TODORČEVIĆ'S AXIOM $\mathcal{K}_2$ AND LADDER SYSTEM COLORINGS

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In this article, it is proved that if every c.c.c. partition $K \subseteq [\omega_1]^2$ has an uncountable homogeneous set, then every ladder system coloring on ω_1 can be σ-uniformized. This improves a previous result of the second author [14].

1. Introduction

In [11], Todorčević and Veličković showed that $\mathsf{MA}_{\aleph_1}$ is equivalent to a Ramsey-theoretic assertion about partial orders. This led to the study of related but *a priori* weaker Ramsey theoretic assertions $\mathcal{K}_n$. Recall that if $K \subseteq [\omega_1]^n$, then a set $H \subseteq \omega_1$ is *K-homogeneous* if $[H]^n \subseteq K$. The axiom $\mathcal{K}_n$ is the assertion that if $K \subseteq [\omega_1]^n$, then either there is an uncountable K-homogenous set or else there is an uncountable collection of finite K-homogenous sets, the union of any two of which is not K-homogeneous.[*1]

All of these axioms are consequences of $\mathsf{MA}_{\aleph_1}$ and for all $n \geq 2$, $\mathcal{K}_{n+1}$ implies $\mathcal{K}_n$. It is a longstanding open problem whether any of these implications can be reversed. Many of the consequences of $\mathsf{MA}_{\aleph_1}$ are known to be consequences of $\mathcal{K}_n$ for some n [4], [5], [7, §7] [8] [11]. The purpose of this report is to establish the uniformization property of ladder system colorings using the weakest of these axioms, $\mathcal{K}_2$.

Recall that a *ladder system on $E \subseteq \omega_1 \cap \mathsf{Lim}$* is a sequence $\vec{C} = \langle C_\alpha : \alpha \in E\rangle$ such that, for each $\alpha \in E$, C_α is an unbounded subset of α and the order type of C_α is ω. A *coloring* of a ladder system $\langle C_\alpha : \alpha \in E\rangle$ is a sequence $\vec{f} = \langle f_\alpha : \alpha \in E\rangle$ such that, for each $\alpha \in E$, f_α is a function from C_α into ω.

If $\vec{f} = \langle f_\alpha : \alpha \in E\rangle$ is a coloring of a ladder system $\vec{C}$, a function φ from ω_1 into ω *uniformizes* $\vec{f}$ if for every $\alpha \in E$, f_α and $\varphi \restriction C_\alpha$ are *almost equal* — that is, the set

$$\{\xi \in C_\alpha : f_\alpha(\xi) \neq \varphi(\xi)\}$$

is finite.

For a subset $\mathcal{S}$ of the power set of $\omega_1 \cap \mathsf{Lim}$, $\mathsf{U}(\mathcal{S})$ is the assertion that, for any coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$, there exist $S \in \mathcal{S}$ and a function from ω_1 into ω which uniformizes the restricted coloring $\langle f_\alpha : \alpha \in S\rangle$. If $\mathcal{S} = \{\omega_1 \cap \mathsf{Lim}\}$, we will write U for $\mathsf{U}(\mathcal{S})$.

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[*1]The notation $\mathcal{K}_n$ is sometimes used to denote the formally stronger hypothesis that every c.c.c. partial order has Property K_n. While it is asserted in [11] that this is equivalent to the above assertion about partitions, it is an open problem whether this equivalence holds in ZFC (if the countable chain condition is productive, they are equivalent). When there is a need to draw a distinction, the notation $\mathcal{K}'_n$ is sometimes used for the weaker statement about partitions, as it is in [12, 13, 14].

Finally, σ-U is the assertion that, for any coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$, there exists $\bigcup_{n\in\omega} J_n = \omega_1 \cap \mathsf{Lim}$ such that, for each $n \in \omega$, $\langle f_\alpha : \alpha \in J_n\rangle$ can be uniformized.

In [1], Devlin and Shelah introduced U, and proved it implies both that $2^{\aleph_0} = 2^{\aleph_1}$ and that there is a non-free Whitehaed group of cardinality $\aleph_1$. Moreover, Eklof and Shelah showed that the existence of a non-free Whitehead group of cardinality $\aleph_1$ is equivalent to the existence of a ladder system $\vec{C} = \langle C_\xi : \xi \in E\rangle$ indexed by a stationary set $E \subseteq \omega_1$ such that every coloring of $\vec{C}$ can be uniformized [2, Ch. XIII] (see [3, §6]). In [14], it is proved that $\mathcal{K}_4$ implies $\mathsf{U}(\mathsf{club})$, and $\mathcal{K}_3$ implies $\mathsf{U}(\mathsf{stat})$.

2. $\mathcal{K}_2$ implies σ-U

We will now prove that $\mathcal{K}_2$ implies σ-U. The proof is closely related to Todorčević's proof that $\mathcal{K}_2$ implies that all Aronszajn trees are special [8]. Fix a ladder system $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ for the remainder of the proof. Fix a sequence $\vec{e} = \langle e_\alpha : \alpha \in \omega_1\rangle$ such that:

- for each $\alpha \in \omega_1$, $e_\alpha : \alpha \to \omega$ is an injective function and
- for each $\alpha, \beta \in \omega_1$ with $\alpha < \beta$, the set
$$\{\xi \in \alpha : e_\beta(\xi) \neq e_\alpha(\xi)\}$$
is finite.

Such a sequence can be defined explicitly from $\vec{C}$ — see [6, 9, 10]. Let $r_\alpha \in {}^\omega 2$ denote the characteristic function of the range of e_α. We notice that, for any $\alpha, \beta \in \omega_1$, if $\alpha + \omega \leq \beta$, then $\Delta(r_\alpha, r_\beta) < \omega$. For each $r, s \in {}^\omega 2$, define
$$\Delta(r, s) := \min\{n \in \omega : r(n) = s(n)\}.$$

For each $\delta \in \omega_1 \cap \mathsf{Lim}$ and $\beta \in \omega_1$ with $\delta < \beta$, define $I(\delta, \beta)$ to be the open interval (δ', δ) where δ' is the least ordinal below δ such that $e_\beta(\xi) > e_\beta(\delta)$ for every ξ in the open interval (δ', δ).

Now suppose that $\vec{f} = \langle f_\alpha \mid \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ is a coloring of $\vec{C}$. Define $K_{\vec{f}} \subseteq [\omega_1]^2$ to consist of all $\{\alpha, \beta\}$ such that $\Delta(r_\alpha, r_\beta) < \omega$, and, whenever $\gamma \in \alpha$ and $\delta \in \beta$ are limit ordinals and $e_\alpha(\gamma) = e_\beta(\delta) < \Delta(r_\alpha, r_\beta)$, then $(f_\gamma \restriction I(\gamma, \alpha)) \cup (f_\delta \restriction I(\delta, \beta))$ is a function. We are finished once we prove the following claims.

Proposition 2.1. *If there is an uncountable $H \subseteq \omega_1$ such that $[H]^2 \subseteq K_{\vec{f}}$, then $\vec{f}$ has a σ-uniformization.*

Proof. Suppose that H is an uncountable K-homogeneous subset of ω_1. For each $s \in {}^{<\omega}2$ and $n < |s|$, define $J_{s,n}$ to be the set of limit ordinals δ in ω_1 such that there exists $\beta \in H$ such that $s \subseteq r_\beta$ and $e_\beta(\delta) = n$. Observe that $\bigcup_{s\in {}^{<\omega}2} \bigcup_{n<|s|} J_{s,n}$ is all limit ordinals in ω_1. For each $s \in {}^{<\omega}2$ and $n < |s|$, let $\varphi_{s,n}$ be the union of functions of the form $f_\delta \restriction I(\delta, \beta)$ such that $\beta \in H$, $\delta \in \beta \cap \mathsf{Lim}$, $s \subseteq r_\beta$, and $e_\beta(\delta) = n$. Since H is homogeneous, $\varphi_{s,n}$ is a function. Clearly $\varphi_{s,n}$ uniformizes $\vec{f}$ on $J_{s,n}$. □

Proposition 2.2. *For every $\vec{f}$, $K_{\vec{f}}$ is c.c.c..*

Proof. Let X be an uncountable set of finite $K_{\vec{f}}$-homogeneous sets. By performing a Δ-system argument and removing the root, we may assume that X consists of pairwise disjoint sets of uniform cardinality n. If $x \in X$ and $i < |x|$, let $x(i)$ denote the ith least element of x.

Take a countable elementary submodel M of $H(2^{\aleph_1 +})$ such that $\vec{C}, \vec{e}, \vec{r}, X \in M$. Define $\varepsilon := \omega_1 \cap M$. Using the pigeonhole principle, find $a \neq a' \in X \setminus M$ such that:

- $\Delta(r_{a(i)}, r_{a(j)}) < l$ whenever $i, j < n$,
- $\Delta(r_{a(i)}, r_{a'(i)}) \geq l$ whenever $i < n$,
- $e_{a(i)}(\varepsilon) = e_{a'(i)}(\varepsilon) < l$ whenever $i < n$, and
- if $i < n$, $\gamma < a(i)$, $\gamma' < a'(i)$ and $e_{a(i)}(\gamma) = e_{a'(i)}(\gamma') < e_{a(i)}(\varepsilon)$, then $f_\gamma \restriction \varepsilon = f_{\gamma'} \restriction \varepsilon$.

Fix an m such that $\Delta(r_{a(i)}, r_{a'(i)}) < m$ for all $i < n$. Let $\bar{\varepsilon} < \varepsilon$ be such that for all $i < n$ and limit ordinals $\gamma < a(i)$:

- if $\gamma < \varepsilon$ and $e_{a(i)}(\gamma) < m$, then $\gamma < \bar{\varepsilon}$ and
- if $\gamma > \varepsilon$ and $e_{a(i)}(\gamma) < m$, then $C_\gamma \cap \varepsilon < \bar{\varepsilon}$.

Let N be a countable elementary submodel of $H(\aleph_2)$ with $N \in M$ and $\vec{C}, \vec{e}, \vec{r}, X, \bar{\varepsilon} \in N$. By elementarity of N, there is a $b \in X \cap N$ such that for all $i < n$:

- $\Delta(r_{a'(i)}, r_{b(i)}) \geq m$,
- if $\gamma < \varepsilon$ and $e_{a'(i)}(\gamma) < m$, then $e_{a'(i)}(\gamma) = e_{b(i)}(\gamma)$,
- if $\varepsilon < \gamma < a'(i)$, $\delta < b(i)$, and $e_{a'(i)}(\gamma) = e_{b(i)}(\delta) < m$, then $f_\gamma \restriction \bar{\varepsilon} = f_\delta \restriction \bar{\varepsilon}$.
- if $\delta < b(i)$ with $e_{b(i)}(\delta) = e_{a'(i)}(\varepsilon)$, then $C_\varepsilon \cap N$ is an initial part of C_δ and $f_\varepsilon \restriction N$ is a restriction of f_δ.

Notice that this implies in particular that whenever $i < n$:

(1) $l \leq \Delta(r_{a(i)}, r_{b(i)}) < m$,
(2) if $\gamma < a(i)$, $\delta < b(i)$ and $e_{a(i)}(\gamma) = e_{b(i)}(\delta) \leq e_{a(i)}(\varepsilon)$, $f_\gamma \cup f_\delta$ is a function.

We claim that $a \cup b$ is $K_{\vec{f}}$-homogeneous. Toward this end, suppose that $i, j < n$, $\gamma < a(i)$, $\delta < b(j)$ and $e_{a(i)}(\gamma) = e_{b(j)}(\delta) < \Delta(r_{a(i)}, r_{b(j)})$.

If $i \neq j$, then by (1), $\Delta(r_{a(i)}, r_{b(j)}) = \Delta(r_{a(i)}, r_{a(j)}) < l$. In particular, $e_{a(i)}(\gamma) = e_{b(j)}(\delta) < l$. Since $r_{a(j)}$ and $r_{b(j)}$ are the characteristic function of the ranges of $e_{a(j)}$ and $e_{b(j)}$, respectively, and $\Delta(r_{a(j)}, r_{b(j)}) \geq l$, we also have that there is a $\gamma' < a(j)$ such that $e_{a(j)}(\gamma') = e_{b(j)}(\delta)$. By our choices of $\bar{\varepsilon}$ and b,

$$f_{\gamma'} \restriction \varepsilon = f_{\gamma'} \restriction \bar{\varepsilon} = f_\delta \restriction \bar{\varepsilon}.$$

Since $[a]^2 \subseteq K_{\vec{f}}$, we know that

$$(f_\gamma \restriction I(\gamma, a(i))) \cup (f_{\gamma'} \restriction I(\gamma', a(j)))$$

is a function. Since $C_\gamma \cap C_\delta$ is contained in $\bar{\varepsilon}$, we have that $C_\gamma \cap C_\delta = C_\gamma \cap C_{\gamma'}$. Since $f_{\gamma'} \restriction C_\gamma \cap C_\delta = f_\delta \restriction C_\gamma \cap C_\delta$,

$$(f_\gamma \restriction I(\gamma, a(i))) \cup (f_\delta \restriction I(\delta, b(j)))$$

is a function. This concludes the case $i \neq j$.

Next suppose that $i = j$. If $e_{a(i)}(\gamma) = e_{b(i)}(\delta) \leq e_{a(i)}(\varepsilon)$, then by our observation (2), $f_\gamma \cup f_\delta$ is a function and in particular

$$f_\gamma \restriction I(\gamma, a(i)) \cup f_\delta \restriction I(\delta, b(i))$$

is a function. The remaining possibility to consider is that

$$e_{a(i)}(\varepsilon) < e_{a(i)}(\gamma) = e_{b(i)}(\delta) < \Delta(r_{a(i)}, r_{b(i)}) < m.$$

This implies that either $\gamma = \delta < \bar{\varepsilon}$ or else $\varepsilon < \gamma$. In the former case, $f_\gamma = f_\delta$. In the later case, $\varepsilon < I(\gamma, a(i))$ and hence $I(\gamma, a(i))$ is disjoint from $I(\delta, b(i))$. In either case

$$f_\gamma \restriction I(\gamma, a(i)) \cup f_\delta \restriction I(\delta, b(i))$$

is a function. □

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