Chang's conjecture and mouse reflection

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We discuss a particular roadblock in a hypothetical proof of PD from $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$, the question of mouse reflection. Unfortunately, other issues keep us from properly testing our ideas. Thus we will change focus to a related property (related to "$\aleph_\omega$ is Jónsson") to make our case. We hope these ideas will some day help us to better understand the strength of Chang's Conjecture.

1 Introduction

Let $\lambda, \lambda', \kappa, \kappa'$ be regular cardinals with $\lambda > \kappa$ and $\lambda' > \kappa'$. We take $(\lambda, \kappa) \twoheadrightarrow (\lambda', \kappa')$ to mean: for any structure (in a countable language) on λ there exists a substructure X with $\mathrm{Card}(X) = \lambda'$ but $\mathrm{Card}(X \cap \kappa) = \kappa'$.

Originating in Model Theory this seemingly innocuous property has significant large cardinal strength. Its most basic form (often known simply as *the* Chang conjecture), $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ in our notation, is equiconsistent with an ω_1-Erdős cardinal.

The exact consistency strength of the analogous property $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$ has proved more elusive. Though it is known to be consistent relative to the existence of a huge cardinal by an argument of Kunen's [5].

In this article we want to explore an approach towards establishing lower bounds for this property. For this purpose inner model theorists have developed a general method called the *Core Model Induction*. In the section following this introduction we will lay out the general framework that a typical core model induction would follow. As part of this framework we will introduce the concept of mouse reflection which will be the focus of our efforts in this article.

In the third section we will go through the first few steps of a core model induction on $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$ and present an idea of how mouse reflection might work in this context. The serious flaw of this part lies in the fact that core model induction is usually attempted for properties which are already known to imply the existence of inner models with Woodin cardinal. This is not known for $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$.

The author would feel ashamed were he to leave the reader with naught but castles in the sky. Hence we will in Section 4 discuss a related property which ,by work of the author, is known to imply the existence of inner model with Woodin cardinals. This other property thus provides an environment in which we can test our ideas about mouse reflection in the hope that these might one day be applicable to the Chang's Conjecture as progress is made.

We will finish the article with one last property which appears to be significantly weaker but admits the same general approach.

While we are discussing a rather big topic in the field of inner model theory we would hope that at least the beginning parts of this article may be comprehensible to the lay man and/or woman. The only things we really assume as a base line is a general knowledge of Descriptive Set Theory ([4] is a good primer) and the basics of inner model theory (rudimentary function, the basic properties of L).

2 The Core Model Induction

Core model induction is a method developed by Hugh Woodin to mine consistency strength from a given hypothesis (the interested might consult [8]). A rough outline of the method is as follows.

(1) One is given a determined scaled pointclass Γ_0.

(2) Using $\mathrm{Det}(\Gamma_0)$ (and possibly the hypothesis) produce a next scaled pointclass Γ_1.

(3) Using the hypothesis show that Γ_1 is determined. (The loop then resets to (1) with Γ_1 replacing Γ_0.

Moving from (1) to (2) generally relies heavily on the use of descriptive set theory. For example in the realm of projective sets one would appeal to Moschovackis's Periodicity Theorems. Moving from (2) to (3) involves the core models from which this technique takes its name.

Let us explain: from Γ_1 one would extract an operator F with the following properties.

- F is defined on a cone, i.e. there is $a \in H_{\aleph_1}$ such that $\mathrm{dom}(F) = \{b \in H_{\aleph_1} | a \in L_1(b)\}$;
- $F(b)$ is a mouse (see below) for all $b \in \mathrm{dom}(F)$;
- $F(b)$ is minimal for all $b \in \mathrm{dom}(F)$, i.e. $F(b)$ satisfies $\psi(a, b)$ (for some Σ_1 formula ψ in the language of premice) but no initial segment of it does;
- $F(b)$ is sound for all $b \in \mathrm{dom}(F)$, i.e. every element of $F(b)$ is Σ_1-definable from some parameter in $L_1(b)$;

- $F(b)$ "captures" Γ_0.

A mouse for our purposes is a model taking the form $(M; \in, \vec{B}, \vec{E}, G)$ where M is generated by repeatedly closing under rudimentary functions, $x \mapsto x \cap \vec{B}$, and $y \mapsto y \cap \vec{E}$, so M is constructed using a variant of Jensen's constructible hierarchy. G is an amenable predicate.

Two mice M, N satisfy comparison, i.e. there exists partial maps $i : M \to M^*$ and $j : N \to N^*$ such that

- $M^* \trianglelefteq N^*$, i is total and Σ_1-elementary, or
- $N^* \trianglelefteq M^*$, j is total and Σ_1-elementary.

Remark: The maps i and j are generated from extenders coded into the sequences associated with the mice. M is an initial segment of N (written $M \trianglelefteq N$) if the associated predicate sequences are.)

Note then that by comparison $F(b)$ is uniquely determined by its properties of soundness and minimality.

Given our operator F the path forward then goes through a core model dichotomy theorem. A core model dichotomy is a statement of the form: if $\mathrm{Det}(\Gamma_1)$ fails there must exist for some $b \in H_{\omega_1}$ an F-closed core model $K^F(b)$ over b.

What do we mean by core model? We cannot go into much detail here but stated simply, a core model is a mouse that is both maximal in the mouse order (generally restricted to some subset of mice, say those closed under the operator F) and sound. Note this notion of soundness is necessarily different from that satisfied by some $F(b)$, the core model in our case must be an uncountable model so not every element can be definable from some countable set of parameters. Be that as it may, this alternative notion of soundness serves fundamentally the same purpose, namely to identify a minimal model within an equivalence class of mice.

Remark: During the course of this article we might use language similar to "show the F-closed core model does not exist". As inner model theorists we would like to think that, of course, there is some "true" core model that reflects the true nature of the universe. Such an object being universal should be F-closed.

What we are actually saying when we use this language then is that some specific procedure involving the operator F fails to produce a core model. That does not mean that another more sophisticated procedure couldn't produce the core model instead.

So it would not be surprising to find that such a core model would be uniquely determined. Additionally, we would expect such a model to exemplify properties similar to those known from Jensen's covering theorem.

Theorem 2.1 (Jensen): *Exactly one of the following is true:*

- $0^\#$ *exists;*

- $\text{cof}((\alpha^+)^L) \geq \text{Card}(\alpha)$ *for all* $\alpha \geq \aleph_2$.

The core model would indeed have these properties. And this should in theory give us great leverage in our attempt to show $\text{Det}(\Gamma_1)$ holds, but there is a problem. An F-closed core model for an operator F only defined on hereditarily countable sets could not contain any uncountable sets, but the covering lemma can only apply at points above $\aleph_2$.

So we need to devise some way to extend the operator F to act on uncountable sets. This is the purpose of *mouse reflection*. We say mouse reflection holds at (κ, λ) (here κ, λ are uncountable cardinals) if and only if any "nice" operator F defined on a cone in H_κ extends to a "nice" operator F' defined on a cone in H_λ.

Remark: We will not define "nice", but keep two things in mind. One, all operators appearing in the course of the core model induction are "nice". Two, "nice" extensions of "nice" operators are uniquely determined. We will generally use the same notation for the original operator and its extension.

Unfortunately, mouse reflection is not a ZFC-theorem. Generally, we utilize our given hypothesis to prove versions of mouse reflection appropriate to our situation.

Using mouse reflection we can thus envision a more accurate idea of core model induction.

(1) One is given a determined scaled pointclass Γ_0.

(2) Using $\text{Det}(\Gamma_0)$ (and possibly the hypothesis) produce a next scaled pointclass Γ_1.

(3) Use the scale property of Γ_1 and $\text{Det}(\Gamma_0)$ to produce a mouse operator F defined on a cone of $H_{\aleph_1}$.

(4) Use the hypothesis to prove that mouse reflection holds at $(\aleph_1, \kappa)$ for some carefully chosen κ; extend then F to act on a cone of H_κ.

(5) Utilize the hypothesis to show that there cannot be an F-closed core model.

(6) Conclude that by core model dichotomy Γ_1 is determined. (The loop then resets to (1) with Γ_1 replacing Γ_0.)

There are indeed core model inductions that use this framework. ([10]) But often the situation is more complicated. The problem is that often we cannot prove mouse reflection holds at $(\aleph_1, \kappa)$ for any κ greater than $\aleph_1$, but perhaps we are able to prove mouse reflection holds at (κ, λ) for some κ, λ greater than $\aleph_1$.

Of course, determinacy as a notion does (so far) have no natural extension to the realm of the uncountable. But there is a rather elegant solution to this conundrum. Work in $V[g]$ where $g \subset \text{Col}(\omega, {<}\kappa)$. Consider then pointclasses in $V[g]$.

This solves a lot of our problems but also introduces some new ones. We start the argument with a hypothesis about our original universe. That hypothesis is quite possibly false in the forcing extension. So we need to consider ways we can relate operators F defined on a cone of $H_{\aleph_1}^{V[g]}$ to some operator $\dot{F}$ defined on a cone of H_κ.

The way this is done depends on the operator F but note that in many cases, certainly those appearing later in this article, the operator is naturally defined over the original universe. With this final complication we arrive at a complete picture of core model induction.

(1) One is given a determined scaled pointclass Γ_0 in $V[g]$ where $g \subset \mathrm{Col}(\omega, {<}\kappa)$ is generic over V, some carefully chosen κ.

(2) Using $\mathrm{Det}(\Gamma_0)$ (and possibly the hypothesis) produce a next scaled pointclass Γ_1 in $V[g]$.

(3) Use the scale property of Γ_1 and $\mathrm{Det}(\Gamma_0)$ to produce a mouse operator F defined on a cone of $H_{\aleph_1}^{V[g]}$. Pull back F to some operator $\dot{F}$ defined on a cone of H_κ

(4) Use the hypothesis to prove that mouse reflection holds at (κ, λ) for some carefully chosen λ; extend then $\dot{F}$ to act on a cone of H_λ (this will also extend F to act on $H_\lambda^{V[g]}$).

(5) Utilize the hypothesis to show that there cannot be an F-closed core model above any $b \in H_{\aleph_1}^{V[g]}$.

(6) Conclude that by core model dichotomy Γ_1 is determined. (The loop then resets to (1) with Γ_1 replacing Γ_0.)

Let us now dispense with theory and take a look at how this method might work in practice.

3 CMI and Chang's Conjecture

Recall the hypothesis we are most interested in is $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$. Looking back to our core model induction framework the most important decision we have to make is to carefully pick κ and λ, two uncountable cardinals such that we can reflect mouse operators from one to the other.

With this particular property the choice almost forces itself upon us. We shall pick $\kappa = \aleph_2$ and $\lambda = \aleph_3$. Let us see if this works.

Work in $V[g]$ where $g \subset \mathrm{Col}(\omega, \aleph_1)$ is generic over V. The first pointclass in any core model induction are the Borel sets. By a theorem of Martin this is a determined pointclass, so (1) is satisfied.

The next scaled pointclass is $\mathbf{\Pi}_1^1$. Our operator is simply $[b \mapsto J_1(b)]$ and the core model $K^F(b) = L(b)$. (Note: Of course, $L(b)$ always exists, but it is not always a core model. So, if our framework tells us to prove that "$K^F(b)$ does not exist", this translates to "$L(b)$ is not a core model".)

$L(b)$ is not a core model if and only if $b^\#$ does exist. ($b^\#$ is a witness that $L(b)$ is not universal.) By a theorem of Martin if $b^\#$ exists for all $b \in H_{\aleph_1}$ then $\text{Det}(\mathbf{\Pi}^1_1)$. So this is our core model dichotomy theorem.

The task then is clear.

Lemma 3.1: *Assume* $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$, *then* $b^\#$ *exists for all* $b \in H_{\aleph_2}$.

PROOF: Let $b \in H_{\aleph_2}$. We can take some $X \prec (H_{\aleph_3}; \in, L_{\aleph_3}(b))$ with $b \in X$ and $\text{Card}(X) = \aleph_2$ but $\text{Card}(X \cap \aleph_2) = \aleph_1$. We can choose X such that $\text{otp}(X \cap \aleph_3) = \aleph_2$.

Let $\pi_X : H_X \to X$ be an isomorphism where H_X is transitive. We do have $\pi_X(b) = b$, so by condensation $\pi_X^{-1}{}'' [L_{\aleph_3}(b)] = L_{\aleph_2}(b)$.

As $\aleph_2$ is a cardinal $\pi_X \restriction L_{\aleph_2}(b)$ then generates the required indiscernibles. ([3, 18.27]) $\dashv$

We can thus conclude that $\text{Det}(\mathbf{\Pi}^1_1)$ holds in $V[g]$. (Or can we? We do actually need that $b^\#$ exists for all $b \in H^{V[g]}_{\aleph_1}$. Consider such a b. There exists then a name $\dot{b}$ for it in $H_{\aleph_2}$. We can then compute $b^\#$ from $(\dot{b})^\# [g]$.) This means we have successfully completed the first loop in our core model induction framework. Note that in this particular case mouse reflection was trivially true. Yet, keep in mind that we did need to consider $L(b)$ up to its $\aleph_3$'s stage in the above argument, so its use was in some way required.

Moving on we can identify our next pointclass $\mathbf{\Sigma}^1_2$ and operator $\left[b \mapsto b^\#\right]$. We have just shown that our operator is defined over $H^{V[g]}_{\aleph_1}$ so no further work is required in that regard. Finally, we have to consider the question of mouse reflection. We do so in two steps.

Lemma 3.2: *Assume* $H_{\aleph_2}$ *is closed under sharps. Let* $b \in H_{\aleph_3}$, *let* $\beta := (\text{Card}(b)^+)^{L(b)}$. *If* $\text{cof}(\beta) < \aleph_2$, *then* $b^\#$ *exists.*

This is a general lemma (used in many core model inductions e.g. [9]) that does not require the use of our hypothesis in any way, so we shall skip the proof. The proof utilizes long ultrapowers to lift up structures, a common technique whose use goes at least as far back as the proof of the covering lemma.

Lemma 3.3: *Assume* $H_{\aleph_2}$ *is closed under sharps. Let* $b \in H_{\aleph_3}$, *let* $\beta := (\text{Card}(b)^+)^{L(b)}$. *If* $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$, *then* $\text{cof}(\beta) < \aleph_2$ *(and thus* $b^\#$ *exists).*

PROOF: Assume towards a contradiction that $\text{cof}(\beta) \geq \aleph_2$. Let $X \prec (H_{\omega_3}; \in, L_{\aleph_3}(b))$ with $b \in X$ and $\text{Card}(X) = \aleph_2$ but $\text{Card}(X \cap \aleph_2) = \aleph_1$. We can choose X such that $\text{otp}(X \cap \aleph_3) = \aleph_2$.

Let $\pi_X : H_X \to X$ be an isomorphism where H_X is transitive. Let $\pi_X(\bar{b}) = b$, so by condensation $\pi_X^{-1}{}'' [L_{\aleph_3}(b)] = L_{\aleph_2}(\bar{b})$.

Crucially, this implies that $\bar{\beta} := \pi_X^{-1}(\beta)$ is the real L-successor of $\text{Card}(\bar{b})^{L(\bar{b})}$. Also, by elementarity its cofinality is uncountable.

But $\bar{b} \in H_{\aleph_2}$, so $\bar{b}^\#$ exists. But this implies that $\text{cof}(\bar{\beta}) = \omega$ ($\bar{b}^\# = \bigcup_{n<\omega} f_n{}'' \left[\bar{b}^{m_n}\right]$ where $f_n : \bar{b}^{m_n} \to L(\bar{b})$ and $f_n \in L(\bar{b})$). Contradiction! $\dashv$

Let us now consider our core model dichotomy.

Lemma 3.4: *Assume* $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$*, let* $b \in H_{\aleph_3}$*. Then* $K^{\#}(b)$ *exists or there exists a sharp for an inner model with a Woodin cardinal containing* b *called* $M_1^{\#}(b)$*.*

By a result of Neeman's ([6]) the existence of $M_1^{\#}(b)$ for all $b \in H_{\aleph_1}$ implies $\mathrm{Det}(\mathbf{\Sigma}^1_2)$ as required.

Let $b \in H_{\aleph_2}$. The task is to show that $K^{\#}(b)$ does not exist. Assume it does.

Let $X \prec (H_{\omega_3}; \in, K^{\#}(b))$ with $b \in X$ and $\mathrm{Card}(X) = \aleph_2$ but $\mathrm{Card}(X \cap \aleph_2) = \aleph_1$. We can choose X such that $\mathrm{otp}(X \cap \aleph_3) = \aleph_2$.

Let $\pi_X : H_X \to X$ be an isomorphism where H_X is transitive. We have $\pi_X(\bar{b}) = b$. Write π_X^{-1}" $\left[K^{\#}(b)\right] := \bar{K}(b)$.

The idea is to compare $\bar{K}(b)$ with $K^{\#}(b)$ which we can do as they are both mice above a common set. (This is the general idea of the covering argument.) We would expect the following two facts to hold.

- $K^{\#}(b)$ is universal, so it "wins" the comparison; we also expect it to "drop" to a size $<\aleph_2$-mouse in the iteration;
- $\bar{K}(b)$ remains idle in the comparison, i.e. if M^* is the last model produced by the iteration on $K^{\#}(b)$, then $\bar{K}(b) \trianglelefteq M^*$.

Unfortunately, so far Inner Model Theorists have failed to produce $M_1^{\#}$ from the Chang's Conjecture. Previous attempts have fallen short ([1],[7]). But let us, just for a moment, pretend we could show that $M_1^{\#}(b)$ exists for all $b \in H_{\aleph_2}$.

Our next task would be to show that $M_1^{\#}(b)$ exists for all $b \in H_{\aleph_3}$. So let us take some such b. Assume that $M_1^{\#}(b)$ does not exist. Then we do have a core model $K^{\#}(b)$. Let us now take a hull of the right type, i.e. a witness to the Chang's Conjecture containing relevant objects.

$K^{\#}(b)$ then collapses to some $\bar{K}(\bar{b})$. Note that we cannot compare these two model as these are potentially mice above two different sets. Instead, we have $\bar{b} \in H_{\aleph_2}$ so $M_1^{\#}(\bar{b})$ exists. Let us then compare $\bar{K}(b)$ and $M_1^{\#}$.

$M_1^{\#}(\bar{b})$ must win this comparison. This is because by assumption $K^{\#}(b)$ fell short of $M_1^{\#}(b)$ and so by elementarity does $\bar{K}(\bar{b})$. Also $M_1^{\#}(\bar{b})$ has size $<\aleph_2$.

So in some ways $M_1^{\#}(\bar{b})$ fulfills the same role that the core model did previously. We might hope that the (pretend) solution to our previous problem could in some slightly modified form help us reach the desired contradiction.

This idea so far stands on illusory ground. But we might consider that there are perhaps other properties similar to Chang's Conjecture in which the same idea could find solid footing.

4 Strenghtenings of Chang's Conjecture

Consider $(\aleph_2, \aleph_3, \aleph_4, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$, i.e. the property that there are stationarily many $X \subset \aleph_\omega$ such that $\mathrm{Card}(X \cap \aleph_{n+1}) = \aleph_n$ for all $n \geq 1$.

Lemma 4.1: *Assume* $(\aleph_2, \aleph_3, \aleph_4, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$*, then* $b^\#$ *exists for all* $b \in H_{\aleph_\omega}$.

PROOF: Proof by induction. Let $b \in H_{\aleph_2}$, note that we have $(\aleph_3, \aleph_2) \twoheadrightarrow$ so then $b^\#$ exists by Lemma 3.1. Now let $n \geq 1$ and assume that $H_{\aleph_{n+1}}$ is closed under sharps. Let $b \in H_{\aleph_{n+2}}$, note that we have $(\aleph_{n+2}, \aleph_{n+1}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ and so then $b^\#$ exists by Lemma 3.2 and Lemma 3.3. ⊣

So far everything from before adapts beautifully. The crucial difference lies in the next lemma.

Lemma 4.2: *Assume* $(\aleph_2, \aleph_3, \aleph_4, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$ *and* $2^{\aleph_0} = \aleph_1$. *Then for any* $b \in H_{\aleph_2}$, $K^\#(b)$ *does not exist.*

PROOF (SKETCH): Fix b. Assume $K^\#(b)$ exists. Let $X \prec (H_{\aleph_\omega}; \in, K^\#(b))$ with $b \in X$ such that $\mathrm{otp}(X \cap \aleph_{n+1}) = \aleph_n$ for all $n \geq 1$.

Let $\pi_X : H_X \to X$ be an isomorphism where H_X is transitive. Let $\bar{K}(b) := \pi_X^{-1}{}''\left[K^\#(b)\right]$. We compare $\bar{K}(b)$ and $K^\#(b)$. We know two things.

- By work of Schindler we can assume that $K^\#(b)$ does drop to a mouse of size $<\aleph_2$.
- $\bar{K}(b)$ does not move.

(This uses a weak form of countable closure of X which follows from the fact that X contains all the reals.) As a consequence of the former we know that there are threads going up to $\aleph_n$ $(n \geq 2)$, i.e. there exists a club $C_n \subset \aleph_n$ such that $\iota_{\alpha,\aleph_n}(\alpha) = \aleph_n$ for all $\alpha \in C_n$. Here $\iota_{\alpha,\beta} : M_\alpha \to M_\beta$ is the iteration embedding between the α-th and β-th model in the iteration.

This implies that $\aleph_n$ $(n \geq 2)$ is a limit cardinal in $\bar{K}(b)$. We can then show that each of those $\aleph_n$'s is a cutpoint of the iteration. Assume not, say the β-th extender $(\beta \geq \aleph_n)$ gets applied to the α-th model $(\alpha < \aleph_n)$. We then have $(\mathrm{crit}(E_\beta)^+)^{\bar{K}(b)} < \aleph_n$ as the latter is a limit, so then clearly $\mathrm{cof}(\mathrm{lh}(E)) = \mathrm{cof}((\mathrm{crit}(E_\beta)^+)^{\bar{K}(b)} < \aleph_n$. But $\mathrm{lh}(E) \geq \aleph_n$ is a successor cardinal in $\bar{K}(b)$ so this contradicts the covering lemma. (This uses that $\bar{K}(b)$ does not move in the iteration, as otherwise $\mathrm{lh}(E)$ would only be a successor cardinal in an iterate of that model where covering might fail.)

As iteration embeddings must be continuous at successor cardinals, we then have $\mathrm{cof}((\aleph_n^+)^{M_{\aleph_n}})) < \aleph_n$. On the other hand because of the covering lemma, we have $\mathrm{cof}((\aleph_n^+)^{\bar{K}(b)}) \geq \aleph_n$. The two models clearly disagree on the successor of $\aleph_n$. So, this means whenever we move out of $M_{\aleph_n}$ during the iteration, a drop must ensue.

As $\aleph_n$ $(n \geq 2)$ are cutpoints the final branch (moving up to $M_{\aleph_\omega}$) moves through all of them. Therefore infinitely many drops occur along this branch. Contradiction! ⊣

So this lemma fixes the gap in our previous induction. We conclude that $M_1^\#(b)$ exists for all $b \in H_{\aleph_2}$. Can we lift this up to $H_{\aleph_3}$?

We apply the idea from before. Let $b \in H_{\aleph_3}$ be such that $M_1^\#(b)$ does not exist. So we have $K^\#(b)$. Let us take $X \prec (H_{\aleph_\omega}; \in, K^\#(b))$ with $b \in X$ such that $\mathrm{otp}(X \cap \aleph_{n+1}) = \aleph_n$ for all $n \geq 2$.

Let $\bar{K}(\bar{b}) := \pi_X^{-1}{}''\left[K^\#(b)\right]$. We have $\bar{b} \in H_{\aleph_2}$ and so $M_1^\#(\bar{b})$ exists. We compare $\bar{K}(\bar{b})$ and $M_1^\#(\bar{b})$. $M_1^\#(\bar{b})$ must win this iteration and it is a small ($<\aleph_2$) mouse. This was one of the ingredients in the preceding proof, the other is that $\bar{K}(\bar{b})$ has covering which it has as it is the preimage of a core model.

The last requirement is that $\bar{K}(\bar{b})$ does not move in the comparison. The author admits that he has not fully investigated this but he thinks that it is not unreasonable to assume that this will turn out to be correct. If it is we can extend this core model induction beyond the projective level.

5 Weakenings of Chang's Conjecture?

We have seen that a property like $(\aleph_2, \aleph_3, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \ldots)$ is in many ways a good candidate for a core model induction, but it has one major flaw. It is not known to be consistent. (In fact, as far as the author is aware not even $(\aleph_2, \aleph_3, \aleph_4, \aleph_5) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \aleph_4)$ is known to be consistent.)

Let us instead consider $(\aleph_{\omega+2}, \aleph_{\omega+3}, \aleph_{\omega+4}, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$. This is known to be consistent relative to large cardinals below a supercompact cardinal ([2]). Curiously, it seems that this property could be weaker than $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$ which we know is consistent relative to a huge cardinal. (The above mentioned paper actually improves this bound but not significantly.)

Could this property work instead of $(\aleph_2, \aleph_3, \aleph_4, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$? We claim it does.

Theorem 5.1: *Assume* $(\aleph_{\omega+2}, \aleph_{\omega+3}, \aleph_{\omega+4}, \ldots) \twoheadrightarrow (\aleph_1, \aleph_2, \aleph_3, \ldots)$, CH *and* $2^{\aleph_{\omega+1}} = \aleph_{\omega+2}$. *Then* $M_1^\#(b)$ *exists for all* $b \in H_{\aleph_2}$.

Proof: We adapt previous arguments. Lemma 4.1 adapts in a straightforward fashion. For Lemma 4.2 we have to consider the core of the argument. This core is that we compare a small ($<\aleph_2$) mouse with a large one ($\geq\aleph_\omega$) that has covering properties yet the small mouse outstrips the large one in the comparison. Both of these things are given, the large mouse is the pullback of the core model under some suitable chosen hull X a witness to our hypothesis, the small mouse is an initial segment of the core model. (The continuum function restrictions are there so we can assume that $(X \cap H_{\aleph_{\omega+1}})^\omega \subset X$.) $\dashv$

Similarly, the same considerations carry over when considering the problem of mouse reflection for $M_1^\#$'s. This gives us a good shot at projective determinacy from this property. Could we go farther? Likely, but mouse reflection would keep being a problem. The approach we have presented here works at succesor points. In essence, we use the core model dichotomy from stage α to prove mouse reflection for station $\alpha + 1$. Limit points in the induction will require some other idea.

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