THE $\Pi^1_1\downarrow$ LÖWENHEIM-SKOLEM-TARSKI PROPERTY OF STATIONARY LOGIC

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Abstract. Fuchino-Maschio-Sakai [7] proved that the Löwenheim-Skolem-Tarski (LST) property of Stationary Logic is equivalent to the Diagonal Reflection Principle on internally club sets (DRP_{IC}) introduced in [4]. We prove that the restriction of the LST property to (downward) reflection of Π^1_1 formulas, which we call the $\Pi^1_1\downarrow$-LST property, is equivalent to the *internal* version of DRP from [2]. Combined with results from [2], this shows that the $\Pi^1_1\downarrow$-LST Property for Stationary Logic is strictly weaker than the full LST Property for Stationary Logic, though if CH holds they are equivalent.

Contents

1. Introduction

Stationary Logic is a relatively well-behaved fragment of Second Order Logic introduced by Shelah [12], and first investigated in detail by Barwise et al [1]. Stationary Logic augments first order logic by introducing a new second order quantifier *stat*; we typically interpret "$\text{stat}Z\ \phi(Z,\dots)$" to mean that there are stationarily many countable Z such that $\phi(Z,\dots)$ holds.[1] The quantifier *aa* stands for "almost all" or "for club many"; so

$$\text{aa}Z\ \phi(Z,\dots)$$

is an abbreviation for

$$\neg\ \text{stat}Z\ \neg\ \phi(Z,\dots).$$

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[1]Other interpretations, e.g. for uncountable Z, or for filters other than the club filter, are often considered too.

Section 2 provides more details.

By *structure* we will always mean a first order structure in a countable signature. The question of whether every structure has a "small" elementary substructure in Stationary Logic was raised already in [1]. One cannot hope to always get countable elementary substructures; e.g. if κ is regular and uncountable, then $(\kappa, \in)$ satisfies "$\in$ is a linear order and

$$\text{aa}Z\ \exists x\ x \text{ is an upper bound of } Z\text{"},$$

but no countable linear order can satisfy that sentence. In a footnote in [1], it was observed that even the statement

(LST) "Every structure has an elementary (w.r.t. Stationary Logic) substructure of size $\leq \omega_1$"

carries large cardinal consistency strength.[2] The quoted statement above is now typically called the *Löwenheim-Skolem-Tarski* (LST) property of Stationary Logic.[3]

Fuchino et al. recently proved that LST is equivalent to a version of the Diagonal Reflection Principle introduced in Cox [4]:

Theorem 1.1 (Fuchino-Maschio-Sakai [7])**.** *LST is equivalent to the **Diagonal Reflection Principle on internally club sets** (DRP_{IC}).*

The purpose of the present note is to prove the following variant of Theorem 1.1 involving Π^1_1 formulas in Stationary Logic (defined in Section 2 below) and the principle $\text{DRP}_{\text{internal}}$ from [2]:

Theorem 1.2. *The $\Pi^1_1{\downarrow}$-LST property of Stationary Logic (see Definition 2.2) is equivalent to the principle $DRP_{internal}$.*

Cox [2] proved that DRP_{IC} is strictly stronger than $\text{DRP}_{\text{internal}}$. This was obtained by forcing over a model of a strong forcing axiom in a way that preserved $\text{DRP}_{\text{internal}}$ while killing DRP_{IC} (in fact killing RP_{IC}; the argument owed much to Krueger [10]). Furthermore, if CH holds, then DRP_{IC} is equivalent to $\text{DRP}_{\text{internal}}$. Combining those results with Theorem 1.2 immediately yields:

Corollary 1.3. *The LST property of Stationary Logic is **strictly** stronger than the $\Pi^1_1{\downarrow}$-LST property of Stationary Logic.*

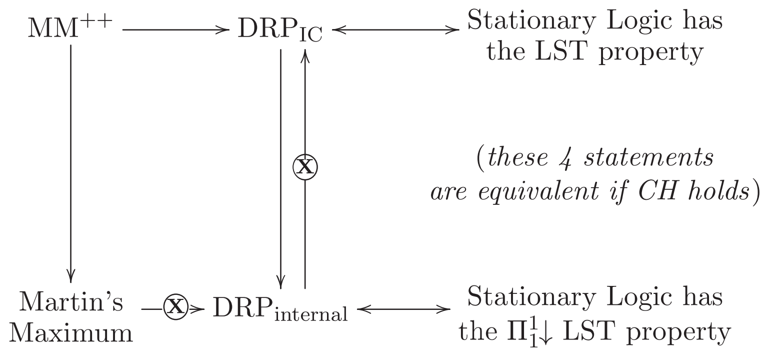
However, if the Continuum Hypothesis holds, they are equivalent.[4]

[2]See Definition 2.2 for precisely what is meant by "elementary substructure" in this context.

[3]The weaker assertion that every consistent theory (in Stationary Logic) has a model of size ω_1, on the other hand, is a theorem of ZFC, as proven in [1].

[4]One doesn't actually need the full continuum hypothesis for this equivalence to hold, but rather a variant of Shelah's Approachability Property, namely that the class of internally stationary sets is the same (mod NS) as the class of internally club sets. See Cox [2] for more details.

Figure 1. An arrow indicates an implication, an arrow with an X indicates a non-implication



We note that while the technical strengthening MM^{++} of Martin's Maximum implies DRP$_{\mathrm{IC}}$ (see [4]), recent work of Cox-Sakai [6] shows that Martin's Maximum alone does not imply even the weakest version of DRP. Figure 1 summarizes the relevant implications and non-implications discussed in this introduction.

Section 2 covers the relevant preliminaries, and Section 3 proves Theorem 1.2. Section 4 ends with some concluding remarks.

2. Preliminaries

Recall that $S \subseteq [A]^\omega$ is stationary if it meets every closed, unbounded subset of $[A]^\omega$ (in the sense of Jech [9]). By Kueker [11] this is equivalent to requiring that for every $f : [A]^{<\omega} \to A$ there is an element of S that is closed under f.

In what follows, we will use uppercase letters to denote second order variables/parameters, and lowercase letters to denote first order variables/parameters. We will also use some standard abbreviations; e.g. if our language includes the $\in$ symbol, v is a first order variable, and Z is a second order variable, "$v = Z$" is short for

$$\forall x \; x \in v \iff Z(x).$$

Given a structure $\mathfrak{A} = (A, \dots)$ (which we always assume to have a countable signature), the satisfaction relation in Stationary Logic is defined recursively by:

$$\mathfrak{A} \models \ \mathrm{stat} Z \ \phi(Z, U_1, \dots, U_\ell, p_1, \dots, p_k)$$

$$\iff$$

$$\big\{ Z \in [A]^\omega \ : \ \mathfrak{A} \models \ \phi(Z, U_1, \dots, U_\ell, p_1, \dots, p_k) \big\} \text{ is stationary in } [A]^\omega.$$

We define a hierarchy of formulas in Stationary Logic that mimics the usual hierarchy in Second Order Logic. Since

$$\mathrm{aa}Z\ \phi(Z,\dots)$$

roughly translates as

$$\exists C\ \ C \text{ is club and } \forall Z \in C\ \phi(Z,\dots),$$

the *aa* quantifier will correspond to the existential second order quantifier when constructing the hierarchy. Similarly, since

$$\mathrm{stat}Z\ \phi(Z,\dots)$$

roughly translates as

$$\forall C\ \ C \text{ is club } \implies \exists Z \in C\ \phi(Z,\dots),$$

the *stat* quantifier will correspond to the universal second-order quantifier.

Definition 2.1. *A formula in Stationary Logic without second order quantifiers will be denoted by* $\mathbf{\Sigma^1_0}$ *or* $\mathbf{\Pi^1_0}$. *For* $n > 0$, *a formula of the form*

$$statZ_1\ \dots\ statZ_k\ \ \phi(Z_1,\dots,Z_k,\dots)$$

where ϕ *is* Σ^1_{n-1} *will be called a* $\mathbf{\Pi^1_n}$ *formula, and a formula of the form*

$$aaZ_1\ \dots\ aaZ_k\ \ \psi(Z_1,\dots,Z_k,\dots)$$

where ψ *is* Π^1_{n-1} *will be called a* $\mathbf{\Sigma^1_n}$ *formula.*

For example, if $\phi(Z_0, Z_1, v_1,\dots,v_\ell)$ has no *stat* or *aa* quantifers, then

$$\mathrm{stat}Z_0\ \ \mathrm{aa}Z_1\ \ \phi(Z_0,Z_1,v_1,\dots,v_\ell)$$

is a Π^1_2 formula.

Definition 2.2. *We say that the* ***LST property holds for Stationary Logic*** *iff for every structure* $\mathfrak{A} = (A,\dots)$[5] *there exists a* $W \subseteq A$ *of size* $\le \omega_1$ *such that for all formulas* ϕ *in Stationary Logic with no free occurrences of second order variables, and all first order parameters* $p_1,\dots,p_k \in W$,

$$\mathfrak{A} \models \phi[\vec{p}] \ \ \textit{if and only if}\ \mathfrak{A}|W \models \phi[\vec{p}].$$

We say that the $\mathbf{\Pi^1_1}\!\!\downarrow$ ***LST property holds for Stationary Logic*** *iff for every structure* $\mathfrak{A} = (A,\dots)$ *there exists a* $W \subseteq A$ *of size* $\le \omega_1$ *such that for all* Π^1_1 *formulas* ϕ *in Stationary Logic with no free occurrences of second order variables, and all first order parameters* $p_1,\dots,p_k \in W$,

$$\textbf{\textit{if}}\ \mathfrak{A} \models \phi[\vec{p}],\ \textbf{\textit{then}}\ \mathfrak{A}|W \models \phi[\vec{p}].$$

Remark 2.3. *Note that in the definition of the* $\Pi^1_1\!\!\downarrow$ *LST property, we only require that* Π^1_1 *formulas reflect* ***downward****. If there is always an* ω_1 *sized substructure that reflects* Π^1_1 *formulas both upward and downward, then the full LST property holds. This issue is discussed further in Section 4.*

[5]Recall we always assume countable signature, though for everything discussed in this paper an ω_1-sized signature would still be fine.

We consider variants of the **Diagonal Reflection Principle** introduced in Cox [4] and [2]. We use the following definition, which by Cox-Fuchs [5] is equivalent to the definitions from [4] and [2]:

Definition 2.4. *$DRP_{internal}$ asserts that for every sufficiently large regular θ, there are stationarily many $W \in \wp_{\omega_2}(H_\theta)$ such that:*

- *$|W| = \omega_1 \subset W$; and*
- *Whenever $A \in W$ is uncountable and $S \in W$ is a stationary subset of $[A]^\omega$, the set $S \cap W \cap [W \cap A]^\omega$ is stationary in $[W \cap A]^\omega$.*

The "internal" part of the definition refers to the fact that we require that $S \cap W \cap [W \cap A]^\omega$ is stationary, not merely that $S \cap [W \cap A]^\omega$ is stationary. Definition 2.4 is simply the diagonal version of an internal variant of WRP introduced in Fuchino-Usuba [8] (see Cox [2] for a discussion).

3. Proof of Theorem 1.2

We prove a slightly stronger variant of Theorem 1.2. The proof below is strongly influenced by Fuchino et al [7].

Theorem 3.1. *The following are equivalent:*

(1) *$DRP_{internal}$.*

(2) *For every structure $\mathfrak{A} = (A, \dots)$, there is a $W \subseteq A$ of size at most ω_1 such that for every finite list $p_1, \dots, p_k \in W \cap A$ and every formula ϕ without 2nd order quantifiers,*

$$\Big(\mathfrak{A} \models \ stat Z \ \phi[Z, \vec{p}]\Big) \implies \Big(\mathfrak{A}|W \models stat Z \ \phi[Z, \vec{p}]\Big).$$

(3) *The $\Pi^1_1{\downarrow}$-LST property holds of Stationary Logic (as in Definition 2.2);*

(4) *For every structure $\mathfrak{A} = (A, \dots)$, there is a $W \subseteq A$ of size at most ω_1 such that for every formula ψ in 2nd order prenex form with no free occurrences of second order variables, and every finite list $p_1, \dots, p_k$ of elements of W, **if***

$$\mathfrak{A} \models \psi[\vec{p}]$$

then, letting $\hat{\psi}$ be the formula obtained from ψ by changing all aa *quantifiers to* stat *quantifiers,*

$$\mathfrak{A}|W \ \models \ \hat{\psi}[\vec{p}].$$

Before proving the theorem, we remark that in parts 2, 3, and 4 of Theorem 3.1, we only mentioned first order parameters from $W \cap A$. If the structure $\mathfrak{A}$ is sufficiently rich then it often makes sense to also speak of second-order parameters that are elements of W. But in general (e.g. when $\mathfrak{A}$ is a group) it is more natural to only speak of first order parameters from $W \cap A$.

Proof. (of Theorem 3.1): (4) trivially implies (3), since if ψ is represented as a prenex Π^1_1 formula, then $\hat{\psi} = \psi$ (because there are no *aa* quantifiers in the original formula at all). Similarly, (3) trivially implies (2) because if ϕ has no second order quantifiers,

$$\text{stat} Z \ \phi$$

is obviously a Π^1_1 formula.

To see that (2) implies (1), assume (2) and suppose θ is a regular cardinal $\geq \omega_2$. We need to find a $W \prec (H_\theta, \in)$ such that $|W| = \omega_1 \subset W$ and for every $s \in W$ that is a stationary collection of countable sets,

$$s \cap W \cap \left[W \cap \bigcup s\right]^\omega \text{ is stationary.}$$

Consider $\mathfrak{A} = (H_\theta, \in)$. Let $W \subset H_\theta$ be as in the statement of (2). Fix any $s \in W$ that is a stationary collection of countable sets. Then

$$\mathfrak{A} \models \ \text{stat} Z \ \exists p \ p = Z \cap \bigcup s \text{ and } p \in s$$

and hence, since $s \in W$ and the only second order quantifier in the (prenex) formula above is a *stat* quantifier,

$$\mathfrak{A}|W \ \models \ \text{stat} Z \ \exists p \ p = Z \cap \bigcup s \text{ and } p \in s.$$

Unravelling the definition of the satisfaction relation, this means that

$$\left\{Z \in [W]^\omega \ : \ Z \cap \bigcup s \in W \cap s\right\} \text{ is stationary in } [W]^\omega$$

and it follows that $W \cap s \cap \left[W \cap \bigcup s\right]^\omega$ is stationary in $\left[W \cap \bigcup s\right]^\omega$.

To see that $\omega_1 \subset W$, it suffices to show that $W \cap \omega_1$ is uncountable (since by first-order elementarity of W in $(H_\theta, \in)$, $W \cap \omega_1$ is transitive). Now

$$\mathfrak{A} \models \text{stat} Z \ \ \exists p \ \exists \alpha \ \ \big(p = Z \cap \omega_1, \ \alpha < \omega_1, \text{ and } \alpha \text{ is an upper bound of } p\big),$$

so by assumption on W, this statement is also satisfied by $\mathfrak{A}|W$ (note that the parameter ω_1 is an element of W because ω_1 is first-order definable in $\mathfrak{A}$ and W is at least first-order elementary in $\mathfrak{A}$). If $W \cap \omega_1$ were countable, say $W \cap \omega_1 = \delta < \omega_1$, it would follow that for stationarily many $Z \in W \cap [W]^\omega$, there is an $\alpha < W \cap \omega_1 = \delta$ such that α is an upper bound of $Z \cap \delta$. This would be a contradiction, since due to the countability of δ, the set of $Z \in [W]^\omega$ such that $\delta \subseteq Z$ is a club.

Finally, to prove that (1) implies (4): fix a structure $\mathfrak{A} = (A, \dots)$ and let θ be a sufficiently large regular cardinal with $\mathfrak{A} \in H_\theta$. By (1) there is a $W \prec (H_\theta, \in, \mathfrak{A})$ witnessing $\text{DRP}_{\text{internal}}$. We prove by induction on complexity of formulas ψ in 2nd order prenex form that if $p_1, \dots, p_k \in W \cap A$ and

$$\mathfrak{A} \models \psi[\vec{p}]$$

then, letting $\hat{\psi}$ be the result of replacing all *aa* quantifiers with *stat* quantifiers,

$$\mathfrak{A}|(W \cap A) \models \hat{\psi}[\vec{p}].$$

We actually need to inductively prove a slightly stronger statement: namely, that whenever ψ is a 2nd order prenex formula, $p_1, \ldots, p_k \in W \cap A$, and $Z_1, \ldots, Z_\ell \in W \cap [A]^\omega$,

$$\text{(1)} \qquad \mathfrak{A} \models \psi[\vec{Z}, \vec{p}] \implies \mathfrak{A}|(W \cap A) \models \hat{\psi}[\vec{Z}, \vec{p}].$$

So suppose

$$\text{(2)} \qquad \mathfrak{A} \models QZ\ \phi[Z, U_1, \ldots, U_k, p_1, \ldots, p_\ell]$$

where Q is either the *aa* or *stat* quantifier, $U_1, \ldots, U_k$ are each elements of $W \cap [A]^\omega$, $p_1, \ldots, p_\ell \in W \cap A$, and the inductive hypothesis holds of the formula ϕ.

Now regardless of whether Q is the *aa* or *stat* quantifier,

$$\widehat{QZ\ \phi} \equiv \text{stat}Z\ \hat{\phi}.$$

and by (2) (since the *aa* quantifier is stronger than the *stat* quantifier)

$$\mathfrak{A} \models \text{stat}Z\ \phi[Z, U_1, \ldots, U_k, p_1, \ldots, p_\ell].$$

Hence, by the definition of the stationary logic satisfaction relation,

$$s := \left\{ Z \in [A]^\omega\ :\ \mathfrak{A} \models \phi[Z, \vec{U}, \vec{p}\,] \right\} \text{ is stationary in } [A]^\omega.$$

Note that since $\vec{U}$, $\vec{p}$, ϕ, and $\mathfrak{A}$ are elements of W, it follows that $s \in W$. Since W is internally diagonally reflecting,

$$s \cap W \cap [W \cap A]^\omega \text{ is stationary in } [W \cap A]^\omega.$$

Consider for the moment an arbitrary $Z \in s \cap W \cap [W \cap A]^\omega$. Then

$$\mathfrak{A} \models \phi[Z, \vec{U}, \vec{p}]$$

and it follows by the induction hypothesis (and that Z, $\vec{U}$, and $\vec{p}$ are each elements of W) that:

$$\mathfrak{A}|(W \cap A) \models \hat{\phi}[Z, \vec{U}, \vec{p}].$$

Hence, we have shown that

$$s \cap W \cap [W \cap A]^\omega \subseteq \left\{ Z \in [W \cap A]^\omega\ :\ \mathfrak{A}|(W \cap A) \models \hat{\phi}[Z, \vec{U}, \vec{p}] \right\}.$$

Since the set on the left side is stationary, the set on the right side is too. So by the definition of the satisfaction relation,

$$\mathfrak{A}|(W \cap A) \models \text{stat}Z\ \hat{\phi}[Z, \vec{U}, \vec{p}\,].$$

This completes the proof of the (1) $\implies$ (4) direction.

□

4. Concluding remarks

We remark that it is straightforward to show, in ZFC alone, that:

Lemma 4.1. *For every structure $\mathfrak{A} = (A, \dots)$ there exists a $W \subseteq A$ of size at most ω_1 such that*

$$\mathfrak{A}|W \prec_{\downarrow}^{\Sigma^1_1} \mathfrak{A}$$

(i.e. such that Σ^1_1 formulas satisfied by $\mathfrak{A}$ are also satisfied by $\mathfrak{A}|W$).

In fact, if θ is a regular cardinal such that $\mathfrak{A} \in H_\theta$, and

$$W \prec_{\text{1st order}} (H_\theta, \in, \mathfrak{A})$$

is such that $|W| = \omega_1$ and

$$W \cap [W \cap A]^\omega \text{ contains a club in } [W \cap A]^\omega \tag{3}$$

(this always holds for stationarily many W, e.g. for those W that are internally approachable), then

$$\mathfrak{A}|(W \cap A) \prec_{\downarrow}^{\Sigma^1_1} \mathfrak{A}.$$

We briefly sketch the proof of the lemma; more details, and other related results, can be found in Cox [3]. One proves by induction on complexity of formulas, making use of (3), that if ϕ is Σ^1_1, $p_1, \dots, p_k \in W \cap A$, and $Z_1, \dots, Z_\ell \in W \cap [A]^\omega$, then

$$\textbf{if } \mathfrak{A} \models \phi[\vec{Z}, \vec{p}] \textbf{, then } \mathfrak{A}|(W \cap A) \models \phi[\vec{Z}, \vec{p}].$$

This was basically part of the proof from Fuchino et al [7] that $\mathrm{DRP}_{\mathrm{IC}}$ implied the LST for Stationary Logic. See [3] for some other related ZFC theorems.

So by Lemma 4.1 one can always get an ω_1 sized substructure that reflects all Σ^1_1 statements downward. And if $\mathrm{DRP}_{\mathrm{internal}}$ holds, one can *also* get an ω_1 sized substructure that reflects all Π^1_1 statements downward. But it is consistent that both of these are true, yet no *single* ω_1-sized substructure downward reflects all Π^1_1 *and* all Σ^1_1 statements. In particular, in any model where $\mathrm{DRP}_{\mathrm{internal}}$ holds and $\mathrm{DRP}_{\mathrm{IC}}$ fails, Theorem 1.2 tells us that there is a structure such that no ω_1-sized substructure reflects all Π^1_1 and all Σ^1_1 statements (though there are structures that reflect one or the other).

Another way to view this phenomenon, in terms of DRP-like principles, is that $\mathrm{DRP}_{\mathrm{internal}}$ yields stationarily many $W \in \wp_{\omega_2}(H_\theta)$ such that the transitive collapse H_W of W is "correct about stationary sets"; i.e. whenever $s \in H_W$ and $H_W \models$ "s is a stationary set of countable sets", then V believes this too. However, if W is not internally club, it is possible (by [2]) that H_W is correct about stationary sets, but is *not* correct about clubs; i.e. there can be a $c \in H_W$ such that $H_W \models$ "c is a club of countable sets", but V does not believe this. If, on the other hand, W witnesses $\mathrm{DRP}_{\mathrm{IC}}$, then H_W is correct about *both* stationarity *and* clubness.

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