CONVERGENCE OF MEASURES ON BOOLEAN ALGEBRAS AND CARDINAL CHARACTERISTICS OF THE CONTINUUM

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Abstract. This is a concise survey of the author's results concerning relations between cardinal characteristics of the continuum and convergence of sequences of finitely additive measures on Boolean algebras.

1. Introduction

Let us start with establishing basic notions from measure theory. Recall that every Boolean algebra $\mathcal{A}$ is isomorphic to the Boolean algebra of clopen subsets of its Stone space $St(\mathcal{A})$ and a totally disconnected compact Hausdorff space is extremely disconnected if and only if its algebra of clopen subsets is complete. Every (finitely additive finite signed) measure μ on a Boolean algebra $\mathcal{A}$ may be uniquely extended to a (regular Borel σ-additive finite signed) measure $\widehat{\mu}$ on the Stone space $St(\mathcal{A})$. A sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is *bounded* if $\sup_{A\in\mathcal{A}}\sup_{n\in\omega}\left|\mu_n(A)\right|<\infty$, and it is *pointwise null* if $\mu_n(A)\to 0$ for every $A\in\mathcal{A}$. We also say that a sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is *weakly* null* if it is bounded and pointwise null, and that $\langle\mu_n\colon\ n\in\omega\rangle$ is *weakly null* if $\widehat{\mu}_n(B)\to 0$ for every Borel subset B of the space $St(\mathcal{A})$. Naturally, every weakly null sequence is weakly* null and every weakly* null sequence is pointwise null, but the converse may not hold in either case—e.g. if the Stone space of a Boolean algebra contains a non-trivial convergent sequence, then there is a sequence of measures which is weakly* null but not weakly null as well as there is a sequence which is pointwise null but not weakly* null.

Nikodym [34] however proved that if a Boolean algebra $\mathcal{A}$ is σ-complete, then every pointwise null sequence of measures on $\mathcal{A}$ is weakly* null (cf. also [12]). The result turned out to be fundamental in modern vector measure theory, see e.g. [16] and [15]. Besides σ-complete Boolean algebras many other classes of Boolean algebras were proved to have the same property—we will present several examples in Section 2. Let us thus introduce the following name.

Definition 1.1. A Boolean algebra $\mathcal{A}$ has *the Nikodym property* if every pointwise null sequence of measures on $\mathcal{A}$ is weakly* null.

Similarly, Grothendieck [24] proved that if a Boolean algebra $\mathcal{A}$ is σ-complete, then every weakly* null sequence of finitely additive measures on $\mathcal{A}$ is automatically weakly null. The result have found many applications and generalizations in the context of Banach spaces and vector measure theory, see e.g. [14] and [15]. Later, many other classes of Boolean algebras were also recognized as having the same property, c.f. Section 2. Analogously to the case of the Nikodym property, we introduce the following name.

Definition 1.2. A Boolean algebra $\mathcal{A}$ has *the Grothendieck property* if every weakly* null sequence of measures on $\mathcal{A}$ is weakly null.

It is not an easy task to find any example of a Boolean algebra having only one of the properties. Schachermayer [36] proved that the Jordan algebra $\mathcal{J}$, i.e. the Boolean algebra of Jordan measurable

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subsets of the unit interval $[0,1]$, has the Nikodym property but lacks the Grothendieck property. For similar results, see e.g. [23] and [2]. On the other hand, assuming the Continuum Hypothesis, Talagrand [46] constructed a Boolean algebra with the Grothendieck property but without the Nikodym property. The existence of such an algebra in ZFC only is an open question.

In this concise survey we present several selected results concerning another problem which appeared also very much set-theoretic in nature. Namely, we will focus on the issue whether there exists an infinite Boolean algebra with any of the two properties and of cardinality strictly less than the continuum $\mathfrak{c}$. It appears that the answer depends strongly on a current set-theoretic setting, in particular, on values of several classical cardinal characteristics of the continuum. To describe the situation better we will introduce two new cardinal characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$, show their relations with the standard ones (e.g. those appearing in Cichoń's diagram or van Douwen's diagram) and study their values in various forcing extensions. We finish the paper with the list of open problems.

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1.2. **Preliminaries.** Our notation and terminology is standard and similar to those of the classical books of Bartoszyński and Judah [4], Halmos [26], Givant and Halmos [22] and Diestel [15]. Regarding Cichoń's and van Douwen's diagrams, we follow Blass' survey [7] with one exception: we denote the σ-ideals of meager and Lebesgue null subsets of the Cantor space 2^ω by $\mathcal{M}$ and $\mathcal{N}$, respectively. For information on the values of the classical cardinal characteristics of the continuum in various models of set theory, we refer the reader again to [7, Section 11.9].

Towards the end of the paper we also make the following reasonable assumption:

All the considered Boolean algebras and compact spaces are assumed to be infinite.

2. (Non)-σ-complete Boolean algebras

Let $\mathcal{A}$ be a Boolean algebra and I a set of indices. A sequence $\langle a_i \in \mathcal{A}\colon\ i \in I\rangle$ is *an antichain* in $\mathcal{A}$ if $a_i \wedge a_j = 0$ for every $i \neq j \in I$. Recall that a Boolean algebra $\mathcal{A}$ is *σ-complete* if every antichain $\langle a_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ has supremum $\bigvee_{n\in\omega} a_n$ in $\mathcal{A}$. As stated in Introduction, every σ-complete Boolean algebra has both the Nikodym property as well as the Grothendieck property. These two results have been generalized by many authors who introduced various properties of Boolean algebras weaker than the σ-completeness but still implying both of the properties. Those new properties may be divided into two main groups, *completeness properties* and *separation properties* (also known as *interpolation properties*). To the first group belong those properties which give supremas of some subantichains of given antichains (much like in the case of the σ-completeness), and to the second one—those which give upper bounds (so not necessarily supremas) of subantichains of given antichains and additionally the upper bounds are disjoint from the rest of the antichains. Usually, every completeness property has a corresponding weaker interpolation property.

The main completeness properties are *the property (E)* (Schachermayer [36]), *the Up–Down Semi-Completeness* (in short *UDSC*; Dashiell [13]), *the Subsequential Completeness Property* (*SCP*; Haydon [27]), and *the Weak Subsequential Completeness Property* (*WSCP*; Aizpuru [1]). To the group of separation/interpolation properties belong primarily *the Interpolation Property* (or *the property (I)*; Seever [37]), *the property (f)* (Moltó [33]), *the Subsequential Interpolation Property* (*SIP*; Freniche [18]), *the Weak Subsequential Interpolation Property* (*WSIP*; Freniche [19]), and *the Subsequential Separation Property* (*SSP*; Haydon [28]). It is known that all the listed properties but the WSCP, WSIP and SSP imply both the Nikodym and Grothendieck properties—regarding the latter three they yield certainly the Grothendieck property and it is unknown if they give also the Nikodym property. To give the reader at least a flavour of those all properties let us provide the definitions of the SCP and SSP, both due to Haydon—for the definitions and other information concerning the remaining ones we refer the reader to the proper sources.

Definition 2.1 (Haydon [27]). A Boolean algebra $\mathcal{A}$ has *the Subsequential Completeness Property* (*SCP*) if and only if for every infinite countable antichain $\langle a_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ there is an infinite set $M \subseteq \omega$ such that the supremum $\bigvee_{n\in M} a_n$ is in $\mathcal{A}$.

Haydon constructed a special Boolean algebra with the SCP in order to obtain the first known example of a non-reflexive Grothendieck Banach space without any embedded copy of the space ℓ_∞.

Definition 2.2 (Haydon [28]). A Boolean algebra $\mathcal{A}$ has *the Subsequential Separation Property* (*SSP*) if and only if for every infinite countable antichain $\langle a_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ there are an infinite set $M \subseteq \omega$ and an element $a \in \mathcal{A}$ such that $a_{2n} \leq a$ and $a_{2n+1} \wedge a = 0$ for every $n \in M$.

It is immediate that σ-completeness $\Rightarrow$ SCP $\Rightarrow$ SSP. Koszmider and Shelah [31] introduced yet another even weaker separation property of Boolean algebras. Their notion turned out to be crucial for our further research.

Definition 2.3 (Koszmider and Shelah [31]). A Boolean algebra $\mathcal{A}$ has *the Weak Subsequential Separation Property* (*WSSP*) if and only if for every infinite countable antichain $\langle a_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ there is an element $a \in \mathcal{A}$ such that both of the sets

$$\big\{n \in \omega\colon\ a_n \leq a\big\} \quad \text{and} \quad \big\{n \in \omega\colon\ a_n \wedge a = 0\big\}$$

are infinite.

The WSSP implies the Grothendieck property for *non-negative* measures, i.e. if $\langle \mu_n\colon\ n \in \omega\rangle$ is a weakly* convergent sequence of non-negative finitely additive measures on a Boolean algebra with the WSSP, then $\langle \mu_n\colon\ n \in \omega\rangle$ is weakly convergent ([31, Proposition 2.4]). The full Grothendieck property may not hold—there is a Boolean algebra with the WSSP but without the Grothendieck property (and without the Nikodym property; see [31, Proposition 2.5]).

Recall that a subset $\mathcal{I}$ of a Boolean algebra $\mathcal{A}$ is *independent* if for every disjoint finite non-empty sets $F, G \subseteq \mathcal{I}$ we have:

$$\Big(\bigwedge_{a\in F} a\Big) \wedge \Big(\bigwedge_{a\in G} -a\Big) \neq 0.$$

Koszmider and Shelah proved the following fundamental theorem.

Theorem 2.4 (Koszmider and Shelah [31, Theorem 1.4]). *If a Boolean algebra $\mathcal{A}$ has the WSSP, then $\mathcal{A}$ contains an independent family of size $\mathfrak{c}$.*

It follows immediately that every Boolean algebra with the WSSP has cardinality $\geq \mathfrak{c}$. All the properties introduced earlier in this section imply the WSSP, so in particular if a Boolean algebra $\mathcal{A}$ has any of those properties, then $|\mathcal{A}| \geq \mathfrak{c}$, too. Since no other essentially different algebraic or structural properties of Boolean algebras forcing the Nikodym or Grothendieck property were commonly known in the beginning of the 21st century, the following question appeared to be natural.

Question 2.5. *Is it consistent that there exists a Boolean algebra $\mathcal{A}$ of size strictly less than $\mathfrak{c}$ and with the Grothendieck property or the Nikodym property?*

Since the Stone space of every countable Boolean algebra has a countable base and thus is metrizable, it follows that a Boolean algebra with either the Nikodym property or the Grothendieck property cannot be countable—otherwise its Stone space would contain a non-trivial convergent sequence, which is, as we already know from Introduction, impossible.

As we will see in the next section, the question though has a positive answer.

3. First consistency results

Brech [10] provided the first positive answer to Question 2.5. Namely, she proved that in the side-by-side Sacks model there is a Boolean algebra with the Grothendieck property and of size ω_1 whereas the continuum may be arbitrarily large. In fact, her result is a preservation-like result.

Theorem 3.1 (Brech [10])**.** *Let κ be a regular cardinal in a model V of set theory and let $\mathbb{S}(\kappa)$ denote the countable support κ-product of Sacks forcing. Let $\mathcal{A} \in V$ be a Boolean algebra. If $\mathcal{A}$ is σ-complete in V, then for every $\mathbb{S}(\kappa)$-generic filter G over V the algebra $\mathcal{A}$ has the Grothendieck property in $V[G]$.*

Recall that if $\mathbb{S}$ denotes the Sacks forcing, then for a cardinal number κ the side-by-side Sacks forcing $\mathbb{S}(\kappa)$ is defined as follows: $p \in \mathbb{S}(\kappa)$ if and only if $p \in \mathbb{S}^\kappa$ and the set $\operatorname{supp}(p) = \big\{\xi < \kappa \colon p(\xi) \neq 1_{\mathbb{S}}\big\}$ is countable; the ordering on $\mathbb{S}$ is defined coordinatewise, i.e. for $p, q \in \mathbb{S}(\kappa)$ we have $p \leq q$ if $p(\xi) \leq q(\xi)$ for every $\xi < \kappa$. $\mathbb{S}(\kappa)$ does not collapse ω_1 and if κ is a regular cardinal in a ground model V, then $\mathfrak{c} = \kappa$ in any $\mathbb{S}(\kappa)$-extension of V. For more details on the side-by-side Sacks forcing, see e.g. Baumgartner [5].

Corollary 3.2. *Let V be a model of set theory satisfying the Continuum Hypothesis, i.e. $V \models \mathfrak{c} = \omega_1$. If κ is a regular cardinal in V such that $\kappa \geq \omega_2$ and V' is an $\mathbb{S}(\kappa)$-generic extension of V, then, in V', $\wp(\omega)^V$ has the Grothendieck property and $\big|\wp(\omega)^V\big| = \omega_1 < \mathfrak{c} = \kappa$.*

Brech's result has an important and weighty meaning. So far, by Theorem 2.4 of Koszmider and Shelah, every known algebraic property of Boolean algebras, introduced in order to obtain a characterization of the Grothendieck property or at least to move closer towards obtaining such a characterization, had implied also that a Boolean algebra having this property must have cardinality at least $\mathfrak{c}$. Brech's result shows that such an approach cannot be effective—every characterization of the Grothendieck property must be more intricate and sophisticated (than merely dealing with antichains and its subantichans) in order to work also for Boolean algebras of size strictly less than $\mathfrak{c}$.

Theorem 3.1 inspired the author and Zdomskyy to prove a similar result for the Nikodym property.

Theorem 3.3 (Sobota and Zdomskyy [43])**.** *Let κ be a regular cardinal in a model V of set theory and let $\mathbb{S}(\kappa)$ denote the countable support κ-product of Sacks forcing. Let $\mathcal{A} \in V$ be a Boolean algebra. If $\mathcal{A}$ is σ-complete in V, then for every $\mathbb{S}(\kappa)$-generic filter G over V the algebra $\mathcal{A}$ has the Nikodym property in $V[G]$.*

In particular, if $\kappa \geq \omega_2$ is a regular cardinal in V, a model of set theory satisfying the Continuum Hypothesis, and V' is an $\mathbb{S}(\kappa)$-generic extension of V, then, in V', $\wp(\omega)^V$ has the Nikodym property and $\big|\wp(\omega)^V\big| = \omega_1 < \mathfrak{c} = \kappa$.

Since no countable Boolean algebra has the Grothendieck or Nikodym property and in any model of set theory there is always a Boolean algebra of size $\mathfrak{c}$ and having either of the properties, namely $\wp(\omega)$, Theorems 3.1 and 3.3 suggest that it is reasonable to introduce the following two cardinal characteristics of the continuum.

Definition 3.4. *The Nikodym number* $\mathfrak{nik}$ is the smallest cardinality of a Boolean algebra with the Nikodym property.

Definition 3.5. *The Grothendieck number* $\mathfrak{gr}$ is the smallest cardinality of a Boolean algebra with the Grothendieck property.

It follows that $\omega_1 \leq \mathfrak{gr}, \mathfrak{nik} \leq \mathfrak{c}$. Theorems 3.1 and 3.3 may be now stated simply as follows.

Corollary 3.6. *If V is a model satisfying the Continuum Hypothesis and $\kappa \in V$ is a cardinal, then $\mathfrak{nik} = \mathfrak{gr} = \omega_1$ in any $\mathbb{S}(\kappa)$-generic extension of V.*

In the next sections we will study the lower and upper bounds for $\mathfrak{nik}$ and $\mathfrak{gr}$ as well as we present consistency results concerning these cardinal characteristics. In particular, we show that consistently $\omega_1 < \mathfrak{nik} = \mathfrak{gr} < \mathfrak{c}$.

4. Lower bounds for $\mathfrak{nik}$ and $\mathfrak{gr}$

We start the section with the following definition. Let $\mathcal{K}$ denote the class of all compact totally-disconnected Hausdorff spaces which do not contain any non-trivial convergent sequences.

Definition 4.1. *The convergence number* $\mathfrak{z}$ is the smallest possible weight $w(K)$ of a space $K \in \mathcal{K}$, that is, $\mathfrak{z} = \min\{w(K)\colon\ K \in \mathcal{K}\}$.[1]

Since $\wp(\omega) \in \mathcal{K}$ and every second countable compact space is metrizable (and hence it contains non-trivial convergent sequences), we have $\omega_1 \le \mathfrak{z} \le \mathfrak{c}$.

Recall that if a Boolean algebra $\mathcal{A}$ has the Nikodym or Grothendieck property, then its Stone space $St(\mathcal{A})$ does not have any non-trivial convergent sequences, i.e. $St(\mathcal{A}) \in \mathcal{K}$. It follows immediately that $\mathfrak{z} \le \mathfrak{nik}$ and $\mathfrak{z} \le \mathfrak{gr}$, thus if a cardinal characteristics is a lower bound for $\mathfrak{z}$, then it is also one for $\mathfrak{nik}$ and $\mathfrak{gr}$.

Recall that a family $\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if for $A \in [\omega]^\omega$ there is $B \in \mathcal{F}$ such that both $A \cap B$ and $A \setminus B$ are infinite. *The splitting number* $\mathfrak{s}$ is the minimal cardinality of a splitting family. Booth [8] provided the following characterization of $\mathfrak{s}$ in terms of sequentially compact spaces:

$$\mathfrak{s} = \min\{w(K)\colon\ K \text{ is a compact, not sequentially compact space}\}.$$

If K is a compact space such that $w(K) < \mathfrak{s}$, then it is sequentially compact and hence it contains a non-trivial convergent sequence. Thus, $\mathfrak{s} \le \mathfrak{z}$ and hence $\mathfrak{s} \le \mathfrak{nik}, \mathfrak{gr}$.

Now, recall that *the covering number* $\operatorname{cov}(\mathcal{M})$ *of* $\mathcal{M}$ denotes the minimal cardinality of a family of meager subsets of the Cantor space 2^ω covering the whole space, i.e. $\operatorname{cov}(\mathcal{M}) = \min\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq \mathcal{M} \text{ and } \bigcup\mathcal{F} = 2^\omega\}$. It is an easy fact that if a compact space K is *scattered* (i.e. each closed subset of K has an isolated point in the topology inherited from K), then K contains a non-trivial convergent sequence. In [20] Geschke proved that if a compact space K is non-scattered and such that $w(K) < \operatorname{cov}(\mathcal{M})$, then there exist a (closed) perfect subset L of K and a point $x \in L$ such that that x is a $\mathbb{G}_\delta$-point in L, i.e. there is a family $\{G_n\colon\ n \in \omega\}$ of open subsets of L such that $\{x\} = \bigcap_{n\in\omega} G_n$. It yields that there is a non-trivial sequence in L convergent to x. It follows that $\operatorname{cov}(\mathcal{M}) \le \mathfrak{z}$ and hence $\operatorname{cov}(\mathcal{M}) \le \mathfrak{nik}, \mathfrak{gr}$.

Let us note here that the convergence number $\mathfrak{z}$ was studied also by Brian and Dow in [9], where it was proved, i.a., that if κ is a cardinal number satisfying the condition:

$$\max\big(\mathfrak{b}, \operatorname{non}(\mathcal{N})\big) \le \kappa = \operatorname{cof}\big([\kappa]^\omega, \subseteq\big),$$

then $\mathfrak{z} \le \kappa$. Here $\mathfrak{b}$ denotes *the bounding number*, i.e., the minimal cardinality of a subfamily of ω^ω which is unbounded in the sense of the order $\le^*$ defined as follows: given $f, g \in \omega^\omega$, $f \le^* g$ if and only if $f(n) \le g(n)$ for almost all $n \in \omega$. *The uniformity number of* $\mathcal{N}$, denoted by $\operatorname{non}(\mathcal{N})$, is the minimal cardinality of a subset $X \subseteq 2^\omega$ such that $X \notin \mathcal{N}$. Finally, for every cardinal κ by $\big([\kappa]^\omega, \subseteq\big)$ we denote the family of all countable subsets of κ ordered by the inclusion $\subseteq$ and by $\operatorname{cof}\big([\kappa]^\omega, \subseteq\big)$ we mean the smallest cardinality of a subfamily $\mathcal{F} \subseteq [\kappa]^\omega$ which is cofinal in the sense of $\subseteq$. Basic information concerning the number $\operatorname{cof}\big([\kappa]^\omega, \subseteq\big)$ one can find in [4, Section 1.3.B]—e.g, we always

[1] The letter $\mathfrak{z}$, i.e. Gothic z, comes from the Polish word *zbieżność* meaning *convergence.*

have $\operatorname{cof}\left([\kappa]^\omega, \subseteq\right) \geq \kappa$ and there are such numbers κ for which we have $\operatorname{cof}\left([\kappa]^\omega, \subseteq\right) > \kappa$, but $\operatorname{cof}\left([\omega_n]^\omega, \subseteq\right) = \omega_n$ for every $n \in \omega$.

The bounding number $\mathfrak{b}$ plays here also a special role for the Nikodym number $\mathfrak{nik}$. Namely, using the celebrated Josefson–Nissenzweig theorem (see e.g. Diestel [15, Chapter XII]), one can prove that if a Boolean algebra has cardinality strictly smaller than $\mathfrak{b}$, then it does not have the Nikodym property—see Sobota [41, Proposition 3.2]. Thus, the following corollary holds.

Corollary 4.2. $\max\big(\mathfrak{b}, \mathfrak{s}, \operatorname{cov}(\mathcal{M})\big) \leq \mathfrak{nik}$.

However, we do not know whether the inequality $\mathfrak{b} \leq \mathfrak{gr}$ holds in ZFC, cf. Section 8.

Corollary 4.3. $\max\big(\mathfrak{s}, \operatorname{cov}(\mathcal{M})\big) \leq \mathfrak{gr}$.

Both Corollaries 4.2 and 4.3 imply that under Martin's axiom $\mathfrak{nik} = \mathfrak{gr} = \mathfrak{c}$.

5. Upper bounds for $\mathfrak{nik}$ and $\mathfrak{gr}$

Since $\wp(\omega)$ has both the Nikodym and Grothendieck properties, $\mathfrak{nik} \leq \mathfrak{c}$ and $\mathfrak{gr} \leq \mathfrak{c}$. Obtaining upper bounds for both the characteristics stronger than $\mathfrak{c}$ is however a much more intricate issue than that of finding lower bounds, since no simple condition for Boolean algebras implying any of the properties and possibly satisfied by algebras of cardinality strictly smaller than $\mathfrak{c}$ is known. For instance, the σ-completeness—or any of the properties introduced in Section 2—of a given Boolean algebra implies by Theorem 2.4 that the algebra has immediately the cardinality at least $\mathfrak{c}$. Yet we present here the author's results asserting that under some natural conditions concerning the cofinality $\operatorname{cof}(\mathcal{N})$ of Lebesgue null ideal $\mathcal{N}$ (like, e.g., $\operatorname{cof}(\mathcal{N}) = \omega_n$ for some $n \in \omega$), it is possible to obtain the inequalities $\mathfrak{nik} \leq \operatorname{cof}(\mathcal{N})$ and $\mathfrak{gr} \leq \operatorname{cof}(\mathcal{N})$.

Recall first that *the cofinality* $\operatorname{cof}(\mathcal{N})$ of the Lebesgue null ideal $\mathcal{N}$ is the minimal cardinality of a family $\mathcal{F} \subseteq \mathcal{N}$ such that for every $A \in \mathcal{N}$ there is $B \in \mathcal{F}$ such that $A \subseteq B$. In Section 4 we presented the theorem of Brian and Dow stating that if $\max\big(\mathfrak{b}, \operatorname{non}(\mathcal{N})\big) \leq \kappa = \operatorname{cof}\left([\kappa]^\omega, \subseteq\right)$ for some κ, then $\mathfrak{z} \leq \kappa$. A similar theorem was obtained earlier by the author relating $\mathfrak{nik}$ and $\operatorname{cof}(\mathcal{N})$.

Theorem 5.1 (Sobota [41])**.** *If κ is such a cardinal number that $\operatorname{cof}(\mathcal{N}) \leq \kappa = \operatorname{cof}\left([\kappa]^\omega\right)$, then there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$ and $\mathcal{A}$ has the Nikodym property. In particular, $\mathfrak{nik} \leq \kappa$.*

Corollary 5.2. *If $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$, then $\mathfrak{nik} = \omega_1 < \mathfrak{c}$, too. More generally, if $\operatorname{cof}(\mathcal{N}) = \omega_n$ for some $n \in \omega$, then $\mathfrak{nik} \leq \operatorname{cof}(\mathcal{N})$.*

Let us elaborate more on the above results. The main idea of the proof of Theorem 5.1 is similar to the one presented by Darst [12] who proved that σ-complete Boolean algebras do have the Nikodym property, i.e. to assume that a given σ-complete Boolean algebra $\mathcal{A}$ does not have the property, so there is a "bad" sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ (that is, pointwise null but not bounded), to construct then a special antichain in $\mathcal{A}$ related to $\langle \mu_n\colon\ n \in \omega\rangle$ and by taking its supremum to obtain an element in $\mathcal{A}$ on which $\langle \mu_n\colon\ n \in \omega\rangle$ is not convergent, which ultimately yields a contradiction. In the proof of Theorem 5.1 we follow a similar way, i.e., we construct inductively an increasing ω_1-sequence of Boolean subalgebras of $\wp(\kappa)$ of size κ in such a way that in each step we "kill" bad sequences of measures on Boolean subalgebras from the previous steps, i.e. those sequences which could possibly contradict the Nikodym property of the final algebra. To do this, in each step we construct a special family of antichains of elements from the previous subalgebras and attach their carefully chosen supremas. Naturally, in each step we have to ensure that the constructed families of antichains and their supremas are not too big, i.e., they have cardinality at most κ. We do this actually separately from the main proof by introducing and studying two new properties of $[\omega]^\omega$ and $Fr(\omega)$, the free Boolean algebra on countably many generators, which would give us such "small" families of antichains and supremas. This two new properties lead us to two new cardinal characteristics of the continuum, which we call *the anti-Nikodym number* and *the Nikodym extracting number*, denoted by $\mathfrak{n}_a$ and $\mathfrak{n}_e$, respectively. Since the definitions of $\mathfrak{n}_a$ and $\mathfrak{n}_e$ are rather technical and long, we refer the reader to [41] for more details. It is worth here however to mention that those two characteristics satisfy the following inequalities: $\mathfrak{b} \le \mathfrak{n}_a \le \operatorname{cof}(\mathcal{N})$ ([41, Corollary 6.7]) and $\operatorname{cov}(\mathcal{M}) \le \mathfrak{n}_e \le \mathfrak{d}$ ([41, Corollary 6.16]). Recall that *the dominating number* $\mathfrak{d}$ is the minimal cardinality of a family $\mathcal{F}$ of functions $f \in \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $g \le^* f$. Since $\mathfrak{d} \le \operatorname{cof}(\mathcal{N})$ we obtain that $\max\big(\mathfrak{n}_a, \mathfrak{n}_e\big) \le \operatorname{cof}(\mathcal{N})$. Since $\kappa \ge \operatorname{cof}(\mathcal{N})$, these inequalities allow us indeed to attach at most κ new elements in each step of the construction which would kill bad sequences in the final algebra.

Note that it follows from the above discussion that we in fact prove the following stronger version of Theorem 5.1 which does not mention $\operatorname{cof}(\mathcal{N})$.

Theorem 5.3. *If κ is such a cardinal number that $\max\big(\mathfrak{n}_a, \mathfrak{n}_e\big) \le \kappa = \operatorname{cof}\big([\kappa]^\omega\big)$, then there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$ and $\mathcal{A}$ has the Nikodym property. In particular, $\mathfrak{nik} \le \kappa$.*

Recently, the author obtained also a theorem concerning $\mathfrak{gr}$ and $\operatorname{cof}(\mathcal{N})$, very much similar to Theorem 5.1.

Theorem 5.4 (Sobota [42])**.** *If κ is such a cardinal number that $\operatorname{cof}(\mathcal{N}) \le \kappa = \operatorname{cof}\big([\kappa]^\omega\big)$, then there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$ and $\mathcal{A}$ has the Grothendieck property. In particular, $\mathfrak{gr} \le \kappa$.*

Corollary 5.5. *If $\operatorname{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$, then $\mathfrak{gr} = \omega_1 < \mathfrak{c}$, too. More generally, if $\operatorname{cof}(\mathcal{N}) = \omega_n$ for some $n \in \omega$, then $\mathfrak{gr} \le \operatorname{cof}(\mathcal{N})$.*

The idea standing behind the proof of Theorem 5.4 is very similar to the one of Theorem 5.1, that is, we inductively construct an increasing ω_1-sequence of Boolean subalgebras of $\wp(\kappa)$ of size κ in such a way that in each step we "kill" "bad" sequences of measures on the Boolean algebras from the previous steps that would contradict the Grothendieck property of the ultimate algebra. This time, however, we need three auxiliary properties of $[\omega]^\omega$ and $Fr(\omega)$ that are related to the Phillips lemma, Rosenthal lemma and Dieudonné–Grothendieck characterization of weak compactness of families of measures on compact Hausdorff spaces. Similarly as before, from those new properties we derive new cardinal characteristics which we call the Phillips number, the Rosenthal number and the Dieudonné–Grothendieck number, denoted $\mathfrak{phil}$, $\mathfrak{ros}$ and $\mathfrak{dg}$, respectively. The role of $\mathfrak{phil}$ and $\mathfrak{ros}$ in the proof of Theorem 5.4 is basically the same, they just yield two possible variants of the proof and hence two possibly different upper bounds of $\mathfrak{gr}$. Let us discuss the three characteristics more carefully, as they are interesting on their own.

The Dieudonné–Grothendieck theorem in its particular form asserts that a bounded sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is not weakly convergent if and only if there are an antichain $\big\langle a_k\colon\ k \in \omega\big\rangle$ in $\mathcal{A}$, an increasing sequence $\big\langle n_k\colon\ k \in \omega\big\rangle$ in ω and $\varepsilon > 0$ such that $\big|\mu_{n_k}\big(a_k\big)\big| \ge \varepsilon$ for every $k \in \omega$. The number $\mathfrak{dg}$ is defined in the following related to the theorem way.

Definition 5.6. *The Dieudonné–Grothendieck number* $\mathfrak{dg}$ is the minimal cardinality of a family $\mathcal{F}$ of antichains in $Fr(\omega)$ such that for every sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ of measures on $Fr(\omega)$ which is weakly* null but not weakly null there are an antichain $\big\langle a_k\colon\ k \in \omega\big\rangle \in \mathcal{F}$, an increasing sequence $\big\langle n_k \in \omega\colon\ k \in \omega\big\rangle$ and $\varepsilon > 0$ such that $\big|\mu_{n_k}\big(a_k\big)\big| \ge \varepsilon$ for every $k \in \omega$.

It was showed in [38] and [42] that $\mathfrak{b} \leq \mathfrak{dg} \leq \operatorname{cof}(\mathcal{N})$. Thus, consistently, to prove that a given weakly* null sequence of measures on the Cantor space 2^ω is (not) weakly null, it is enough to check its values on antichains from a special family $\mathcal{F}$ of size strictly less than $\mathfrak{c}$.

The next number is related to the Phillips lemma asserting that for every sequence $\langle \mu_n \colon n \in \omega \rangle$ of measures on the Boolean algebra $\wp(\omega)$ if $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \wp(\omega)$, then $\lim_{n\to\infty} \sum_{k\in\omega} |\mu_n(\{k\})| = 0$ (i.e., $\mu_n \to 0$ in the norm on $\ell_1(\omega)$).

Definition 5.7. *The Phillips number* $\mathfrak{phil}$ is the minimal size of a family $\mathcal{F} \subseteq [\omega]^\omega$ such that for every sequence $\langle \mu_n \colon n \in \omega \rangle$ of measures on the Boolean algebra $\wp(\omega)$ if $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{F}$, then $\lim_{n\to\infty} \sum_{k\in\omega} |\mu_n(\{k\})| = 0$.

In his PhD thesis [38], the author showed that $\mathfrak{p} \leq \mathfrak{phil} \leq \operatorname{cof}(\mathcal{N})$ and in [39, Proposition 2.5] that $\mathfrak{p} < \mathfrak{phil} = \mathfrak{c}$ consistently holds. We refer the reader to [39] for information concerning possible analytic applications of those inequalities.

The last number which we want to discuss here is the Rosenthal number related to Rosenthal's lemma stating that for every antichain $\langle a_n \colon n \in \omega \rangle$ in a Boolean algebra $\mathcal{A}$, sequence $\langle \mu_k \colon k \in \omega \rangle$ of non-negative measures on $\mathcal{A}$ of norm 1, and $\varepsilon > 0$, there exists an infinite set $A \in [\omega]^\omega$ such that for every $k \in A$ the inequality $\sum_{\substack{n\in A\\ n\neq k}} \mu_k(a_n) < \varepsilon$ holds. For analytic and measure-theoretic applications of the lemma, see e.g. [15, Chapter VII] and [16, Section I.4].

The lemma may be actually also stated in a much simpler form: for every infinite matrix $\langle m_n^k \colon n, k \in \omega \rangle$ of non-negative real numbers such that $\sum_{n\in\omega} m_n^k \leq 1$ and $a_k^k = 0$ for every $k \in \omega$, and every $\varepsilon > 0$ there is $A \in [\omega]^\omega$ such that $\sum_{n\in A} m_n^k < \varepsilon$ for every $k \in A$.

Definition 5.8. *The Rosenthal number* $\mathfrak{ros}$ is the minimal cardinality of a family $\mathcal{F} \in [\omega]^\omega$ such that for every infinite matrix $\langle m_n^k \colon n, k \in \omega \rangle$ of non-negative real numbers such that $\sum_{n\in\omega} m_n^k \leq 1$ and $a_k^k = 0$ for every $k \in \omega$, and every $\varepsilon > 0$ there is $A \in \mathcal{F}$ such that $\sum_{n\in A} m_n^k < \varepsilon$ for every $k \in A$.

The number $\mathfrak{ros}$ was studied by the author in [40] where it was proved that $\operatorname{cov}(\mathcal{M}) \leq \mathfrak{ros}$ and that for every selective ultrafilter (if any exists) its base has cardinality at least $\mathfrak{ros}$, so $\mathfrak{ros} \leq \mathfrak{u}_{sel}$, where $\mathfrak{u}_{sel}$ denotes the minimal cardinality of a base of a selective ultrafilter, if any exists, or $\mathfrak{c}$, otherwise. It was also showed there that under Martin's axiom there exists a filter which is not selective but satisfies the (Rosenthal's lemma) condition from Definition 5.8. Koszmider and Martínez-Celis [30] proved in ZFC that actually every ultrafilter on ω satisfies this condition and that $\mathfrak{ros} = \mathfrak{r}$. Let us note that the inequality $\mathfrak{ros} \geq \mathfrak{r}$ was obtained independently by Repický [35].

After we defined all the necessary additional cardinal characteristics and having in mind that $\mathfrak{ros} = \mathfrak{r}$, we are in the position to recall the following stronger variant of Theorem 5.4.

Theorem 5.9. *If κ is such a cardinal number that $\max\big(\mathfrak{dg}, \min(\mathfrak{phil}, \mathfrak{ros})\big) \leq \kappa = \operatorname{cof}\big([\kappa]^\omega\big)$, then there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$ and $\mathcal{A}$ has the Grothendieck property. In particular, $\mathfrak{gr} \leq \kappa$.*

Theorems 5.1 and 5.4 yield an important conclusion concerning the characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$ and the almost disjointness number $\mathfrak{a}$. Recall that a family $\mathcal{F}$ of infinite subsets of ω is *almost disjoint* if $A \cap B$ is finite for any two distinct elements of $\mathcal{F}$. *The almost disjointness number* $\mathfrak{a}$ is defined as the minimal cardinality of a maximal almost disjoint family. It is a basic result that $\mathfrak{a} = \omega_1$ in the Cohen model, so consistently $\mathfrak{a} < \mathfrak{nik}$ and $\mathfrak{a} < \mathfrak{gr}$ (as $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ in this model). On the other hand, Brendle [11, Proposition 4.7] showed that it consistently holds $\omega_2 = \operatorname{cof}(\mathcal{N}) < \mathfrak{a} = \omega_3 = \mathfrak{c}$, so in this model we also have $\mathfrak{nik} < \mathfrak{a}$ and $\mathfrak{gr} < \mathfrak{a}$.

Corollary 5.10. *Neither of the characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$ is comparable in ZFC with $\mathfrak{a}$.*

6. Other consistency results

Let us start the section with noticing in which standard models of set theory the characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$ are equal to $\mathfrak{c}$. This may be easily derived from Corollaries 4.2 and 4.3, having on hand Table 4 in [7, page 92]. Indeed, we have the following equalities in the corresponding models: $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ in the Cohen model, $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ in the Mathias model, and $\mathfrak{b} = \operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ in the Hechler model; so in these mentioned models it holds $\mathfrak{nik} = \mathfrak{gr} = \mathfrak{c}$. In the Laver model, however, we have $\mathfrak{b} = \mathfrak{c}$ only, so $\mathfrak{nik} = \mathfrak{c}$ in this model and we do not know the value of $\mathfrak{gr}$ therein, see Section 8.

Let us now study a more interesting case, i.e., when $\mathfrak{nik} = \mathfrak{gr} < \mathfrak{c}$. A main set-theoretic tool used in the proofs of Theorems 3.1 and 3.3 is Axiom (A) of the Sacks forcing, in particular, the existence of fusions for fusion sequences and the Sacks property—see [21] for definitions. Since many other notions of forcing satisfy Axiom A or have similar properties, the author and Zdomskyy looked more carefully on the proofs of the theorems and in [44] generalized the results to much broader class of forcings. To explain this, let us recall the following two standard definitions.

Definition 6.1. Let V be a model of set theory. A notion of forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, functions $f \in \omega^\omega \cap V$ and $h \in \omega^\omega \cap V[G]$ such that $h \leq^* f$ there is a function $H\colon \omega \to [\omega]^{<\omega}$ in V such that $h(n) \in H(n)$ and $|H(n)| \leq n + 1$ for every $n \in \omega$.

Standard notions of forcing having the Laver property are, e.g., Sacks, side-by-side Sacks, Laver, Mathias, Miller, Silver (-like), and their countable support iterations, see [44] for references.

Definition 6.2. Let V be a model of set theory. A notion of forcing $\mathbb{P} \in V$ *preserves the ground model reals non-meager* if for every $\mathbb{P}$-generic filter G over V the set $\mathbb{R} \cap V$ is non-meager in $V[G]$.

Among the notions listed after Definition 6.1, only Sacks, side-by-side Sacks, Miller, Silver (-like), and their countable support iterations, preserve ground model reals non-meager.

We are in the position to state the main theorem of [44].

Theorem 6.3 (Sobota and Zdomskyy [44]). *Let V be a model of set theory and $\mathcal{A}$ a σ-complete Boolean algebra in V. If a notion of forcing $\mathbb{P} \in V$ is proper, preserves the ground model reals non-meager and has the Laver property, then in any $\mathbb{P}$-generic extension $V[G]$ the algebra $\mathcal{A}$ has the Nikodym and Grothendieck properties.*

We immediately get the corollary analogous to Corollary 3.6.

Corollary 6.4. *Let V be a model of set theory satisfying the Continuum Hypothesis. If a notion of forcing $\mathbb{P} \in V$ is proper, preserves the ground model reals non-meager and has the Laver property, then in any $\mathbb{P}$-generic extension $V[G]$ we have $\omega_1 = \mathfrak{nik} = \mathfrak{gr}$.*

Since in the side-by-side Sacks forcing all the standard characteristics are equal to ω_1, so far, the only strict inequalities we have had between $\mathfrak{nik}$, $\mathfrak{gr}$ and these characteristics followed from Corollaries 4.2 and 4.3 together with the well-known fact that there is no ZFC inequality between $\mathfrak{b}$, $\mathfrak{s}$ and $\operatorname{cov}(\mathcal{M})$. Theorem 6.3 and Corollary 6.4 yield us some new situations where $\omega_1 = \mathfrak{nik} = \mathfrak{gr} < \mathfrak{c} = \omega_2$ and $\mathfrak{c}$ is equal to some other standard cardinal characteristics of the continuum. Let us thus look at two particularly interesting situations, namely, those of the Miller and Silver models, and check what new inequalities between the characteristics we obtain therein. Note that by Corollary 6.4 both in the Miller model as well as in the Silver (-like) model we obtain that $\omega_1 = \mathfrak{nik} = \mathfrak{gr} < \omega_2 = \mathfrak{c}$.

Two important cardinal characteristics of the continuum which have value $\mathfrak{c}$ in the Miller model are the dominating number $\mathfrak{d}$ and the groupwise density number $\mathfrak{g}$. It is well known that $\mathfrak{d} \geq \max\big(\mathfrak{b}, \mathfrak{s}, \operatorname{cov}(\mathcal{M})\big)$, so, in the context of Corollaries 4.2 and 4.3, $\mathfrak{d}$ was another natural candidate for a cardinal characteristic $\mathfrak{x}$ such that $\mathfrak{x} \leq \mathfrak{nik}$ and $\mathfrak{x} \leq \mathfrak{gr}$ in ZFC, however, as we see, this is not the case.

Corollary 6.5. *It is consistent that* $\omega_1 = \mathfrak{nik} = \mathfrak{gr} < \mathfrak{d} = \mathfrak{c} = \omega_2$.

The definition of *the groupwise density number* $\mathfrak{g}$ is a bit complicated and would take too much space, therefore we refer the reader to [7, Definition 2.26] and [6]. Let us note here that $\mathfrak{g} = \omega_1$ in the Cohen model, therefore both $\mathfrak{gr}$ and $\mathfrak{nik}$ are independent of $\mathfrak{g}$ in the sense that there is no provable in ZFC inequality between either of the former characteristics and $\mathfrak{g}$.

Among cardinal characteristics which have value $\mathfrak{c}$ in the Silver model are the reaping number $\mathfrak{r}$ and the ultrafilter number $\mathfrak{u}$, see [25, Page 379]. *The reaping number* $\mathfrak{r}$ is the smallest cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ for which there is no $A \in [\omega]^\omega$ such that for every $B \in \mathcal{F}$ the sets $B \cap A$ and $B \setminus A$ are infinite. *The ultrafilter number* $\mathfrak{u}$ is the minimal size of a base of a non-principal ultrafilter in $\wp(\omega)$.

Note that it is consistent that $\omega_1 = \mathfrak{r} = \mathfrak{u} < \mathfrak{s} = \mathfrak{c}$, cf. [7, Section 11.11], therefore we consistently get that $\omega_1 = \mathfrak{r} = \mathfrak{u} < \mathfrak{nik} = \mathfrak{gr} = \mathfrak{c}$ and hence the following corollary is true (cf. Corollary 5.10).

Corollary 6.6. *If* $\mathfrak{x} \in \{\mathfrak{r}, \mathfrak{u}, \mathfrak{g}\}$ *and* $\mathfrak{y} \in \{\mathfrak{nik}, \mathfrak{gr}\}$*, then there is no inequality between* $\mathfrak{x}$ *and* $\mathfrak{y}$ *provable in ZFC.*

The independence number $\mathfrak{i}$ is the minimal size of a maximal family $\mathcal{I} \subseteq [\omega]^\omega$ such that for every finite disjoint subsets $F, G \subseteq \mathcal{I}$ the intersection $\bigcap_{A \in F} A \cap \bigcap_{A \in G} (\omega \setminus A)$ is infinite. The value of $\mathfrak{i}$ in the Silver model is also $\mathfrak{c}$, so consistently $\mathfrak{nik} < \mathfrak{i}$ and $\mathfrak{gr} < \mathfrak{i}$. However, we do not now if any of the inequalities $\mathfrak{i} < \mathfrak{nik}$ and $\mathfrak{i} < \mathfrak{gr}$ holds in some model of set theory.

Corollary 6.7. *It is consistent that* $\mathfrak{nik} < \mathfrak{i}$ *and* $\mathfrak{gr} < \mathfrak{i}$.

6.1. **Adding one new real.** In the previous section we studied several forcing notions in the context of preserving the Nikodym and Grothendieck properties of σ-complete Boolean algebras. In this section we study the converse, i.e., when adding one new real to the ground model destroys the properties of all ground model (not necessarily) σ-complete Boolean algebras.

In Section 4 we recalled the result of Geschke asserting that if a non-scattered compact space K has weight strictly less than $\operatorname{cov}(\mathcal{M})$, then there is a perfect subset $L \subseteq K$ and a point $x \in L$ which is a $\mathbb{G}_\delta$-point in L, hence K contains a non-trivial convergent sequence. It appears that the proof of the result may be actually rewritten in terms of the Cohen forcing as follows (see [45]).

Proposition 6.8. *Let* $\mathcal{A}$ *be a Boolean algebra in the ground model* V*. If* c *is a Cohen real over* V*, then, in* $V[c]$*, the Stone space* $St(\mathcal{A})$ *has a non-trivial convergent sequence.*

The above fact was first observed by Dow and Fremlin [17, Introduction] as being implied by the results of Koszmider [29]. Since a Boolean algebra with neither the Nikodym property nor the Grothendieck property may have a non-trivial convergent sequence in its Stone spaces, Proposition 6.8 yields that adding a Cohen real to the ground model kills either of the properties of ground model Boolean algebras.

Note that adding a Hechler real to the ground model adds also a Cohen real, so the Hechler forcing adds non-trivial convergent sequences to the Stone spaces of ground model Boolean algebras, too.

Adapting the result of Booth mentioned in Section 4 and stating that if a compact Hausdorff space K has weight strictly less than the splitting number $\mathfrak{s}$, then K is sequentially compact, one can obtain an analogon of Proposition 6.8 for unsplit reals. Recall that $x \in [\omega]^\omega$ is *an unsplit real* over the ground model V if for no $A \in [\omega]^\omega \cap V$ both sets $B \cap A$ and $B \setminus A$ are infinite. Every Mathias real is an example of an unsplit real.

Proposition 6.9 ([45])**.** *Let* $\mathcal{A}$ *be a Boolean algebra in the ground model* V*. If* u *is an unsplit real over* V*, then, in* $V[u]$*, the Stone space* $St(\mathcal{A})$ *has a non-trivial convergent sequence.*

Thus, similarly as before, adding an unsplit real to the ground model kills the Nikodym and Grothendieck properties of ground model Boolean algebras.

Also the fact that if a Boolean algebra has cardinality strictly less than the bounding number $\mathfrak{b}$, then it does not have the Nikodym property, may be adapted to the forcing setting concerning adding a dominating real and thus killing the Nikodym property of ground model Boolean algebras.

Proposition 6.10 ([45]). *Let $\mathcal{A}$ be a Boolean algebra in the ground model V. If d is a dominating real over V, then, in $V[d]$, the Boolean algebra $\mathcal{A}$ does not have the Nikodym property.*

We do not know if Proposition 6.10 holds true with the Grothendieck property instead of the Nikodym property. A natural forcing to check this seems to be the Laver forcing (as it has the Laver property, see the previous section).

We left the most interesting case of a notion of forcing to the very end of the section, namely, the case of the random forcing. Dow and Fremlin [17] proved the following preservation result. Recall that a compact Hausdorff space K is *an F-space* if every two disjoint open $\mathbb{F}_\sigma$-subsets of K have disjoint closures; in particular, every extremely disconnected space (or, equivalently, the Stone space of every complete Boolean algebra) is an F-space.

Theorem 6.11 (Dow and Fremlin [17]). *Let $\mathcal{A}$ be a Boolean algebra in the ground model V such that the Stone space $St(\mathcal{A})$ is an F-space. If r is a random real over V, then, in $V[r]$, the Stone space $St(\mathcal{A})$ does not have any non-trivial convergent sequences.*

Theorem 6.11 cannot be generalized in any way to the case of the Nikodym or Grothendieck property. Namely, the author and Zdomskyy proved that adding a random real to the ground model kills either of the properties of ground model Boolean algebras.

Theorem 6.12 ([45]). *Let $\mathcal{A}$ be a Boolean algebra in the ground model V. If r is a random real over V, then, in $V[r]$, the Boolean algebra $\mathcal{A}$ has neither the Nikodym property nor the Grothendieck property.*

As a corollary we obtain that in the model obtained by forcing with the Boolean algebra $Bor(2^\kappa)/\mathcal{N}(\kappa)$, i.e. the algebra of Borel subsets of 2^κ modulo the ideal of null sets of the standard product measure on 2^κ, the characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$ are equal to $\mathfrak{c}$ while the dominating number $\mathfrak{d}$ is ω_1.

Corollary 6.13. *It is consistent that $\omega_1 = \mathfrak{d} < \mathfrak{nik} = \mathfrak{gr} = \mathfrak{c} = \omega_2$.*

Together with Corollary 6.5 we get the following result.

Corollary 6.14. *There is no provable in ZFC inequality between $\mathfrak{d}$ and either of the characteristics $\mathfrak{nik}$ and $\mathfrak{gr}$.*

Another corollary is related to the result of Dow and Fremlin and the convergence number $\mathfrak{z}$ defined in Section 4.

Corollary 6.15. *It is consistent that $\omega_1 = \mathfrak{z} < \mathfrak{nik} = \mathfrak{gr} = \mathfrak{c} = \omega_2$.*

7. The cofinalities of $\mathfrak{nik}$ and $\mathfrak{gr}$

Schachermayer [36, Proposition 4.6] proved that if a Boolean algebra $\mathcal{A}$ can be written as a countable increasing union of its proper subalgebras, i.e. $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$ where $\mathcal{A}_n$ is a proper subalgebra of $\mathcal{A}$ and $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each $n \in \omega$, then $\mathcal{A}$ does not have the Nikodym property or the Grothendieck property. It follows immediately that the cofinality of $\mathfrak{nik}$ and $\mathfrak{gr}$ is uncountable.

Corollary 7.1. $\mathrm{cf}(\mathfrak{nik}) \geq \omega_1$ *and* $\mathrm{cf}(\mathfrak{gr}) \geq \omega_1$.

Under the Continuum Hypothesis $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c} = \mathfrak{nik} = \mathfrak{gr}$, so both $\mathfrak{nik}$ and $\mathfrak{gr}$ may be consistently regular. On the other hand, in a model obtained by adding, e.g., $\aleph_{\omega_1}$ many Cohen reals to a model of the Generalized Continuum Hypothesis we have $\mathfrak{nik} = \mathfrak{gr} = \mathfrak{c} = \aleph_{\omega_1} > \omega_1 = \mathrm{cf}\left(\aleph_{\omega_1}\right)$ (see [32, Lemma 5.14]), so $\mathrm{cf}(\mathfrak{nik}) < \mathfrak{nik}$ and $\mathrm{cf}(\mathfrak{gr}) < \mathfrak{gr}$ in this model and hence neither $\mathfrak{nik}$ nor $\mathfrak{gr}$ is regular. Note that in this way we may obtain actually any uncountable cofinality of $\mathfrak{nik}$ and $\mathfrak{gr}$ with both characteristics being still equal to $\mathfrak{c}$.

Corollary 7.2. *The regularity of neither* $\mathfrak{nik}$ *nor* $\mathfrak{gr}$ *is decidable in ZFC.*

8. Open problems

In this final section of the survey we present several most important open questions concerning the Nikodym and Grothendieck properties and their set-theoretic aspects. We start with the following one concerning Talagrand's result mentioned in the introductory section and stating that under the assumption of the Continuum Hypothesis there exists a Boolean algebra with the Grothendieck property but without the Nikodym property.

Question 8.1. *Does there exist in ZFC a Boolean algebra with the Grothendieck property but without the Nikodym property?*

A weaker variant of Question 8.1 could be the following: *assuming Martin's axiom (or the Proper Forcing Axiom, etc.), does every Boolean algebra with the Grothendieck property have the Nikodym property?*

In Section 2 we discussed several properties of Boolean algebras that are weaker than the σ-completeness but still imply at least one of the properties. We presented there the result of Koszmider and Shelah yielding that all those properties imply also that every Boolean algebra having any of them must be of size at least $\mathfrak{c}$. In view of the results presented in Sections 3, 5 and 6 we ask the following question.

Question 8.2. *Is there any (algebraic or structural) property of Boolean algebras that imply either the Nikodym property or the Grothendieck property and is satisfied also by algebras of size strictly less than* $\mathfrak{c}$*?*

The following question appears to be the most important one concerning the relation between $\mathfrak{nik}$ and $\mathfrak{gr}$.

Question 8.3. *May any of the inequalities* $\mathfrak{nik} < \mathfrak{gr}$ *and* $\mathfrak{nik} > \mathfrak{gr}$ *consistently hold?*

In Section 4 we stated that the inequality $\mathfrak{b} \le \mathfrak{nik}$ holds in ZFC and it is unknown to us whether $\mathfrak{b} \le \mathfrak{gr}$ holds, too.

Question 8.4. *Does the inequality* $\mathfrak{b} \le \mathfrak{gr}$ *hold in ZFC?*

A natural model to check whether the inequality $\mathfrak{gr} < \mathfrak{b}$ may consistently hold (and thus $\mathfrak{gr} < \mathfrak{nik}$) is the Laver model (in which $\mathfrak{b} = \mathfrak{nik} = \omega_2 = \mathfrak{c}$).

Question 8.5. *What is the value of* $\mathfrak{gr}$ *in the Laver model?*

Recall that the Laver forcing adds dominating reals and thus "kills" the Nikodym property of ground model Boolean algebras (Proposition 6.10). Since we do not know if adding dominating reals automatically kills the Grothendieck property, a question related to Questions 8.3–8.5 and being in the spirit of Theorem 6.3 and Proposition 6.10 could be thus stated as follows.

Question 8.6. *Does there exists a notion of forcing adding dominating reals and preserving the Grothendieck property of ground model* σ*-complete Boolean algebras?*

Two strict inequalities between $\mathfrak{nik}$ or $\mathfrak{gr}$ and a characteristic from van Douwen's diagram about which we do not know whether can be consistently true are $\mathfrak{nik} > \mathfrak{i}$ and $\mathfrak{gr} > \mathfrak{i}$—note that the reverse strict inequalities hold in the Silver model (Corollary 6.7).

Question 8.7. *Does any of the inequalities $\mathfrak{nik} > \mathfrak{i}$ and $\mathfrak{gr} > \mathfrak{i}$ consistently hold?*

Concerning Cichoń's diagram the only strict inequalities that are unknown to hold consistently are listed in the following question.

Question 8.8. *Let $\mathfrak{x} \in \{\mathfrak{nik}, \mathfrak{gr}\}$. Does any of the following inequalities consistently hold:*

- $\mathfrak{x} < \operatorname{cov}(\mathcal{N})$,
- $\mathfrak{x} < \operatorname{non}(\mathcal{M})$,
- $\mathfrak{x} < \operatorname{non}(\mathcal{N})$,
- $\operatorname{cof}(\mathcal{M}) < \mathfrak{x}$?

Having analyzed the proof of Theorem 6.3, it seems that the theorem holds not only for ground model σ-complete Boolean algebras but also for algebras with the SCP (Definition 2.1). However, we do not know whether a similar theorem could be proved for algebras with some weaker property mentioned in Section 2.

Question 8.9. *In Theorem 6.3, can we exchange the σ-completeness with some weaker property and still preserve either the Nikodym property or the Grothendieck property?*

The last question concerns weakening the assumptions of Theorems 5.1 and 5.4.

Question 8.10. *In Theorems 5.1 and 5.4, can we drop the assumption that $\kappa = \operatorname{cof}\left([\kappa]^\omega, \subseteq\right)$?*

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