Devil's infinite chessboard puzzle under a weaker choice principle

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1 Devil's chessboard puzzle

As a usual convention in set theory, we identify a natural number n and the set $\{0, \ldots, n-1\}$ of natural numbers less than n, and ω denotes the set of all natural numbers. For a set S, ${}^S2$ denotes the set of all functions from S to $2 = \{0, 1\}$, whereas by 2^n we will mean the usual arithmetic exponentiation. We will often regard the set $2 = \{0, 1\}$ as the two-element cyclic group $\mathbb{Z}_2 = (\mathbb{Z}/2\mathbb{Z}, +)$, and for $f, g \in {}^S2$, $f + g$ denotes the usual coordinatewise addition in ${}^S(\mathbb{Z}_2)$.

Devil's chessboard puzzle, also known as *life or death problem*, is a mathematical puzzle which can be formulated as follows. Fix a natural number $b \in \omega$. Alice wants to send Bob a b-bit message $m \in {}^b2$ under the following conditions:

(1) The only medium available to Alice is a given 2^b-bit sequence $\sigma \in {}^{({}^b2)}2$ which Bob cannot see.
(2) Alice is allowed only to flip (change 0 to 1 or the other way round) exactly one place of the sequence σ and to send Bob the resulting sequence.
(3) Alice and Bob can share a strategy in advance (before Alice sees σ).

The question is to find a strategy with which Alice can successfully send Bob a message. The word "chessboard" comes from the special case when $b = 6$ (and hence $2^b = 64 = 8 \times 8$). It is known that there *is* such a strategy for each $b \in \omega$ (folklore; see [1] for example).

In the present paper we will generalize this question to infinity.

First, we just put any cardinal κ (either finite or infinite) into b, that is, Alice sends Bob a function $\mu \in {}^\kappa 2$ using a given function $\sigma \in {}^{({}^\kappa 2)}2$ as a medium. We will employ the concept of *parity functions*, which was suggested by Geschke, Lubarsky

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and Rahn [2], to generalize a standard strategy for a finite case to infinite cases.

Second, we replace ${}^{b}2$ by ω, that is, we consider the situation that Alice sends Bob a natural number $m \in \omega$ using a given $\sigma \in {}^{\omega}2$ as a medium.

2 Parity function

Geschke, Lubarsky and Rahn [2] introduced a notion of *parity functions* to investigate "infinite hat guessing games". We say, for a set S, a function p from ${}^{S}2$ to 2 is a *parity function on* S if it has the following property.

For $f, g \in {}^{S}2$, if $f(x) \neq g(x)$ holds for exactly one $x \in S$, then $p(f) \neq p(g)$.

Clearly, if S is finite, then the function p determined by $p(f) = \sum_{x \in S} f(x)$, where $\sum$ is taken in $\mathbb{Z}_2$, is a parity function on S. On the other hand, the existence of a parity function p on ω cannot be proved under ZF alone, since the set $p^{-1}(\{1\}) \subseteq {}^{\omega}2$ would be Lebesgue nonmeasurable and fail to have the Baire property [2, Theorem 10].

The following theorem assures the existence of a parity function on ω under AC.

Theorem 2.1. [2, Lemma 6] *There is a parity function p on ω.*

The following proof, which is called "the E_0-transversal proof" in [2], is essentially the proof of Lenstra's theorem presented in [3]. What we actually need in the proof is a selection of representatives of the quotient set $2^{\omega}/E_0$, where E_0 denotes the equality modulo finitely many places. We may regard the existence of a set of representatives of $2^{\omega}/E_0$ as a weaker choice principle. See [2, Section 3] for more information.

Proof. Let A be a set of representatives for the quotient set $2^{\omega}/E_0$. Define a function p from ${}^{\omega}2$ to 2 in the following way. For $s \in {}^{\omega}2$, let t be the unique element of A with $s \, E_0 \, t$, and let $p(s) = 1$ if $|\{n \in \omega : s(n) \neq t(n)\}|$ is an odd number and $p(s) = 0$ otherwise. It is easily checked that this p works. □

It is easy to generalize the theorem above to the one asserting the existence of a parity function on λ for any infinite cardinal λ.

3 Strategies

This section is devoted to the construction of successful strategies in Devil's infinite chessboard puzzles.

Let κ be a cardinal, either finite or infinite, and we deal with the case when Alice sends Bob a message $\mu \in {}^{\kappa}2$ using a $\sigma \in {}^{({}^{\kappa}2)}2$ as a medium. We call such a puzzle a ${}^{\kappa}2$-*chessboard puzzle.*

Theorem 3.1. *For any cardinal κ, there is a successful strategy for a ${}^{\kappa}2$-chessboard puzzle.*

Proof. Fix a cardinal κ and a parity function p on ${}^{\kappa}2$. For a function $\tau \in {}^{({}^{\kappa}2)}2$, we define a function $\pi_\tau \in {}^{\kappa}2$ in the following way. For each $\alpha \in \kappa$, define $[\![\tau]\!]_\alpha \in {}^{({}^{\kappa}2)}2$ by letting, for each $\eta \in {}^{\kappa}2$,

$$[\![\tau]\!]_\alpha(\eta) = \begin{cases} \tau(\eta) & \text{if } \eta(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then define $\pi_\tau \in {}^{\kappa}2$ by letting $\pi_\tau(\alpha) = p([\![\tau]\!]_\alpha)$ for each $\alpha \in \kappa$. Observe that, if two functions $\tau, \tau' \in {}^{({}^{\kappa}2)}2$ take different values only at one point $\zeta \in {}^{\kappa}2$, then $\pi_\tau(\alpha) \neq \pi_{\tau'}(\alpha)$ if and only if $\zeta(\alpha) = 1$. This property will help Alice find the right place to flip.

Suppose that Alice has a medium $\sigma \in {}^{({}^{\kappa}2)}2$ and wants to send Bob a message $\mu \in {}^{\kappa}2$.

Let $\zeta_{\sigma,\mu} = \pi_\sigma + \mu$, and σ_μ be the function which is obtained from σ by flipping the value at $\zeta_{\sigma,\mu}$. By the observation, we have $\pi_{\sigma_\mu}(\alpha) = \mu(\alpha)$ for all $\alpha \in \kappa$.

Therefore, the following strategy is successful: Alice and Bob share a parity function p on 2^κ in advance. Alice calculates σ_μ and send it to Bob, and Bob regains μ by calculating $\pi_{\sigma_\mu}(\alpha)$ for all $\alpha \in \kappa$. □

Now we turn to the case when Alice sends Bob a message $m \in \omega$ using a medium $\sigma \in {}^{\omega}2$. We call this an *ω-chessboard puzzle.*

Theorem 3.2. *There is a successful strategy for an ω-chessboard puzzle.*

We will present two proofs. The first proof, due to Shohei Tajiri (in a private communication), uses a selection of representatives of the quotient set $2^\omega/E_0$. The second proof only uses a parity function on ω.

First proof. In the beginning Alice and Bob share a set A of representatives of $2^\omega/E_0$.

For $f, g \in {}^{\omega}2$ with $f\, E_0\, g$, let $N(f,g) = \min\{N < \omega : f(n) = g(n) \text{ for all } n \geq N\}$.

Suppose that Alice has a message $m \in \omega$ and a given medium $\sigma \in {}^{\omega}2$. Find the unique $h \in A$ with $h\, E_0\, \sigma$, and $l = N(h,\sigma)$. Let $\tilde{\sigma}$ is the function obtained from σ by flipping the value at $N + m$. Note that $N(h, \tilde{\sigma}) = N + m + 1$. Alice sends Bob the function $\tilde{\sigma}$.

Now Bob can decode the message m from $\tilde{\sigma}$ in the following way. Find the unique $h' \in A$ with $h'\, E_0\, \tilde{\sigma}$. Let $l' = N(h', \tilde{\sigma})$. Clearly $h' = h$, and hence Bob can regain σ from $\tilde{\sigma}$ by flipping the value at $l' - 1$, and also find $l = N(h,\sigma)$. Finally Bob obtains $m = (l' - 1) - l$. □

For the second proof we employ the binary expression of natural numbers. For $f \in {}^{\omega}2$ such that $f^{-1}(\{1\})$ is finite, we define $\sharp(f) = \sum_{i\in\omega} f(i)2^i$. For the other way round, for each $n \in \omega$, $\langle n \rangle$ denotes the unique $f \in {}^{\omega}2$ with $n = \sharp(f)$, $\langle n \rangle_i = f(i)$ for each i, and $\mathrm{lh}(n) = \min\{N \in \omega : f^{-1}(\{1\}) \subseteq N\}$.

Second proof. In the beginning Alice and Bob share a parity function p on ω. We set an encoding and decoding scheme which is similar to the one in the proof of the preceding theorem. For a function $\tau \in {}^{\omega}2$, we define a function $\pi_\tau \in {}^{\omega}2$ in the following way. For each $a \in \omega$, define $[\![\tau]\!]_a \in {}^{\omega}2$ by letting, for each $k \in \omega$,

$$[\![\tau]\!]_a(k) = \begin{cases} \tau(k) & \text{if } \langle k \rangle_a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then define $\pi_\tau \in {}^{\omega}2$ by letting $\pi_\tau(a) = p([\![\tau]\!]_a)$ for each $a \in \omega$. If two functions $\tau, \tau' \in {}^{\omega}2$ disagree only at one point $z \in \omega$, then $\pi_\tau(a) \neq \pi_{\tau'}(a)$ if and only if $\langle z \rangle_a = 1$. Note that flipping the value of τ at $z \in \omega$ does not affect values of π_τ at $n \geq \mathrm{lh}(z)$.

Suppose that Alice has a message $m \in \omega$ and a given medium $\sigma \in {}^{\omega}2$. Let $N_m = \mathrm{lh}(m)$ and $\tilde{m} = (2m+1) \cdot 2^{N_m}$. Note that

$$\langle \tilde{m} \rangle_i = \begin{cases} 0 & \text{if } i < N_m, \\ 1 & \text{if } i = N_m, \\ \langle m \rangle_{i-(N_m+1)} & \text{if } N_m + 1 \leq i < 2N_m + 1, \\ 0 & \text{if } 2N_m + 1 \leq i. \end{cases}$$

Alice will embed $\tilde{m}$ into σ in a similar, but slightly different, way as in the proof of the preceding theorem.

Define a function $z_{\sigma,m} \in {}^{\omega}2$ by

$$z_{\sigma,m}(i) = \begin{cases} \pi_\sigma(i) + \langle \tilde{m} \rangle_i & \text{if } i < 2N_m + 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $+$ is calculated in $\mathbb{Z}_2$. Let $\sigma_m \in {}^{\omega}2$ be the one obtained from σ by flipping the value at $\sharp(z_{\sigma,m})$. Alice sends Bob the function σ_m.

Bob calculates $m_a = \pi_{\sigma_m}(a)$ for all $a \in \omega$ and regains $N_m = \min\{a \in \omega : m_a = 1\}$. Then Bob obtains the message m by calculating

$$\sum_{i=N_m+1}^{2N_m} m_i 2^{i-(N_m+1)},$$

which concludes the proof. □

It seems natural to ask, for an infinite cardinal λ, if there is a successful strategy when Alice wants to send Bob a message $\mu \in \lambda$ using a given $\sigma \in {}^{\lambda}2$ as a medium. Theorem 3.1 applies in the case when $\lambda = 2^\kappa$ holds for some cardinal κ. Also, when $2^{<\lambda} = \lambda$ holds, it is not so hard to modify Theorem 3.2 to fit in this case. AC will be used only to ensure the existence a parity function on λ, and Alice and Bob will share two bijections: one is $\psi : \lambda \to {}^{<\lambda}2$, and the other is $\varphi : \kappa \times 2 \to \kappa$ such that, for any $\beta < \kappa$, $\varphi``(\beta \times 2)$ is bounded in κ. Details are left to the reader as an exercise.

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