = 1$.

Let K_p be the twist knot with p full twist. Masbaum [25] gave the cyclotomic expansion $\{h_{K_p,k}(q)\}_k$ for K_p through the linear skein theory for Kauffman bracket. More specifically, he used the m full twist formula and a special expansion of the twist element ω for the Kauffman bracket skein module.

Problem 1.2 (W. Yuasa)**.** *Calculate $\{h_{K,k}(q)\}_k$ explicitly for other knots K, for example, knots with small crossing number, etc.*

In [35, 36], we gave m full twist formula for the A_2 web with one-row coloring.

Problem 1.3 (W. Yuasa)**.** *Give the cyclotomic expansion $\{h_{K_p,k}^{(3)}(q)\}_k$ by using the full twist formula for one-row colored A_2 webs and Masbaum's method.*

In [4], the volume conjecture is also formulated. Other than these conjectures, we can consider the slope conjecture for the quantum $\mathfrak{sl}_N$ invariant. In general, a computation of the quantum $\mathfrak{sl}_N$ invariant for knot K is so hard (except for very special knots and colorings). However, we can calculate $\{J_n^{\mathfrak{sl}_3}(K)\}_n$ explicitly for some knots K by using formulas for A_2 webs in [35, 36].

Problem 1.4 (W. Yuasa). *Calculate $\{J_n^{\mathfrak{sl}_3}(K)\}_n$ for some knots and formulate the slope conjecture for the quantum $\mathfrak{sl}_N$ invariant with one-row colorings.*

2 Johnson-type homomorphisms of mapping class groups, and the LMO invariant

(Anderson Vera)

Let Σ be a compact connected oriented surface of genus g with exactly one boundary component. Denote by $\mathcal{M}$ the mapping class group of Σ, that is the group of isotopy classes of orientation-preserving homeomorphisms $h : \Sigma \to \Sigma$ which are the identity on the boundary $\partial\Sigma$ of Σ.

From the 3-dimensional point of view it is natural to consider Σ as being part of the boundary of a handlebody V of genus g, that is, we consider an embedding $\iota : \Sigma \to V$ as shown in Figure 1.

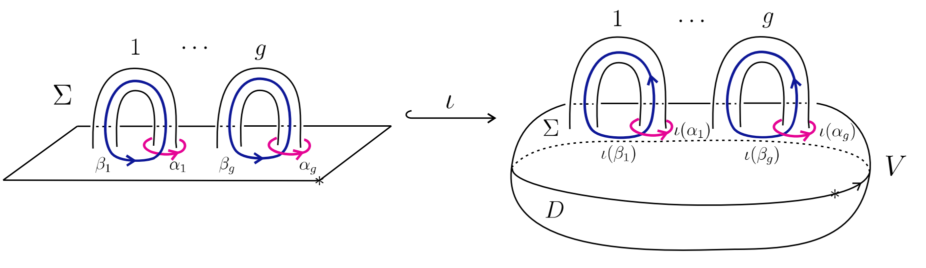

Figure 1: Embedding $\iota : \Sigma \to V$. Here $\partial V = \Sigma \cup D$, where $D \subset \partial V$ is an embedded disk.

Let $* \in \partial\Sigma$ and consider the following notations:

$$
\begin{array}{lll}
\pi = \pi_1(\Sigma, *), & \pi' = \pi_1(V, *), & \mathbb{A} = \ker(\pi \xrightarrow{\iota_\#} \pi'), \\
 & & A = \ker(H \xrightarrow{\iota_*} H'), \\
H = H_1(\Sigma; \mathbb{Z}), & H' = H_1(V; \mathbb{Z}) \ \text{ and } & K_2 = \ker(\pi \xrightarrow{\iota_\#} \pi' \xrightarrow{\mathrm{ab}'} H') = \mathbb{A} \cdot [\pi, \pi].
\end{array}
$$

The group $\mathcal{M}$ acts naturally on H and π. The *Torelli group*, denoted by $\mathcal{I}$, consists of the elements of $\mathcal{M}$ acting trivially on H, that is, $\mathcal{I} = \{h \in \mathcal{M} \mid h_* = \mathrm{Id}_H\}$. We also consider the following subgroups of $\mathcal{M}$. The *handlebody group*

$$\mathcal{H} = \{h \in \mathcal{M} \mid h_\#(\mathbb{A}) \subseteq \mathbb{A}\},$$

the *Lagrangian mapping class group*

$$\mathcal{L} = \{h \in \mathcal{M} \mid h_*(A) \subseteq A\},$$

the *Lagrangian Torelli group*

$$\mathcal{I}^L = \{h \in \mathcal{L} \mid h_{*|A} = \mathrm{Id}_A\},$$

and finally, the *alternative Torelli group*

$$\mathcal{I}^{\mathfrak{a}} = \left\{ h \in \mathcal{L} \;\middle|\; \begin{array}{c} \text{for } x \in \pi : \quad h_\#(x)x^{-1} \in K_2 \\ \text{and for } y \in K_2 : \\ h_\#(y)y^{-1} \in [\pi, [\pi, \pi]] \cdot [\mathbb{A}, \pi] =: K_3 \end{array} \right\}.$$

The group $\mathcal{I}^{\mathfrak{a}}$ can be defined as the subgroup of $\mathcal{M}$, generated by Dehn twists t_γ about curves γ on Σ which are homologically trivial in V. The above subgroups play an important role in the study of homology 3-spheres and the theory of finite-type invariants.

It can be shown (for genus enough large) that the groups $\mathcal{I}^L$ and $\mathcal{I}^{\mathfrak{a}}$ are finitely generated (see [32, Remark 4.15]).

Problem 2.1 (A. Vera)**.** *Find an explicit set of generators for $\mathcal{I}^L$ and $\mathcal{I}^{\mathfrak{a}}$ as it is known for $\mathcal{I}$.*

From Johnson [17] and Morita [27] works we can consider a stepwise approximation of the action of $\mathcal{M}$ on π by considering the action of $\mathcal{M}$ on the nilpotent quotients $\pi/\Gamma_{m+1}\pi$ for $m \geq 1$, where $\{\Gamma_m\pi\}_{m\geq 1}$ is the *lower central series of* π (i.e. $\Gamma_1\pi = \pi$ and $\Gamma_{m+1}\pi = [\pi, \Gamma_m\pi]$ for $m \geq 1$). This gives rise to the so-called *Johnson filtration*

$$\mathcal{I} = J_1\mathcal{M} \supseteq J_2\mathcal{M} \supseteq J_3\mathcal{M} \cdots$$

where $J_m\mathcal{M}$ consists of the elements in $\mathcal{M}$ acting trivially on $\pi/\Gamma_{m+1}\pi$. Each one of the terms $J_m\mathcal{M}$ of the Johnson filtration comes equipped with a group homomorphism $\tau_m : J_m\mathcal{M} \to D_m(H)$ such that $\ker(\tau_m) = J_{m+1}\mathcal{M}$, and taking values in a particular abelian group $D_m(H)$ which can be described in terms of H.

Similar filtrations and homomorphisms were introduced for the groups $\mathcal{I}^L$ and $\mathcal{I}^{\mathfrak{a}}$. We refer to them as *Johnson-type filtrations* and *Johnson-type homomorphisms.* Let us review and state some problems about them.

Levine [23, 24] introduced a filtration for $\mathcal{I}^L$, which we call *Johnson-Levine filtration*,

$$\mathcal{I}^L = J_1^L\mathcal{M} \supseteq J_2^L\mathcal{M} \supseteq J_3^L\mathcal{M} \cdots$$

by using the lower central series $\{\Gamma_m\pi'\}_{m\geq 1}$ of π', more precisely we have

$$J_m^L\mathcal{M} = \{h \in \mathcal{I}^L \mid \iota_\# h_\#(\mathbb{A}) \subseteq \Gamma_{m+1}\pi'\}$$

for $m \geq 1$. Levine also introduced a family of homomorphisms $\tau_m^L : J_m^L\mathcal{M} \to D_m(H')$ such that $\ker(\tau_m^L) = J_{m+1}^L\mathcal{M}$.

Conjecture 2.2 (Levine [24]). *For every $m \geq 1$, we have $J_m^L\mathcal{M} = J_m\mathcal{M} \cdot (\mathcal{H} \cap \mathcal{I}^L)$.*

Levine showed this conjecture for $m \in \{1, 2\}$.

Problem 2.3 (A. Vera). *Prove or disprove that $[J_l^L\mathcal{M}, J_m^L\mathcal{M}] \subseteq J_{l+m}^L\mathcal{M}$ for all $l, m \geq 1$.*

In the case that this property holds, it would be interesting to study the so-called *Andreadakis problem* for $\mathcal{I}^L$, that is, the comparison between the lower central of $\mathcal{I}^L$ and the Johnson-Levine filtration.

Besides, Habiro and Massuyeau [13] introduced a filtration for $\mathcal{I}^\mathfrak{a}$, which we call *alternative Johnson filtration*,

$$\mathcal{I}^\mathfrak{a} = J_1^\mathfrak{a}\mathcal{M} \supseteq J_2^\mathfrak{a}\mathcal{M} \supseteq J_3^\mathfrak{a}\mathcal{M} \cdots$$

by using a different decreasing sequence $\{K_m\}_{m\geq 1}$ of subgroups of π. This sequence is defined by

$$K_1 = \pi, \qquad K_2 = [\pi, \pi] \cdot \mathbb{A} \quad \text{and} \quad K_m = [K_1, K_{m-1}] \cdot [K_2, K_{m-1}] \quad \text{for } m \geq 2.$$

The m-the term of the alternative Johnson filtration is given by

$$J_m^\mathfrak{a}\mathcal{M} = \left\{ h \in \mathcal{L} \;\middle|\; \begin{array}{c} \text{for } x \in \pi : \quad h_\#(x)x^{-1} \in K_{1+m} \\ \text{and for } y \in K_2 : \\ h_\#(y)y^{-1} \in K_{2+m} \end{array} \right\}$$

for $m \geq 1$. Habiro and Massuyeau also introduced the respective family of Johnson-type homomorphisms $\tau_m^\mathfrak{a} : J_m^\mathfrak{a}\mathcal{M} \to D_m(H', A)$ such that $\ker(\tau_m^\mathfrak{a}) = J_{m+1}^\mathfrak{a}\mathcal{M}$, to which we refer as *alternative Johnson homomorphisms.* In this case the abelian group $D_m(H', A)$ can be described by using H' and A.

The alternative Johnson filtration satisfies $[J_l^\mathfrak{a}\mathcal{M}, J_m^\mathfrak{a}\mathcal{M}] \subseteq J_{l+m}^\mathfrak{a}\mathcal{M}$ for all $l, m \geq 1$.

Problem 2.4 (A. Vera). *Compare between the lower central series of $\mathcal{I}^\mathfrak{a}$ and the alternative Johnson filtration.*

It is known that the first Johnson homomorphism τ_1 appears in the computation of $H_1(\mathcal{I}; \mathbb{Z})$, in particular it gives all the non-torsion part [18]. This is also the case for τ_1^L and $H_1(\mathcal{I}^L; \mathbb{Z})$, see [30].

Problem 2.5 (A. Vera). *Determine $H_1(\mathcal{I}^\mathfrak{a}; \mathbb{Z})$, or more particularly, determine whether $\tau_1^\mathfrak{a}$ gives the non-torsion part of $H_1(\mathcal{I}^\mathfrak{a}; \mathbb{Z})$.*

Morita refined the Johnson homomorphisms by defining the so-called *Morita homomorphisms* [28]. Since the Johnson filtration and the alternative Johnson filtration have similar properties, it seems plausible the existence of a refinement of the alternative Johnson homomorphisms. The following problem consists in the definition of such "alternative" version of the Morita homomorphisms for the alternative Johnson filtration.

Problem 2.6 (A. Vera)**.** *If such homomorphisms exist, study their relation with the functorial extension of the Le-Murakami-Ohtsuki invariant following the lines of the previous works [5, 26, 31, 32].*

3 Equivalence of some ribbon 2-knots with isomorphic knot groups

(Taizo Kanenobu)[2]

This problem has already given in [22, Question 7.1]. Let Y43 and Y46 be ribbon 2-knots with handlebody presentations as in Figure 2, which are given in Yasuda [34].

Problem 3.1 (T. Kanenobu, T. Sumi [22, Question 7.1])**.** *Decide whether Y43 and Y46 are equivalent or not.*

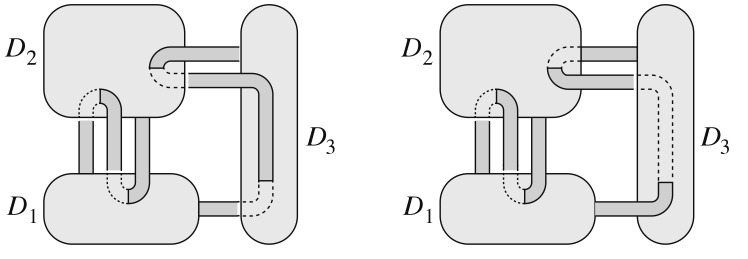

Figure 2: Handlebody presentations of Y43 and Y46.

They have isomorphic knot groups. From Figure 2 we have

$$G(\mathrm{Y43}) = \langle\, x, y, z \mid x(yx) = (yx)y,\ x(zy) = (zy)z \,\rangle,$$
$$G(\mathrm{Y46}) = \langle\, x, y, z \mid x(yx) = (yx)y,\ x(z^{-1}y^{-1}) = (z^{-1}y^{-1})z \,\rangle.$$

We can deform them into ribbon 2-knots of 1-fusion:

$$\mathrm{Y43} \approx R(-1,-1,1,1,1,1,-1,-1),$$
$$\mathrm{Y46} \approx R(1,1,1,-1,-1,1,1,1),$$

which shows that they are positive-amphicheiral. So, from the mirror image of the handlebody presentation of Y46 we obtain:

$$G(\mathrm{Y46}) = G(\mathrm{Y46!}) = \langle\, x, y, z \mid x(y^{-1}x^{-1}) = (y^{-1}x^{-1})y,\ x(zy) = (zy)z \,\rangle.$$

The first relation is the same as $xyx = yxy$, and so $G(\mathrm{Y43}) = G(\mathrm{Y46})$.

[2]Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
`kanenobu@sci.osaka-cu.ac.jp`

4 A geography problem for triangulated tori

(Tamás Kálmán)

In my talk I discuss "trinities" (vertex-three-colorable triangulations) and "Tutte matchings" in two contexts, namely the sphere and the torus. Please refer to [15] or to the talk for the precise definitions.

On the sphere, Tutte matchings generalize Kauffman states. With C. Hine, we proved that Kauffman's Clock Theorem also has a natural generalization. One wonders if other related phenomena, such as the Duality Conjecture (later proved by Gilmer and Litherland) and, of course, state polynomials, have analogues in the context of trinities.

Since I do not have actual conjectures regarding the above, let me ask a concrete question that has to do with toric trinities. For now this is just a combinatorial problem with a topological flavor. On the torus, Tutte matchings are not always related by "triangle moves." In other words, the "state transition graph" is usually disconnected. Its connected components come in two types, acyclic and cyclic. The following question is open and my answer to it is just a guess.

Conjecture 4.1 (T. Kálmán)**.** *Every toric trinity has both cyclic and acyclic components in its state transition graph.*

For example, the simplest toric trinity, shown in Figure 3, has six states. Three of them form a cyclic component of the state transition graph, and the other three are isolated points, which count as acyclic components.

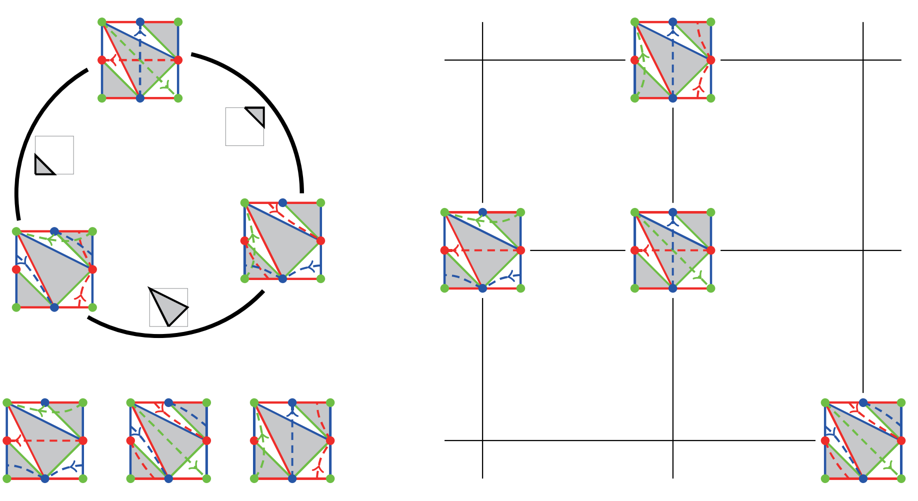

Figure 3

In fact, I suspect that a lot more can be said if we use the group $H_1(T^2) \cong \mathbf{Z}^2$ to establish more structure. For that, let us place a vertex w in each white triangle and represent our Tutte matchings σ by directing a segment from all w to the vertex of the triangle that it is matched to. (I.e., we only keep the last one-third of each arrow that we previously used to represent matches.) Let us denote this 1-chain with C_σ.

Then, for any two Tutte matchings σ and τ, we have a 1-cycle $C_\sigma - C_\tau$, whose homology class is denoted with $\sigma - \tau \in H_1(T^2)$. If two Tutte matchings differ by a single triangle move, then their difference in $H_1(T^2)$ is 0. In particular, there exist well-defined differences, as elements of $H_1(T^2)$, between connected components of the state transition graph.

For instance, if we plot the four connected components of our example above, we see (on the right side of Figure 3) that the isolated points form a triangle and the cyclic component corresponds to the unique interior lattice point of the triangle. From this and other examples, the following question emerges.

Question 4.2 (T. Kálmán). *Is it true that for each toric trinity, there exists a convex lattice polygon whose boundary lattice points correspond to acyclic components of the state transition graph, while the interior lattice points correspond to cyclic components?*

If so, what does the area of the polygon represent? Do all lattice polygons arise from toric trinites, or only some special ones? What can be said about the number and size of the connected components that correspond to each lattice point — is that related to the geometry of the polygon?

For a slightly larger example, the toric trinity shown in Figure 4 has 860 Tutte matchings. It is not hard to find acyclic components (in fact, isolated points) at the lattice points indicated in orange. I can also construct cyclic components at the two purple lattice points shown. I have not yet determined whether components (and of which type) exist at the remaining two interior lattice points, or at any other lattice points of the plane.

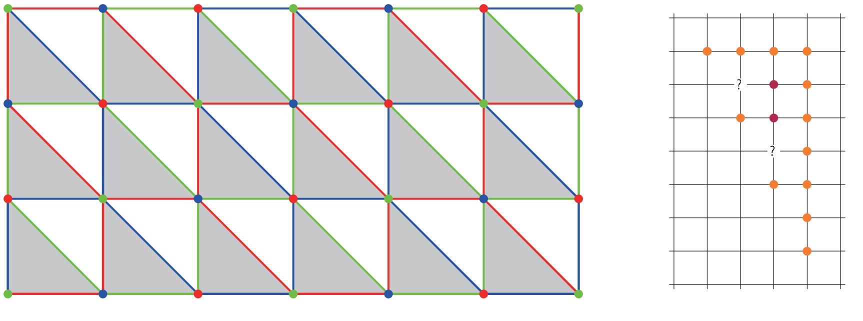

Figure 4

5 On quantum representation of knots via braided Hopf algebra

(Jun Murakami)

This joint work with Roland van der Veen [29] is motivated by the work of Inoue-Kabaya [16] which gives a way to get the complex volume of the knot complement from the conjugate quandle corresponding to the $\mathrm{PSL}(2,\mathbb{C})$ representation of the knot group. It is shown in [6] and [7] that the above Inoue-Kabaya theory relates to the volume potential function of the Kashaev invariant and the colored Jones invariant. The volume potential function is a kind of a limit of such quantum invariants, which is obtained by replacing quantum factorial with the dilogarithm function, and contains information about the A-polynomial.

We just constructed q-analogue of the quandle construction of $\mathrm{PSL}(2,\mathbb{C})$ representation, and there are many problems to construct q-analogues of things relating $\mathrm{PSL}(2,\mathbb{C})$ representations. Among them, I would like to propose the following problems.

Problem 5.1 (J. Murakami)**.** *Find some relation to the representations given by Hikami-Inoue [14] which uses the quantum dilogarithm function, and then construct a q-analogue of the volume potential function.*

Problem 5.2 (J. Murakami)**.** *Here we constructed a q-analogue of the representation space. Refine it to a q-analogue of the character variety.*

Problem 5.3 (J. Murakami)**.** *By introducing an element in the representation space corresponding to the longitude, construct a q-analogue of the A-polynomial and investigate its relation to the AJ-conjecture [8].*

If we can construct the above three objects, they must relate naturally.

6 On the Strong Slope Conjecture for knots

(Kimihiko Motegi)[3]

Jones slopes and Jones surfaces

Let K be a knot in the 3–sphere S^3. The *colored Jones function* of K is a sequence of Laurent polynomials $J_{K,n}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ for $n \in \mathbb{N}$, where $J_{\bigcirc,n}(q) = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$ for the unknot $\bigcirc$ and $\frac{J_{K,2}(q)}{J_{\bigcirc,2}(q)}$ is the ordinary Jones polynomial of K. Since the colored Jones function is q–holonomic [11, Theorem 1], the degrees of its terms are given by *quadratic quasi-polynomials* for suitably large n [10, Theorem 1.1 & Remark 1.1]. For the maximum degree $d_+[J_{K,n}(q)]$, we set its quadratic quasi-polynomial to be

$$\delta_K(n) = a(n)n^2 + b(n)n + c(n)$$

[3]Department of Mathematics, Nihon University, 3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156–8550, Japan
motegi.kimihiko@nihon-u.ac.jp

for rational valued periodic functions $a(n), b(n), c(n)$ with integral period, *i.e.*, $d_+[J_{K,n}(q)] = \delta_K(n)$ when $n \geq n_0$ for some large integer n_0. We call $4a(n)$ a *Jones slope*. A number $p/q \in \mathbb{Q} \cup \{\infty\}$ is a *boundary slope* of a knot K if there exists an essential surface in the knot exterior $E(K) = S^3 - \mathrm{int}N(K)$ with a boundary component representing $p[\mu] + q[\lambda] \in H_1(\partial E(K))$ with respect to the standard meridian μ and longitude λ. Garoufalidis conjectures

Conjecture 6.1 (**Slope Conjecture** [9])**.** *For any knot K in S^3, every Jones slope is a boundary slope.*

Garoufalidis' Slope Conjecture concerns only the quadratic terms of $\delta_K(n)$. Kalfagianni and Tran propose the Strong Slope Conjecture which subsumes the Slope Conjecture and asserts that the topology of the surfaces whose boundary slopes are Jones slopes may be predicted by the linear terms of $\delta_K(n)$.

Conjecture 6.2 (**The Strong Slope Conjecture** [21, 19])**.** *Let K be a knot in S^3. For any Jones slope p/q there exists an essential surface $F_n \subset E(K)$ such that F_n has boundary slope $p/q = 4a(n)$ and $\dfrac{\chi(F_n)}{|\partial F_n| q} = 2b(n)$ for some $n \in \mathbb{N}$.*

We call F_n a *Jones surface*.

Question 6.3 (**Selection principle**)**.**
(1) (Garoufalidis [9]) *Which boundary slope can be a Jones slope?*
(2) (Kalfagianni-Lee [20]) *Which essential surface can be a Jones surface?*

Previously known example has a single Jones slope. So it may be plausible to ask

Question 6.4 (**[20, Question 3.7]**)**.** *Is $a(n)$ constant?*

The max-degree of colored Jones polynomials

In general $d_+[J_{K,n}(q)]$ forms a quadratic quasi-polynomial $\delta_K(n)$ when $n \geq n_0$ for some large integer n_0. We call $\{n \mid n \geq n_0\}$ the *stable range* of $d_+[J_{K,n}(q)]$, and $\{n \mid 1 \leq n < n_0\}$ the *unstable range* of $d_+[J_{K,n}(q)]$. An existence of an unstable range bothers us. In [2, Section 3.3] we construct concrete examples of cabled knots K for which $d_+[J_{K,n}(q)]$ has an unstable range. Moreover this unstable range can be arbitrarily large.

Since our construction uses cabling, and noting that $d_+ = \delta$ for torus knots, it is natural to wonder if any hyperbolic knot exhibits this behavior.

Question 6.5 (K. L. Baker, K. Motegi, T. Takata)**.** *For every hyperbolic knot K, does $d_+[J_{K,n}(q)] = \delta_K(n)$ for all integers $n \geq 1$?*

In our example, even when an unstable range exists, $d_+[J_{K,n}(q)]$ forms another quadratic (quasi-)polynomial in this unstable range. So we would like to ask

Question 6.6 (K. L. Baker, K. Motegi, T. Takata)**.** *Even when $d_+[J_{K,n}(q)] = \delta_K(n)$ only for $n \geq n_0$, is $d_+[J_{K,n}(q)]$ another quadratic quasi-polynomial for $n < n_0$ as well?*

Sign Condition

An existence of an unstable range of $d_+[J_{K,n}(q)]$ causes a difficulty to determine the maximum degree of the colored Jones polynomial of knots obtained by cablings and Whitehead doubles.

To avoid this difficulty we introduced a rather strong condition, the Sign Condition, which requires that the sign $\varepsilon_n(K)$ of the coefficient of the term of the maximum degree of $J_{K,n}(q)$ satisfies $\varepsilon_m(K) = \varepsilon_n(K)$ for $m \equiv n \bmod 2$.

In [1] we show that torus knots, B-adequate knots and knots obtained from theses knots by cablings, Whitehead doublings and connected sums satisfy the Sign Condition, and we asked if every knot satisfies the Sign Condition.

However, a computer experiments suggest that the knots 8_{20}, 9_{43}, and 9_{44} do not satisfy the Sign Condition. The following table gives the sign $\varepsilon_n(K)$ for $K = 8_{20}$, 9_{43}, 9_{44} and $1 \leq n \leq 6$.

K	$\varepsilon_1(K)$	$\varepsilon_2(K)$	$\varepsilon_3(K)$	$\varepsilon_4(K)$	$\varepsilon_5(K)$	$\varepsilon_6(K)$
8_{20}	$-$	$-$	$+$	$-$	$-$	$+$
9_{43}	$-$	$+$	$+$	$-$	$+$	$+$
9_{44}	$+$	$-$	$-$	$-$	$+$	$+$

We computed the colored Jones polynomials for these knots using Mathematica package `KnotTheory‘` and its program `ColouredJones` [3, 33] in order to determine the signs $\varepsilon_n(K)$. In the above examples all knots have $\delta_K(n)$ with period 3. It may be reasonable to ask:

Question 6.7 (K. L. Baker, K. Motegi, T. Takata)**.** *Let K be a knot such that $\delta_K(n)$ has period ≤ 2. Then does K satisfy the Sign Condition?*

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