$\mathcal{R}$-solver and periodic solutions of the Navier-Stokes equations

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1 Introduction

Since Kato-Fujita theory, the Navier-Stokes equations have been stuided by a lot of mathematicians based on analytic semigroup properties of Stokes equations. Since the Navier-Stokes equations are a system of semi-linear parabolic equations, and so the analytic semigroup approach has yielded many fruitful results. On the other hand, in many flow problems, for example a falling drop problem, ocean problem, nuclear power, energy conversion technique, envilonment issues, bood flows..., we meet free boundary problem for the Navier-Stokes equations, which is formulated in an unknown time dependent domain. Under suitable transformation from a time dependent unknown domain with free boundary to a known domain with fixed boundary, the equations become a system of quasilinear parabolic equations with non-homogeneous boundary conditions. The basical tool of proving the local in time existence theorem for such problems is the maximal regularity for the Stokes equations with non-homogeneous boundary or transmission conditions. There are a lot of works have been done by Solonnikov and his colleagues since the early of 1980 in the Hölder spaces, $C^{2+\alpha,1+\alpha/2}$ $(\alpha > 0)$, and Sobolev-Slobodetskii spaces $W_2^{2+\ell,1+\ell/2}$ $(1/2 < \ell < 1)$, by Jan Pruess and his colleagues in the anisotropic $W_p^{2,1}$ space since the early of 2000, and by Shibata and his colleagues in the anisotropic $W_{q,p}^{2,1}$ space also since the early of 2000. From technical point of view, their approaches are different, and in this note I would like to explain an approach based on $\mathcal{R}$-solver of the generlized resolvent equations for the Stokes operator with non-homogeneous free boundary conditions, which gives a systematic study of a system of quasilinear parabolic equations with non-homogeneous boundary conditions.

2 Framework based on $\mathcal{R}$-solvers

I would like to formulate $\mathcal{R}$-solvers for free boundary problem without surface tension in mind. Let me consider an initial-boundary value problem formulated as follows:

$$\dot{u} - Au = f, \quad Bu = g \quad \text{for } t > 0, \quad u|_{t=0} = u_0, \tag{2.1}$$

where t is the time variable, $\dot{u}$ denotes the time derivative of u, and $Bu = f$ denotes a boundary condition.

- Let X and Y be two UMB Banach spaces and $Y \subset X$.

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- Let $Z = (X, Y)_{[1/2]}$ be a complex interpolation space of order 1/2.
- Let $A \in \mathcal{L}(Y, X)$, $\mathcal{L}(Y, X)$ denoting the set of all bounded linear operator from Y into X.
- Let $B \in \mathcal{L}(Y, Z) \cap \mathcal{L}(Z, X)$.

Example 1. The following two sets of equations are typical example in the setting of this note:

$$\dot{v} - \Delta v = f \text{ in } \Omega \times (0, \infty), \quad \frac{\partial v}{\partial \nu} = g \text{ on } \partial\Omega \times (0, \infty), \quad v|_{t=0} = v_0,$$

where Ω is a domain, $\partial\Omega$ is its boundary, Δ denotes the Laplace operator and ν denotes the unit outer normal to $\partial\Omega$ (Neumann operator);

$$\begin{aligned} \dot{\mathbf{v}} - \Delta\mathbf{v} + \nabla\mathfrak{p} = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega \times (0, \infty), \\ (\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\nu = \mathbf{g} \quad \text{on } \partial\Omega \times (0, \infty), \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \end{aligned}$$

In these cases, for $1 < p < \infty$, I set $X = L_p$, $Y = H^2_p$, $Z = H^1_p$.

In what follows, for $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$ $\Sigma_{\epsilon,\lambda_0}$ denotes a subset of $\mathbb{C}$ defined by setting

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon, \quad |\lambda| \geq \lambda_0\}.$$

And, let me consider a generalized resolvent problem corresponding to (2.1) as follows:

$$\lambda u - Au = f, \quad Bu = g \tag{2.2}$$

for $\lambda \in \Sigma_{\epsilon,\lambda_0}$, where "generalized" means the non-homogeneous boundary condition.

A main tool in my approach is Weis's operator valued Fourier multiplier theorem [7], and so I introduce the notion of $\mathcal{R}$ boundedness of operator families.

Definition 2. Let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from X into Y. A family of operators, $\mathcal{T} \subset \mathcal{L}(X, Y)$, is called $\mathcal{R}$-bounded if there exists a constant C and an exponent $p \in [1, \infty)$ such that for all $m \in \mathbb{N}$, $\{T_k\}_{k=1}^m \subset \mathcal{T}$, and $\{x_k\}_{k=1}^m \subset X$, there hold the inequalities:

$$\Big\|\sum_{k=1}^m r_k T_k x_k\Big\|_{L_p((0,1),Y)} \leq C\Big\|\sum_{k=1}^m r_k x_k\Big\|_{L_p((0,1),X)}.$$

Here, the Rademacher function r_k, $k \in \mathbb{N}$, are given by r_k: $[0, 1] \to \{-1, 1\}$, $t \mapsto \operatorname{sign}(\sin(2^k\pi t))$. The smallest such C is called the $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$ in what follows.

In the following, I consider the situation that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $f \in X$ and $g \in X \cap Z$, problem (1) admits a unique solution $u \in Y$ possessing the estimate:

$$\|u\|_Y + \|\lambda u\|_X \leq C(\|f\|_X + \|g\|_Z + \|\lambda^{1/2} g\|_X), \tag{2.3}$$

which has been studied since 1950's as parameter elliptic problems. My concern is to prove the generalized resolvent estimate in terms of $\mathcal{R}$-norms instead of standard norms, $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$.

For any Banach space U, let $\operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, U)$ denote the set of all U valued holomorphic functions defined on $\Sigma_{\epsilon,\lambda_0}$. Below, I assume the existence of an operator family

$$\mathcal{M}(\lambda) : X \times X \times Z \to Y; \quad X \times X \times Z \ni (F_1, F_2, F_3) \mapsto \mathcal{M}(\lambda)(F_1, F_2, F_3) \in Y$$

for every $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_0}$ with

$$\mathcal{M}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X \times X \times Z, Y)), \quad \lambda\mathcal{M}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X \times X \times Z, X))$$

such that

(i) for every $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $f \in X$ and $g \in Z$, $u = \mathcal{M}(\lambda)(f, \lambda^{1/2}g, g)$ is a solution of problem (2.2);

(ii) $\mathcal{M}(\lambda)$ satisfies

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(X\times X\times Z,X)}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{M}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\} \le r_b,\\ \mathcal{R}_{\mathcal{L}(X\times X\times Z,Y)}(\{(\tau\partial_\tau)^\ell\mathcal{M}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\} \le r_b,\end{aligned} \tag{2.4}$$

for $\ell = 0, 1$ with some constant r_b.

Remark 3. Such an operator family $\mathcal{M}(\lambda)$ is called an $\mathcal{R}$-solver for equations (2.3). Since $\mathcal{R}$ boundedness implies the standard boundedness (in the $m = 1$ case in Definition 2), the estimate (2.3) is derived automatically from (2.4).

I now consider the following time dependent problem:

$$\dot{u} - Au = f, \quad Bu = g \quad (t > 0) \quad u|_{t=0} = u_0. \tag{2.5}$$

The compatibility condition is:

$$g|_{t=0} = Bu_0 \tag{2.6}$$

which is obtained from the boundary condition $Bu = g$ at $t = 0$. Set $u = v + w$, where v and w are solutions of the following equations:

$$\dot{v} - Av = f, \quad Bv = g \quad \text{for } t \in \mathbb{R}, \tag{2.7}$$

$$\dot{w} - Aw = 0, \quad Bw = 0 \quad \text{for } t \in (0,\infty), \quad w|_{t=0} = u_0 - v|_{t=0}. \tag{2.8}$$

I first consider equations (2.7). First of all, let f and g be extended to $t < 0$. Since f is not required to be differentiable in time, and so f is extended by 0, that is $f_0 = f$ for $t > 0$ and $f_0 = 0$ for $t < 0$. On the other hand, g is usually required to be differentiablity at least of some fractional order on t and so here it is assumed that g is defined for $t > 0$, and then $g_0(t) = g(t)$ and $g_0(t) = \varphi(t)g(-t)$, where $\varphi(t) \in C^\infty(\mathbb{R})$ which equals one for $t > -1$ and vanishes for $t < -2$. Instead of (2.7), I consider the following equations:

$$\dot{v} - Av = f_0, \quad Bv = g_0 \quad \text{for } t \in \mathbb{R}. \tag{2.9}$$

Applying the Laplace transform to equations (2.9) yields that

$$\lambda\hat{v} - A\hat{v} = \hat{f}_0, \quad B\hat{v} = \hat{g}_0. \tag{2.10}$$

Here, the Laplace transform $\hat{v}$ is defined by setting

$$\hat{v}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t}v(t)\,dt = \int_{-\infty}^{\infty} e^{-i\tau t}e^{-\gamma t}v(t)\,dt = \mathcal{F}[e^{-\gamma t}v](\tau)$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$, where $\mathcal{F}$ denotes the Fourier transform. Using the $\mathcal{R}$-solver $\mathcal{M}(\lambda)$, $\hat{v}$ is represented by $\hat{v} = \mathcal{M}(\lambda)(\hat{f}, \lambda^{1/2}\hat{g}, \hat{g})$. By the Laplace inverse transform,

$$\begin{aligned}v(t) &= \mathcal{L}^{-1}[\mathcal{M}(\lambda)(\hat{f}, \lambda^{1/2}\hat{g}, \hat{g})](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+i\tau)t}\mathcal{M}(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)](\tau)\,d\tau\\ &= e^{\gamma t}\mathcal{F}^{-1}[\mathcal{M}(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2}g, g)](\tau)](t),\end{aligned}$$

where $\Lambda_\gamma^{1/2}g$ is defined by setting

$$\Lambda_\gamma^{1/2}g = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[g](\lambda)](t).$$

I now quote the Weis operator valued Fourier multiplier theorem [7].

Theorem 4. *Let X and Y be two UMD Banach spaces and let $1 < p < \infty$. Let m be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that the following conditions are satisfied:*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{m(\tau), \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{m(\tau), \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

Let an operator T_m acting on elements of $\mathcal{F}^{-1}[\mathcal{D}(\mathbb{R}, X)]$ be defined by setting

$$[T_m f](t) = \mathcal{F}^{-1}[m(\tau)\mathcal{F}[f](\tau)](t) \quad \text{for } f \text{ with } \mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, X),$$

where $\mathcal{D}(\mathbb{R}, X)$ denotes the set of all X-valued $C_0^\infty(\mathbb{R})$ functions. Then, the operator T_m is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$ with norm

$$\|T_m\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

with some constant depending only on p, X, and Y.

Applying Theorem 4 to the formulas:

$$e^{-\gamma t}\dot{v} = \mathcal{F}^{-1}[\lambda M(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2} g, g)]](t),$$
$$e^{-\gamma t}v = \mathcal{F}^{-1}[M(\lambda)\mathcal{F}[e^{-\gamma t}(f_0, \Lambda_\gamma^{1/2} g, g)]](t)$$

yields that

$$\|e^{-\gamma t}\partial_t v\|_{L_p(\mathbb{R},X)} + \|e^{-\gamma t}v\|_{L_p(\mathbb{R},Y)}$$
$$\leq C(\|e^{-\gamma t}f\|_{L_p(\mathbb{R},X)} + \|e^{-\gamma t}\Lambda_\gamma^{1/2} g\|_{L_p(\mathbb{R},X)} + \|e^{-\gamma t}g\|_{L_p(\mathbb{R},Z)}),$$

which is the maximal L_p regularity theorem for problem (2.7).

Problem (2.8) is solved by C^0 analytic semigroup $T(t)$, whose generation is obtained with the help of the $\mathcal{R}$-solver. In fact, the underlysing space $\mathcal{H}$, the operator $\mathcal{A}$ and its domain $\mathcal{D}(\mathcal{A})$ are defined as follows:

$$\mathcal{H} = X_0, \quad \mathcal{D}(\mathcal{A}) = \{x \in Y \mid Bx = 0\}, \quad \mathcal{A}x = Ax \text{ for } x \in \mathcal{D}(\mathcal{A}).$$

Problem (2.8) is formulated by

$$\dot{w} - \mathcal{A}w = 0 \quad (t > 0), \quad w|_{t=0} = u_0 - v|_{t=0}. \tag{2.11}$$

The corresponding resolvent problem to (2.11) is

$$\lambda\hat{w} - A\hat{w} = f, \quad B\hat{w} = 0 \quad (t > 0). \tag{2.12}$$

with $f = u_0 - v|_{t=0}$. Since the $\mathcal{R}$ boundedness implies the boundedness, by the first estimate in (2.4) implies that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ problem (2.12) admits a unique solution $\hat{w} \in Y$ possessing the estimate:

$$|\lambda|\|\hat{w}\|_X + \|\hat{w}\|_Y \leq 2r_b\|f\|_X,$$

Thus, there exists a C^0 analytic semigroup $\{T(t)\}_{t\geq 0}$ such that for any $f \in X$, $w = T(t)f$ gives a unique solution of problem (2.12). Moreover, if $f \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}$ which is an real interpolation space between X and $\mathcal{D}(\mathcal{A})$, then w satisfies the estimate:

$$\|e^{-\gamma t}\dot{w}\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}w\|_{L_p((0,\infty),Y)} \leq C\|f\|_{(X,Y)_{1-1/p,p}}$$

for any $\gamma > \lambda_0$, where C is a constant depending solely on λ_0 and ϵ. By the compatibility condition (2.6), $B(u_0 - v|_{t=0}) = Bu_0 - g|_{t=0} = 0$, and so $f = u_0 - v|_{t=0} \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}$ provided that $u_0 \in (X, \mathcal{D}(\mathcal{A}))_{1-1/p,p}$. In fact, a real interpolation theorem yields that $v \in C([0,\infty), (X,Y)_{1-1/p,p})$ and

$$\sup_{t\in[0,\infty)} \|v(t)\|_{(X,\mathcal{D}(\mathcal{A}))_{1-1/p,p}} \leq C(\|e^{-\gamma t}\dot{v}\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}v\|_{L_p((0,\infty),Y)}).$$

Thus, we have

$$\begin{aligned}&\|e^{-\gamma t}\dot{w}\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}w\|_{L_p((0,\infty),Y)}\\ &\quad \leq C(\|u_0\|_{(X,Y)_{1-1/p,p}} + \|e^{-\gamma t}\dot{v}\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}v\|_{L_p((0,\infty),Y)}\}.\end{aligned}$$

Then, $u = v + w$ is a requred solution to problem (2.5).

Summing up, I have proved the following theorem.

Theorem 5. *Let $1 < p < \infty$ and X, Y and Z be three UMD Banach spaces. If $\mathcal{R}$-solver $\mathcal{M}(\lambda)$ exists for $\lambda \in \Sigma_{\epsilon,\lambda_0}$, then problem (2.5) admits a solution u with*

$$e^{-\gamma t}u \in L_p((0,\infty),Y) \cap H^1_p((0,\infty),X)$$

for any $\gamma > \lambda_0$ possessing the estimate:

$$\begin{aligned}&\|e^{-\gamma t}\dot{u}\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}u\|_{L_p((0,\infty),Y)} \leq C(\|u_0\|_{(X,Y)_{1-1/p,p}}\\ &\quad + \|e^{-\gamma t}f\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}\Lambda^{1/2}_\gamma(\varphi g_0)\|_{L_p(\mathbb{R},X)} + \|e^{-\gamma t}\varphi g_0\|_{L_p(\mathbb{R},Z)}\}.\end{aligned}$$

3 Framework for periodic solutions with the help of $\mathcal{R}$-solver and transference theorem

Now, let me consider periodic solutions of equations:

$$\dot{v} - Av = f, \quad Bv = g \quad \text{for } t \in (0, 2\pi) = \mathbb{T}, \tag{3.1}$$

where it is assumed that $f(t+2\pi) = f(t)$ and $g(t+2\pi) = g(t)$ for $t \in \mathbb{R}$. Let

$$\hat{f}(k) := \mathcal{F}_\mathbb{T}[f](k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}f(t)\,dt, \quad \mathcal{F}^{-1}_\mathbb{T}[(a_k)_{k\in\mathbb{Z}}](t) := \sum_{k\in\mathbb{Z}} e^{ikt}a_k,$$

$$L_{p,\mathrm{per}}((0,2\pi),X) = \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R},X) \mid f(t+2\pi) = f(t) \quad (t\in\mathbb{R})\}.$$

Applying Fourier transform gives that

$$ik\hat{v} - A\hat{v} = \hat{f}(k), \quad B\hat{v} = \hat{g}(k). \tag{3.2}$$

Applying $\mathcal{R}$-solver $\mathcal{M}(\lambda)$ gives that

$$\hat{v}(k) = \mathcal{M}(ik)(\hat{f}(k), (ik)^{1/2}\hat{g}(k), \hat{g}(k))$$

for $|k| \geq \lambda_0$, because $ik \in \Sigma_{\epsilon,\lambda_0}$ for $|k| \geq \lambda_0$. The following theorem was obtained in [2].

Theorem 6 (Either-Kyed-Shibata)**.** *Let X, Y be Banach spaces and $p \in (1, \infty)$. Assume that Y is reflexive. If*

$$T_m[f] = \mathcal{F}^{-1}[m(\xi)\mathcal{F}[f](\xi)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}m(\xi)\mathcal{F}[f](\xi)\,d\xi \quad \text{for } f \text{ with } \mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, X)$$

is a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$, that is

$$\|T_m[f]\|_{L_p(\mathbb{R},Y)} \leq M_p\|f\|_{L_p(\mathbb{R},X)},$$

then

$$T_{m,\mathbb{T}}[g] := \mathcal{F}_{\mathbb{T}}^{-1}[(m|_{\mathbb{Z}}(ik)\mathcal{F}_{\mathbb{T}}[g](ik))_{k\in\mathbb{Z}}] = \sum_{k\in\mathbb{Z}} e^{ikt}m(k)\mathcal{F}_{\mathbb{T}}[g](k) \quad \text{for } g \in L_{p,\mathrm{per}}((0, 2\pi), X)$$

is also a bounded linear operator from $L_{p,\mathrm{per}}((0, 2\pi), X))$ into $L_{p,\mathrm{per}}((0, 2\pi), Y)$ with essentially the same bound. Namely, we have

$$\|T_{m,\mathbb{T}}[g]\|_{L_p((0,2\pi),Y)} \leq C_pM_p\|g\|_{L_p((0,2\pi),X)}$$

for some constant C_p depending solely p.

Let $\varphi(\tau) \in C^\infty(\mathbb{R})$ which equals 1 for $|\tau| \geq \lambda_0 + 1/2$ and 0 for $|\tau| \leq \lambda_0 + 1/4$. Set

$$v_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\mathcal{M}(ik)(\hat{f}(ik), (ik)^{1/2}\hat{g}(k), \hat{g}(k)))_{k\in\mathbb{Z}}]$$

where $\hat{h}(ik) = \mathcal{F}_{\mathbb{T}}[h](k)$. And then, v_φ satisfies the equations:

$$\partial_t v_\varphi - Av_\varphi = \mathcal{F}^{-1}[(\varphi(k)\hat{f}(k))_{k\in\mathbb{Z}}], \quad Bv_\varphi = \mathcal{F}^{-1}[(\varphi(k)\hat{g}(k))_{k\in\mathbb{Z}}].$$

By the transference theorem, Theorem 6,

$$\|\partial_t v_\varphi\|_{L_p((0,2\pi),X)} + \|v_\varphi\|_{L_p((0,2\pi),Y))} \leq C(\|f\|_{L_p((0,2\pi),X)} + \|g\|_{H_p^{1/2}((0,2\pi),X)} + \|g\|_{L_p((0,2\pi),Z)}). \tag{3.3}$$

A solution v of equations (4.29) is given by

$$v = \sum_{|k|\leq\lambda_0+1/2} e^{ikt}v_k + v_\varphi \tag{3.4}$$

where v_k are solutions of the equations:

$$ikv_k - Av_k = \hat{f}(k), \quad Bv_k = \hat{g}(k). \tag{3.5}$$

The part, v_φ, of v in (3.4) is called the high frequency part, and the estimate (3.3) is the maximal L_p regularity of the high frequency part.

4 One phase problem for the Navier-Stokes equations

The material here is taken from my joint paper [1] with Thomas Eiter and Mads Kyed. Free boundary problem for the Navier-Stokes equations is formulated as follows: Let Ω_t be a time dependent domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$), which is unknown. Let Γ_t be the boundary of Ω_t and $\mathbf{n}_t$ the unit outer normal to Γ_t. It is asssumed that Ω_t is occupied by some incompressible viscous fluid of unit mass density whose viscosity coefficient is a positive constant μ. Let $\mathbf{u} = {}^\top(u_1(x, t), \ldots, u_N(x, t))$ be the velocity field and $\mathfrak{p} = \mathfrak{p}(x,t)$ the pressure field and then $\mathbf{u}$ and $\mathfrak{p}$ satisfies the Navier-Stokes equations in Ω_t with free boundary condition as follows:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \mathbf{f} & \text{in } \Omega_t, \\ \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_t, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t & \text{on } \Gamma_t, \\ \quad V_{\Gamma_t} = \mathbf{u}\cdot\mathbf{n}_t & \text{on } \Gamma_t \end{cases} \tag{4.1}$$

for $t \in (0, 2\pi)$. Here, $\mathbf{f} = \mathbf{f}(x,t)$ is a prescribed 2π time periodic external force; $H(\Gamma_t)$ denotes the $N-1$ fold mean curvature of Γ_t which is given by $H(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t} x$ for $x \in \Gamma_t$, where Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t; V_{Γ_t} the evolution speed of Γ_t along $\mathbf{n}_t$; σ a positive constant representing the surface tension coefficient; $\mathbf{D}(\mathbf{u})$ the doubled deformation tensor given by $\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + {}^\top\nabla\mathbf{u}$; and $\mathbf{I}$ the $N\times N$ identity matrix. For any $N\times N$ matrix of functions $\mathbf{K}$ whose $(i,j)^{\rm th}$ component is K_{ij}, $\mathrm{Div}\,K$ denotes an N-vector whose $i^{\rm th}$ component is $\sum_{j=1}^n \partial_j K_{ij}$ and for any N-vector of functions $\mathbf{v} = {}^\top(v_1,\ldots,v_N)$, $\mathbf{v}\cdot\nabla\mathbf{v}$ denotes an N vector of functions whose $i^{\rm th}$ component is $\sum_{j=1}^N v_j\partial_j v_i$, where $\partial_j = \partial/\partial x_j$.

Our problem is to find Ω_t, Γ_t, $\mathbf{u}$ and $\mathfrak{p}$ satisfying the periodic condition:

$$\Omega_t = \Omega_{t+2\pi}, \quad \Gamma_t = \Gamma_{t+2\pi}, \quad \mathbf{u}(x,t) = \mathbf{u}(x,t+2\pi), \quad \mathfrak{p}(x,t) = \mathfrak{p}(x,t+2\pi) \tag{4.2}$$

for any $t \in \mathbb{R}$.

4.1 Assumptions

Let $\mathbf{p}_i = \mathbf{e}_i = {}^\top(0,\ldots,0,\overset{\text{i-th}}{1},0,\ldots,0)$ for $i = 1,\ldots,N$ and $\mathbf{p}_\ell$ ($\ell = N+1,\ldots,M$) be one of $x_i\mathbf{e}_j - x_j\mathbf{e}_i$ ($1 \le i,j \le N$). It is known that an N-vector of functions, $\mathbf{d}$, satisfies $\mathbf{D}(\mathbf{d}) = 0$ if and only if $\mathbf{d}$ is represented as a linear combination of $\mathbf{p}_i$ ($i = 1,\ldots,M$). The unknown domain Ω_t will be constructed such that the following three conditions are satisfied:

$$\det\Big(\int_0^{2\pi} (\mathbf{p}_\ell, \mathbf{p}_m)_{\Omega_t}\,dt\Big)_{\ell,m=1,\ldots,M} \neq 0, \tag{4.3}$$

$$\int_0^{2\pi}\Big(\frac{1}{|\Omega_t|}\int_{\Omega_t} x\,dx\Big)\,dt = 0, \tag{4.4}$$

$$|\Omega_t| = |B_R| \quad \text{for any } t \in (0,2\pi). \tag{4.5}$$

In what follows, the following symbols will be used:

$$H^1_{p,\mathrm{per}}((0,2\pi),X) = \{f(\cdot,t) \in L_{p.\mathrm{loc}}(\mathbb{R},X) \mid \dot f \in L_{p,\mathrm{per}}((0,2\pi),X)\};$$

$$H^{1/2}_{p,\mathrm{per}}((0,2\pi),X) = \{f(\cdot,t) \in L_{p.\mathrm{loc}}(\mathbb{R},X) \mid \mathcal{F}^{-1}_{\mathbb{T}}[((1+k^2)^{1/4}\hat f(k))_{k\in\mathbb{Z}}] \in L_{p,\mathrm{per}}((0,2\pi),X)\};$$

$$\|f\|_{L_p((0,2\pi),X)} := \Big(\int_0^{2\pi} \|f(t)\|_X^p\,dt\Big)^{1/p} < \infty;$$

$$\|f\|_{H^{1/2}_p((0,2\pi),X)} := \|\mathcal{F}^{-1}_{\mathbb{T}}[((1+k^2)^{1/4}\hat f(k))_{k\in\mathbb{Z}}]\|_{L_p((0,2\pi),X)};$$

$$(f,g)_G = \int_G f(x)\cdot\overline{g(x)}\,dx, \quad (f,g)_{\partial G} = \int_{\partial G} f(x)\overline{g(x)}\,d\sigma.$$

Let Ω_t, $\mathbf{u}$ and $\mathfrak{p}$ satisfy equations (4.1) and periodic condition (4.2), and then the divergence theorem of Gauss implies that

$$\begin{aligned} &((\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t, \mathbf{e}_i)_{\Gamma_t} = \sigma(\Delta_{\Gamma_t}x, \mathbf{e}_i)_{\Gamma_t} = -\sigma(\nabla_{\Gamma_t}x, \nabla_{\Gamma_t}\mathbf{e}_i)_{\Gamma_t} = 0; \\ &((\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t, x_i\mathbf{e}_j - x_j\mathbf{e}_i)_{\Gamma_t} = \sigma(\Delta_{\Gamma_t}x, x_i\mathbf{e}_j - x_j\mathbf{e}_i)_{\Gamma_t} \\ &\quad = -\sigma(\nabla_{\Gamma_t}x_j, \nabla_{\Gamma_t}x_i)_{\Gamma_t} + \sigma(\nabla_{\Gamma_t}x_i, \nabla_{\Gamma_t}x_j)_{\Gamma_t} = 0. \end{aligned} \tag{4.6}$$

Since
$$\frac{d}{dt}(\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}), \mathbf{p}_\ell)_{\Omega_t} + (\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t}$$
as follows from the first equation in (4.1) and $\mathrm{div}\,\mathbf{u} = 0$, it follows from (4.6) that
$$\frac{d}{dt}(\mathbf{u}, \mathbf{p}_\ell)_{\Omega_t} = (\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t}. \tag{4.7}$$
Assumption on f. There exists a domain $D \subset \Omega_t$ such that $\mathrm{supp}\,\mathbf{f}(x,t) \subset D$ for any $t \in \mathbb{R}$. □

Thus, the periodic condition (4.2) together with (4.7) yields that
$$\int_0^{2\pi}\Big(\int_D \mathbf{f}(x,\cdot)\cdot\mathbf{p}_\ell(x)\,dx\Big)\,dt = 0 \quad \text{for } \ell = 1,\ldots,M. \tag{4.8}$$
Instead of problem (4.2), we consider the following equations:
$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) + \displaystyle\sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot,t), \mathbf{p}_k)_{\Omega_t}\,dt\,\mathbf{p}_k = \mathbf{f} & \text{in } \Omega_t, \\ \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_t, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t & \text{on } \Gamma_t, \\ \quad V_{\Gamma_t} = \mathbf{u}\cdot\mathbf{n}_t & \text{on } \Gamma_t \end{cases} \tag{4.9}$$
for $t \in (0, 2\pi)$. In fact, if Ω_t, $\mathbf{u}$ and $\mathfrak{p}$ satisfy equations (4.9), the assumption (4.3), and the periodic condition (4.2), then by (4.8) we have
$$(\mathbf{f}, \mathbf{p}_\ell)_{\Omega_t} = \frac{d}{dt}(\mathbf{u}\cdot, t), \mathbf{p}_\ell)_{\Omega_t} + \sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot,t), \mathbf{p}_k)_{\Omega_t}\,dt(\mathbf{p}_k, \mathbf{p}_\ell)_{\Omega_t}.$$
Integrating this formula on $(0, 2\pi)$ and using the periodicity and the assumption on $\mathbf{f}$, (4.8), gives
$$\sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot,t), \mathbf{p}_k)_{\Omega_t}\,dt \int_0^{2\pi} (\mathbf{p}_k, \mathbf{p}_\ell)_{\Omega_t}\,dt = 0,$$
which, combined with (4.3), yields that
$$\int_0^{2\pi} (\mathbf{u}(\cdot,t), \mathbf{p}_k)_{\Omega_t}\,dt = 0. \tag{4.10}$$
Thus, Ω_t, $\mathbf{u}$ and $\mathfrak{p}$ satisfy the first equation in (4.1).

4.2 Hanzawa transform

Since Ω_t is unknown, the problem should be formulated in a fixed domain. For this purpose, the Hanzawa transform is used. Let $\xi(t)$ be the barycenter point of Ω_t defined by setting
$$\xi(t) = \frac{1}{|B_R|}\int_{\Omega_t} x\,dx, \tag{4.11}$$
where the fact that $|\Omega_t| = |B_R|$ has been used, which follows from the assumption (4.5). From the Reynolds transport theorem and $\mathrm{div}\,\mathbf{u} = 0$ it follows that
$$\frac{d}{dt}\xi(t) = \frac{1}{|B_R|}\int_{\Omega_t}(\partial_t x + \mathbf{u}\cdot\nabla x)\,dx = \frac{1}{|B_R|}\int_{\Omega_t}\mathbf{u}(x,t)\,dx, \quad \xi''(t) = \frac{1}{|B_R|}\int_{\Omega_t}\mathbf{f}(x,t)\,dx. \tag{4.12}$$

Let $\rho(y,t)$ be an unknown periodic function with period 2π such that

$$\Gamma_t = \{x = y + R^{-1}\rho(y,t)y + \xi(t) \mid y \in S_R\},$$

where $S_R = \{x \in \mathbb{R}^N \mid |x| = R\}$ and $R^{-1}y$ is the unit outer normal to S_R for $y \in S_R$. Let H_ρ be a suitable extension of ρ to $\mathbb{R}^N$ such that

$$\|H_\rho\|_{H^k_q(B_R)} \leq C\|\rho\|_{W^{k-1/q}_q(S_R)} \quad \text{for } k = 1, 2, 3,$$
$$\|\partial_t H_\rho\|_{H^k_q(B_R)} \leq C\|\partial_t\rho\|_{W^{k-1/q}_q(S_R)} \quad \text{for } k = 1, 2.$$

Let

$$\begin{aligned}\Omega_t &= \{x = y + R^{-1}H_\rho(y,t)y + \xi(t) \mid y \in B_R\},\\ \Gamma_t &= \{x = y + R^{-1}\rho(y,t)y + \xi(t) \mid y \in S_R\}.\end{aligned} \tag{4.13}$$

Let $J(t)$ be the Jacobian of the transformation: $x = \Phi(y,t) = y + R^{-1}H_\rho(y,t)y + \xi(t)$. Assume that

$$\sup_{t\in(0,2\pi)} \|H_\rho(\cdot,t)\|_{H^1_\infty(B_R)} \leq \delta \tag{4.14}$$

with some small constant $\delta > 0$, which is chosen so small that the map $x = \Phi(y,t)$ is injective for any $t \in (0,2\pi)$, and so the inverse map $y = \Phi^{-1}(x,t)$ exists and has the same regularity property as that Φ has.

4.3 Kinematic equations

Let $\mathbf{u}(x,t)$ and $\mathfrak{p}(x,t)$ satisfy equations (4.1), and let $\mathbf{v}(y,t) = \mathbf{u}(\Phi(y,t),t)$ and $\mathfrak{q}(y,t) = \mathfrak{p}(\Phi(y,t),t)$. An equation for $\mathbf{v}$ and ρ is derived from the kinematic condition : $V_{\Gamma_t} = \mathbf{u}\cdot\mathbf{n}_t$ on Γ_t. From the fact that Γ_t is represented by $x = y + R^{-1}\rho(y,t)y + \xi(t)$ it follows that

$$V_{\Gamma_t} = \frac{\partial x}{\partial t}\cdot\mathbf{n}_t = (\frac{\partial\rho}{\partial t}\mathbf{n} + \xi'(t))\cdot\mathbf{n}_t, \tag{4.15}$$

where $\mathbf{n} = R^{-1}y$. To represent $\xi'(t)$, let me represent the Jacobian, $J(t)$, of the map $x = \Phi(y,t)$ as $J(t) = 1 + J_0(t)$ with

$$J_0(t) = \det\big(\delta_{ij} + R^{-1}\frac{\partial}{\partial y_i}(H_\rho(y,t)y_j)\big)_{i,j=1,\ldots,N} - 1.$$

Thus,

$$\xi'(t) = \frac{1}{|B_R|}\int_{\Omega_t}\mathbf{u}\,dx = \frac{1}{|B_R|}\int_{B_R}\mathbf{v}(y,t)\,dy + \frac{1}{|B_R|}\int_{B_R}\mathbf{v}(y,t)J_0(t)\,dy,$$

which, combined with (4.16) and the kinematic condition: $V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_t$, yields that

$$\partial_t\rho - (\mathbf{v} - \frac{1}{|B_R|}\int_{B_R}\mathbf{v}(y,t)\,dy) = d(\mathbf{v},\rho) \tag{4.16}$$

with

$$d(\mathbf{v},\rho) = \frac{1}{|B_R|}\int_{B_R}\mathbf{v}(y,t)J_0(t)\,dy + \frac{\partial\rho}{\partial t}\mathbf{n}\cdot(\mathbf{n}-\mathbf{n}_t) + \mathbf{v}\cdot(\mathbf{n}_t - \mathbf{n}). \tag{4.17}$$

4.4 Mass conservation and Barycenter

The the case where Ω_t is close to B_R is considerd below, and so Δ_{Γ_t} is a small perturbation from Δ_{S_R}, where Δ_{S_R} is the Laplace-Beltrami operator on S_R. Thus,

$$< H(\Gamma_t)\mathbf{n}_t, \mathbf{n}_t >= (\Delta_{S_R} + (N-1)/R^2)\rho - (N-1)/R + \text{ nonlinear terms.}$$

Here, $-(N-1)/R^2$ is the first eigen-value of the Laplace-Beltrami operator Δ_{S_R} on S_R with eigen-functions y_j/R for $y = (y_1, \dots, y_N) \in S_R$. To avoid the zero and first eigen-values of Δ_{S_R} in this linear analysis, the following observation is useful: Since $\Gamma_t = \partial\Omega_t = \{x = y + R^{-1}\rho(y,t)y + \xi(t) \mid y \in S_R\}$,

$$\begin{aligned}
|B_R| = |\Omega_t| &= \int_{S_R} \Big(\int_0^{R+\rho(\omega,t)} r^{N-1}\,dr\Big)\,d\omega = \frac{1}{N}\int_{S_R} (R+\rho(\omega))^N\,d\omega \\
&= |B_R| + \int_{S_R} \rho\,d\omega + \sum_{k=2}^N \frac{{}_NC_k}{N}\int_{S_R} \rho^k\,d\omega = 0,
\end{aligned}$$

and so

$$\int_{S_R} \rho\,d\omega + \sum_{k=2}^N \frac{{}_NC_k}{N}\int_{S_R} \rho^k\,d\omega = 0, \tag{4.18}$$

where $d\omega$ denotes the surface element of S_R. The relations $\Gamma_t = \partial\Omega_t = \{x = y + R^{-1}\rho(y,t)y + \xi(t) \mid y \in S_R\}$ and $\xi(t) = \frac{1}{|B_R|}\int_{\Omega_t} x\,dx$ gives that

$$\begin{aligned}
0 &= \frac{1}{|B_R|}\int_{\Omega_t} (x - \xi(t))\,dx = \frac{1}{|B_R|}\int_{S_R}\Big(\int_0^{R+\rho} r^N\omega\,dr\Big)\,d\omega \\
&= \frac{1}{|B_R|}\frac{1}{N+1}\int_{S_R} (R+\rho)^{N+1}\omega\,d\omega \\
&= \frac{1}{|B_R|}\Big(\int_{S_R} \rho\omega\,d\omega + \sum_{k=2}^{N+1} \frac{{}_{N+1}C_k}{N+1}\int_{S_R} \rho^k\omega\,d\omega\Big),
\end{aligned}$$

from which it follows that

$$\int_{S_R} \rho\omega_j\,d\omega + \sum_{k=2}^{N+1}\sum_{k=2}^{N+1} \frac{{}_{N+1}C_k}{N+1}\int_{S_R} \rho^k\omega_j\,d\omega = 0 \tag{4.19}$$

for $j = 1, \dots, N$.

4.5 New kinematic equation

Using these two formulas (4.18) and (4.19), one can see that the kinematic equation is equivalent to equation:

$$\partial_t\rho + \int_{S_R} \rho\,d\omega + \sum_{k=1}^N \Big(\int_{S_R} \rho\omega_k\,d\omega\Big)y_k - \Big(\mathbf{v} - \frac{1}{|B_R|}\int_{B_R} \mathbf{v}\,dy\Big)\cdot\mathbf{n} = \tilde{d}(\mathbf{v},\rho) \tag{4.20}$$

on $S_R \times (0, 2\pi)$ with

$$\tilde{d}(\mathbf{v},\rho) = d(\mathbf{v},\rho) - \sum_{k=2}^N \frac{{}_NC_k}{N}\int_{S_R} \rho^k\,d\omega - \sum_{k=2}^{N+1} \frac{{}_{N+1}C_k}{N+1}\Big(\int_{S_R} \rho^k\omega\,d\omega\Big)y_k. \tag{4.21}$$

4.6 Linearization Principle

To prove the existence of $(\Omega_t, \mathbf{u}, \mathfrak{p})$ satisfying (4.1), it is enough to prove the existence of periodic solutions to the following equations:

$$\begin{cases} \partial_t \mathbf{v} + \mathcal{L}\mathbf{v}_S - \mathrm{Div}\,(\mu(\mathbf{D}(\mathbf{v}) - \mathfrak{q}\mathbf{I}) = \mathbf{G} + \mathbf{F}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \mathrm{div}\,\mathbf{v} = g(\mathbf{v}, \rho) = \mathrm{div}\,\mathbf{g}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \partial_t \rho + \mathcal{M}\rho - \mathcal{A}\mathbf{v} \cdot \mathbf{n} = \tilde{d}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \\ (\mu \mathbf{D}(\mathbf{v}) - \mathfrak{q})\mathbf{n} - (\mathcal{B}_R \rho)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \end{cases} \tag{4.22}$$

where $\mathbf{G}(y, t) = \nabla\Phi(y, t)\mathbf{f}(\Phi(y, t), t)$, $\mathbf{F}(\mathbf{v}, \rho)$, $g(\mathbf{v}, \rho)$, $\mathbf{g}(\mathbf{v}, \rho)$, and $\mathbf{h}(\mathbf{v}, \rho)$ are nonlinear terms, and we have set

$$\begin{aligned} \mathcal{L}\mathbf{v}_S &= 2\pi \sum_{k=1}^{M} (\mathbf{v}_S, \mathbf{p}_k)_{\mathbb{T}}\, \mathbf{p}_k, \quad \mathbf{v}_S = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{v}(\cdot, s)\, ds, \\ \mathcal{A}\mathbf{v} &= \mathbf{v} - \frac{1}{|B_R|} \int_{B_R} \mathbf{v}\, dy; \quad \mathcal{M}\rho = \int_{S_R} \rho\, d\omega + \sum_{k=1}^{N} \Big(\int_{S_R} \rho \omega_k\, d\omega \Big) y_k; \\ \mathcal{B}_R \rho &= (\Delta_{S_R} + \frac{N-1}{R^2})\rho = R^{-2}(\Delta_{S_1} + (N-1))\rho, \end{aligned} \tag{4.23}$$

where Δ_{S_1} is the Laplace-Beltrami operator on the unit sphere S_1.

4.7 Main Result

Theorem 7. *Let $1 < p, q < \infty$ and $2/p + N/q < 1$. Let D be a domain $\subset B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$. Assume that*

- $\mathbf{f} \in L_{p,\mathrm{per}}((0, 2\pi), L_q(D)^N)$ *and* $\mathrm{supp}\,\mathbf{f}(\cdot, t) \subset D$ *for any* $t \in (0, 2\pi)$,
- $$\int_0^{2\pi} (\mathbf{f}(\cdot, t), \mathbf{p}_\ell)_D\, dt = 0 \quad \text{for } \ell = 1, \ldots, M, \tag{4.24}$$
- *there exists a small* $\epsilon > 0$ *such that* $\|\mathbf{f}\|_{L_p((0,2\pi), L_q(D)^N)} \le \epsilon$.

Then, there exist $\mathbf{v}(y, t)$, $\mathfrak{q}(y, t)$, *and* $\rho(y, t)$ *with*

$$\begin{aligned} &\mathbf{v} \in L_{p,\mathrm{per}}((0, 2\pi), H_q^2(B_R)^N) \cap H^1_{p,\mathrm{per}}((0, 2\pi), L_q(B_R)^N), \\ &\mathfrak{q} \in L_{p,\mathrm{per}}((0, 2\pi), H_q^1(B_R)), \\ &\rho \in L_{p,\mathrm{per}}((0, 2\pi), W_q^{3-1/q}(B_R)^N) \cap H^1_{p,\mathrm{per}}((0, 2\pi), W_q^{2-1/q}(S_R)), \end{aligned} \tag{4.25}$$

such that the correspondence: $x = \Phi(y, t) := y + R^{-1}H_\rho(y, t)y + \xi(t)$ *is an injective map defined on* B_R *and*

$$\begin{gathered} \Omega_t = \{x = \Phi(y, t) \mid y \in B_R\}, \quad \Gamma_t = \{x = y + R^{-1}\rho(y, t)y + \xi(t) \mid y \in S_R\}, \\ \mathbf{u}(x, t) = \mathbf{v}(\Phi^{-1}(x, t), t), \quad \mathfrak{p}(x, t) = \mathfrak{q}(\Phi^{-1}(x, t), t), \end{gathered}$$

where $\Phi^{-1}(x, t)$ *is the inverse map of the correspondence:* $x = \Phi(y, t)$ *for any* $t \in (0, 2\pi)$, *are unique solutions of equations* (4.1) *satisfying the periodic condition* (4.2), *where* $\xi(t)$ *is a* 2π *periodic function which satisfies the equation:*

$$\xi(t) = \frac{1}{|\Omega_t|} \int_{\Omega_t} x\, dx.$$

Moreover, $\mathbf{v}$ *and* ρ *satisfy the estimate:*

$$\begin{aligned}&\|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R))}+\|\partial_t\mathbf{v}\|_{L_p((0,2\pi),L_q(B_R))}\\&\quad+\|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))}+\|\partial_t\rho\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))}+\|\partial_t\rho\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))}\le C\epsilon\end{aligned}\tag{4.26}$$

for some constant C *independent of* ϵ*.*

4.8 Linearized equations

The linearized equations for problem (4.22) are the set of the following equations:

$$\begin{cases}\partial_t\mathbf{v}+\mathcal{L}\mathbf{v}_S-\operatorname{Div}(\mu(\mathbf{D}(\mathbf{v})-\mathfrak{q}\mathbf{I})=\mathbf{f} & \text{in } B_R\times(0,2\pi),\\ \operatorname{div}\mathbf{v}=g=\operatorname{div}\mathbf{g} & \text{in } B_R\times(0,2\pi),\\ \partial_t\rho+\mathcal{M}\rho-\mathcal{A}\mathbf{v}\cdot\mathbf{n}=d & \text{on } S_R\times(0,2\pi),\\ (\mu\mathbf{D}(\mathbf{v})-\mathfrak{q})\mathbf{n}-(\mathcal{B}_R\rho)\mathbf{n}=\mathbf{h} & \text{on } S_R\times(0,2\pi).\end{cases}\tag{4.27}$$

The following theorem is the unique existence of periodic solutions of problem (4.27).

Theorem 8. *Let* $1<p,q<\infty$*. Then, for any*

$$\begin{aligned}&\mathbf{f}\in L_{p,\mathrm{per}}((0,2\pi),L_q(B_R)^N),\\&g\in L_{p,\mathrm{per}}((0,2\pi),H_q^1(B_R))\cap H^{1/2}_{p,\mathrm{per}}((0,2\pi),L_q(B_R)),\\&\mathbf{g}\in H^1_{p,\mathrm{per}}((0,2\pi),L_q(B_R)^N),\quad d\in L_{p,\mathrm{per}}((0,2\pi),W_q^{2-1/q}(S_R)),\\&\mathbf{h}\in L_{p,\mathrm{per}}((0,2\pi),H_q^1(B_R)^N)\cap H^{1/2}_{p,\mathrm{per}}((0,2\pi),L_q(B_R)^N),\end{aligned}$$

problem (4.27) *admits unique solutions* $\mathbf{v}$*,* $\mathfrak{q}$*, and* ρ *with*

$$\begin{aligned}\mathbf{v}&\in L_{p,\mathrm{per}}((0,2\pi),H_q^2(B_R)^N)\cap H^1_{p,\mathrm{per}}((0,2\pi),L_q(B_R)^N),\\\nabla\mathfrak{q}&\in L_{p,\mathrm{per}}((0,2\pi),L_q(B_R)^N),\\\rho&\in L_{p,\mathrm{per}}((0,2\pi),W_q^{3-1/q}(S_R))\cap H^1_{p,\mathrm{per}}((0,2\pi),W_q^{2-1/q}(S_R))\end{aligned}$$

possessing the estimate:

$$\begin{aligned}&\|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R))}+\|\partial_t\mathbf{v}\|_{L_p((0,2\pi),L_q(B_R))}+\|\nabla\mathfrak{q}\|_{L_p((0,2\pi),L_q(B_R))}\\&\quad+\|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))}+\|\partial_t\rho\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))}\\&\qquad\le C\{\|\mathbf{f}\|_{L_p((0,2\pi),L_q(B_R))}+\|d\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))}\\&\qquad\quad+\|\mathbf{g}\|_{H_p^1((0,2\pi),L_q(B_R))}+\|(g,\mathbf{h})\|_{H_p^{1/2}((0,2\pi),L_q(B_R))}+\|(g,\mathbf{h})\|_{L_p((0,2\pi),H_q^1(B_R))}\}.\end{aligned}$$

In what follows, I will give an idea how to prove Theorem 8.

4.9 $\mathcal{R}$-solver and High frequency part

For any periodic function, f, the stationary part f_S and oscillatory part f_{per} are defined by setting

$$f_S=\frac{1}{2\pi}\int_0^{2\pi}f(\cdot,s)\,ds,\quad f_{\mathrm{per}}(\cdot,t)=f(\cdot,t)-f_S(\cdot).$$

And then, problem (4.27) is divided as follows:

$$\begin{cases} \mathcal{L}\mathbf{v}_S - \mathrm{Div}\,(\mu(\mathbf{D}(\mathbf{v}_S) - \mathfrak{q}_S\mathbf{I}) = \mathbf{f}_S & \text{in } B_R, \\ \mathrm{div}\,\mathbf{v}_S = g_S = \mathrm{div}\,\mathbf{g}_S & \text{in } B_R, \\ \mathcal{M}\rho_S - \mathcal{A}\mathbf{v}_S \cdot \mathbf{n} = d_S & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{v}_S) - \mathfrak{q}_S)\mathbf{n} - (\mathcal{B}_R\rho_S)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho)_S & \text{on } S_R \times (0, 2\pi), \end{cases} \tag{4.28}$$

and

$$\begin{cases} \partial_t\mathbf{v}_{\mathrm{per}} - \mathrm{Div}\,(\mu(\mathbf{D}(\mathbf{v}_{\mathrm{per}}) - \mathfrak{q}_{\mathrm{per}}\mathbf{I}) = \mathbf{f}_{\mathrm{per}} & \text{in } B_R \times (0, 2\pi), \\ \mathrm{div}\,\mathbf{v}_{\mathrm{per}} = g_{\mathrm{per}} = \mathrm{div}\,\mathbf{g}_{\mathrm{per}} & \text{in } B_R \times (0, 2\pi), \\ \partial_t\rho_{\mathrm{per}} + \mathcal{M}\rho_{\mathrm{per}} - \mathcal{A}\mathbf{v}_{\mathrm{per}} \cdot \mathbf{n} = d_{\mathrm{per}} & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{v}_{\mathrm{per}}) - \mathfrak{q}_{\mathrm{per}})\mathbf{n} - (\mathcal{B}_R\rho_{\mathrm{per}})\mathbf{n} = \mathbf{h}_{\mathrm{per}} & \text{on } S_R \times (0, 2\pi), \end{cases} \tag{4.29}$$

In this subsection, I consider problem (4.29) for the high frequency part.

According to Sect. 3, I consider the generalized resolvent problem:

$$\begin{aligned} \lambda\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \hat{\mathbf{f}} & \quad \text{in } B_R, \\ \mathrm{div}\,\mathbf{u} = \hat{g} = \mathrm{div}\,\hat{\mathbf{g}} & \quad \text{in } B_R, \\ \lambda\eta + \mathcal{M}\eta - (\mathcal{A}\mathbf{u}) \cdot \mathbf{n} = \hat{d} & \quad \text{on } S_R, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n} - (\mathcal{B}_R\eta)\mathbf{n} = \hat{\mathbf{h}} & \quad \text{on } S_R \end{aligned} \tag{4.30}$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with any $\epsilon \in (0, \pi/2)$ and some large positive number λ_0 depending on ϵ. And then, from the result due to Shibata [4, 5], the following theorem follows.

Theorem 9. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let*

$$\begin{aligned} X_q(B_R) = \{&(\hat{\mathbf{f}}, \hat{d}, \hat{\mathbf{h}}, \hat{g}, \hat{\mathbf{g}}) \mid \hat{\mathbf{f}} \in L_q(B_R)^N,\ \ \hat{d} \in W_q^{2-1/q},\ \ \hat{\mathbf{h}} \in H_q^1(B_R)^N, \\ &\hat{g} \in H_q^1(B_R),\ \ \hat{\mathbf{g}} \in L_q(\mathbb{R}^N)^N\}, \\ \mathcal{X}_q(B_R) = \{&F = (F_1, F_2, \ldots, F_7) \mid F_1, F_3, F_7 \in L_q(B_R)^N,\ \ F_2 \in W_q^{2-1/q}(S_R), \\ &F_4 \in H_q^1(B_R)^N,\ \ F_5 \in L_q(B_R),\ \ F_6 \in H_q^1(\Omega)\}. \end{aligned}$$

Here, F_1, F_2, F_3, F_4, F_5, F_6, and F_7 are corresponding variables to $\hat{\mathbf{f}}$, $\hat{d}$, $\lambda^{1/2}\hat{\mathbf{h}}$, $\hat{\mathbf{h}}$, $\lambda^{1/2}\hat{g}$, $\hat{g}$, and $\lambda\hat{\mathbf{g}}$, respectively.

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{A}(\lambda)$, $\mathcal{P}(\lambda)$, and $\mathcal{H}(\lambda)$ with

$$\begin{aligned} 2\mathcal{A}(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), H_q^2(B_R)^N)), \\ \mathcal{P}(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), H_q^1(B_R))), \\ \mathcal{H}(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(B_R), W_q^{3-1/q}(S_R))), \end{aligned}$$

where $\mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, X)$ denotes the set of all X-valued holomorphic functions defined on $\Sigma_{\epsilon,\lambda_0}$, such that for any $(\hat{\mathbf{f}}, \hat{d}, \hat{\mathbf{h}}, \hat{g}, \hat{\mathbf{g}}) \in X_q(B_R)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v} = \mathcal{A}(\lambda)\mathcal{F}_\lambda$, $\mathfrak{q} = \mathcal{P}(\lambda)\mathcal{F}_\lambda$ and $\eta = \mathcal{H}(\lambda)\mathcal{F}_\lambda$, where

$$\mathcal{F}_\lambda = (\hat{\mathbf{f}}, \hat{d}, \lambda^{1/2}\hat{\mathbf{h}}, \hat{\mathbf{h}}, \lambda^{1/2}\hat{g}, \hat{g}, \lambda\hat{\mathbf{g}}),$$

are unique solutions of equations (4.30), and

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), H_q^{2-m}(B_R)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{m/2}\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le r_b, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), L_q(B_R)^N)}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{P}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le r_b, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(B_R), W_q^{3-n-1/q}(S_R))}(\{(\tau\partial_\tau)^\ell(\lambda^n\mathcal{H}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le r_b \end{aligned} \tag{4.31}$$

for $\ell = 0, 1$, $m = 0, 1, 2$ and $n = 0, 1$ with some constant r_b, where $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_0} \subset \mathbb{C}$.

Let k_0 be a natural number such that $\lambda_0 < k_0$ and φ a $C^\infty(\mathbb{R})$ function which equals one for $|k| \geq k_0+1$ and zero for $|k| \leq k_0 + 1/2$. Let

$$\begin{aligned}
&\mathbf{F}_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{\mathbf{f}}_{\rm per}(ik))_{k\in\mathbb{Z}}], \quad G_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{g}_{\rm per}(ik))_{k\in\mathbb{Z}}], \quad \mathbf{G}_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{\mathbf{g}}_{\rm per}(ik))_{k\in\mathbb{Z}}],\\
&D_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{d}_{\rm per}(ik))_{k\in\mathbb{Z}}], \quad \mathbf{H}_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{\mathbf{h}}_{\rm per}(ik))_{k\in\mathbb{Z}}].
\end{aligned}$$

Let

$$\mathbf{v}_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\mathcal{A}(ik)\mathbf{F}_k)_{k\in\mathbb{Z}}], \quad \mathfrak{q}_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\mathcal{P}(ik)\mathbf{F}_k)_{k\in\mathbb{Z}}], \quad \rho_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\mathcal{H}(ik)\mathbf{F}_k)_{k\in\mathbb{Z}}],$$

where $\mathbf{F}_k = (\hat{\mathbf{f}}_{\rm per}(ik), \hat{d}_{\rm per}(ik), (ik)^{1/2}\hat{\mathbf{h}}_{\rm per}(ik), \hat{\mathbf{h}}_{\rm per}(ik), (ik)^{1/2}\hat{g}_{\rm per}(ik), \hat{g}_{\rm per}(ik), ik\hat{\mathbf{g}}_{\rm per}(ik))$. Then, $\mathbf{v}_\varphi$, $\mathfrak{q}_\varphi$ and ρ_φ are unique solutions of equations:

$$\begin{aligned}
\partial_t\mathbf{v}_\varphi - {\rm Div}\,(\mu\mathbf{D}(\mathbf{v}_\varphi) - \mathfrak{q}_\varphi\mathbf{I}) = \mathbf{F}_\varphi &\quad \text{in } B_R \times (0,2\pi),\\
{\rm div}\,\mathbf{v}_\varphi = G_\varphi = {\rm div}\,\mathbf{G}_\varphi &\quad \text{in } B_R \times (0,2\pi),\\
\partial_t\rho_\varphi + \mathcal{M}\rho_\varphi - (\mathcal{A}\mathbf{v}_\varphi)\cdot\mathbf{n} = D_\varphi &\quad \text{on } S_R \times (0,2\pi),\\
(\mu\mathbf{D}(\mathbf{v}_\varphi) - \mathfrak{q}_\varphi\mathbf{I})\mathbf{n} - (\mathcal{B}_R\rho_\varphi)\mathbf{n} = \mathbf{H}_\varphi &\quad \text{on } S_R \times (0,2\pi),
\end{aligned} \tag{4.32}$$

with

$$\begin{aligned}
&\mathbf{v}_\varphi \in L_{p,\rm per}((0,2\pi), H^2_q(B_R)^N) \cap H^1_{p,\rm per}((0,2\pi), L_q(\mathbb{R}^N)^N), \quad \nabla\mathfrak{q}_\varphi \in L_{p,\rm per}((0,2\pi), L_q(B_R)^N),\\
&\rho_\varphi \in L_{p,\rm per}((0,2\pi), W^{3-1/q}_q(S_R)) \cap H^1_{p,\rm per}((0,2\pi), W^{2-1/q}_q(S_R)).
\end{aligned}$$

Moreover, the following estimate holds:

$$\begin{aligned}
&\|\mathbf{v}_\varphi\|_{L_p((0,2\pi),H^2_q(B_R))} + \|\partial_t\mathbf{v}_\varphi\|_{L_p((0,2\pi),L_q(B_R))} + \|\nabla\mathfrak{q}_\varphi\|_{L_p((0,2\pi),L_q(B_R))}\\
&\quad + \|\rho_\varphi\|_{L_p((0,2\pi),W^{3-1/q}_q(S_R))} + \|\partial_t\rho_\varphi\|_{H^1_p((0,2\pi),W^{2-1/q}_q(S_R))}\\
&\qquad \leq C\{\|\mathbf{F}_\varphi\|_{L_p((0,2\pi),L_q(B_R))} + \|D_\varphi\|_{L_p((0,2\pi),W^{2-1/q}_q(S_R))} + \|(G_\varphi,\mathbf{H}_\varphi)\|_{H^{1/2}_p((0,2\pi),L_q(B_R))}\\
&\qquad\quad + \|(G_\varphi,\mathbf{H}_\varphi)\|_{L_p((0,2\pi),H^1_q(B_R))} + \|\partial_t\mathbf{G}_\varphi\|_{L_p((0,2\pi),L_q(B_R))}\}
\end{aligned} \tag{4.33}$$

for some constant $C > 0$. To estimate the right side of (4.33), we use the inequality:

$$\|\mathcal{F}_{\mathbb{T}}^{-1}[(\varphi(k)\hat{f}(ik))_{k\in\mathbb{Z}}]\|_{L_p((0,2\pi),X)} \leq C_p\|f\|_{L_p((0,2\pi),X)}$$

where X is a UMD Banach space, which follows from Weis' operator valued Fourier multiplier theorem, Theorem 4.

4.10 Low frequency part

I now consider the generalized resolvent problem corresponding to (4.29) for $k \in [-k_0, k_0]$. Namely, I consider the following equations:

$$\begin{aligned}
ik\mathbf{v}_k - {\rm Div}\,(\mu\mathbf{D}(\mathbf{v}_k) - \mathfrak{p}_k\mathbf{I}) = \hat{\mathbf{f}}_{\rm per}(ik) &\quad \text{in } B_R,\\
{\rm div}\,\mathbf{v}_k = \hat{g}(ik) = {\rm div}\,\hat{\mathbf{g}}_{\rm per}(ik) &\quad \text{in } B_R,\\
ik\rho_k + \mathcal{M}\rho_k - (\mathcal{A}\mathbf{v}_k)\cdot\mathbf{n} = \hat{d}_{\rm per}(ik) &\quad \text{on } S_R,\\
(\mu\mathbf{D}(\mathbf{v}_k) - \mathfrak{p}_k\mathbf{I})\mathbf{n} - (\mathcal{B}_R\rho_k)\mathbf{n} = \hat{\mathbf{h}}_{\rm per}(ik) &\quad \text{on } S_R
\end{aligned} \tag{4.34}$$

for $k \in [-k_0, k_0] \setminus \{0\}$. Then, the following theorem holds.

Theorem 10. *Let $1 < q < \infty$ and $k \in \mathbb{Z}$ with $1 \leq |k| \leq k_0$. Then, for any $\hat{\mathbf{f}}_{\rm per}(ik) \in L_q(B_R)^N$, $\hat{g}_{\rm per}(ik) \in H^1_q(B_R)$, $\hat{d}_{\rm per}(ik) \in W^{2-1/q}_q(S_R)$, $\hat{\mathbf{h}}_{\rm per}(ik) \in H^1_q(B_R)^N$, and $\hat{\mathbf{g}}_{\rm per}(ik) \in L_q(B_R)^N$, problem (4.38) admits unique solutions $\mathbf{v}_k \in H^2_q(B_R)^N$, $\mathfrak{q}_k \in H^1_q(B_R)$, and $\eta_k \in W^{3-1/q}_q(S_R)$ possessing the estimate:*

$$\begin{aligned}&\|\mathbf{v}_k\|_{H^2_q(B_R)} + \|\nabla\mathfrak{q}_k\|_{L_q(B_R)} + \|\eta_k\|_{W^{3-1/q}_q(S_R)}\\ &\quad\leq C(\|\hat{\mathbf{f}}_{\rm per}(ik)\|_{L_q(B_R)} + \|\hat{d}_{\rm per}(ik)\|_{W^{2-1/q}_q(S_R)} + \|(\hat{g}_{\rm per}(ik), \hat{\mathbf{h}}(ik))\|_{H^1_q(B_R)} + \|\hat{\mathbf{g}}_{\rm per}(ik)\|_{L_q(B_R)})\end{aligned} \tag{4.35}$$

for some constant $C > 0$ independent of k with $|k| \leq k_0$.

Remark 11. To estimate the right side of (4.39), we use the inequality:

$$\|\hat{f}(ik)\|_X \leq (2\pi)^{-1}\int_0^{2\pi}\|f(s)\|_X\,ds \leq (2\pi)^{-1/p'}\|f\|_{L_p((0,2\pi),X)}$$

for any $f \in L_p((0, 2\pi), X)$, where X is a Banach space and $\|\cdot\|_X$ is its norm.

To prove Theorem 10, in view of the Riesz-Schauder theorem, Fredholm alternative principle, it is sufficien to prove the uniqueness in the L_2 framework. Let $\mathbf{w} \in H^2_2(B_R)^N$, $\mathfrak{q} \in H^1_2(B_R)$ and $\zeta \in W^{3-1/2}_2(S_R)$ satisfy the homogeneous equations:

$$\begin{aligned}ik\mathbf{w} - {\rm Div}\,(\mu\mathbf{D}(\mathbf{w}) - \mathfrak{q}\mathbf{I}) = 0, \quad {\rm div}\,\mathbf{w} = 0 \qquad &\text{in } B_R,\\ ik\zeta + \mathcal{M}\zeta - (\mathcal{A}\mathbf{w})\cdot\mathbf{n} = 0 \qquad &\text{on } S_R,\\ (\mu\mathbf{D}(\mathbf{w}) - \mathfrak{q}\mathbf{I})\mathbf{n} - \sigma(\mathcal{B}_R\zeta)\mathbf{n} = 0 \qquad &\text{on } S_R.\end{aligned} \tag{4.36}$$

Recall: $\mathcal{M}\zeta = \int_{S_R}\zeta\,d\omega + \sum_{k=1}^N\Big(\int_{S_R}\zeta\omega_k\,d\omega\Big)y_k$ and $\mathcal{A}\mathbf{v} = \mathbf{v} - \frac{1}{|B_R|}\int_{B_R}\mathbf{v}\,dy$. We first prove that

$$(\zeta, 1)_{S_R} = 0, \quad (\zeta, x_j)_{S_R} = 0 \quad \text{for } j = 1, \ldots, N. \tag{4.37}$$

Integrating the second equation of equations (4.40) and applying the divergence theorem of Gauss gives that

$$0 = ik(\zeta, 1)_{S_R} + (\zeta, 1)_{S_R}|S_R| - \int_{B_R}{\rm div}\,\mathcal{A}\mathbf{w}\,dx = (ik + |S_R|)(\zeta, 1)_{S_R},$$

where we have set $|S_R| = \int_{S_R}d\omega$ and we have used the fact that ${\rm div}\,\mathbf{w} = 0$ in B_R. Thus, we have $(\zeta, 1)_{S_R} = 0$.

Multiplying the second equation of equations (4.40) with x_j, integrating the resultant formla over S_R and using the divergence theorem of Gauss gives that

$$0 = ik(\zeta, x_k)_{S_R} + (\zeta, x_k)_{S_R}(x_k, x_k)_{S_R} - \int_{B_R}{\rm div}\,(x_k\mathcal{A}\mathbf{w})\,dx,$$

because $(x_j, x_k)_{S_R} = 0$ for $j \neq k$. Since

$$\int_{B_R}{\rm div}\,(x_k\mathcal{A}\mathbf{w})\,dx = \int_{B_R}(\mathbf{w}_k - \frac{1}{|B_R|}\int_{B_R}\mathbf{w}_k\,dx)\,dx = 0,$$

we have $(\zeta, x_k)_{S_R} = 0$, because $(x_k, x_k)_{S_R} = (R^2/N)|S_R| > 0$. Thus, we have proved (4.37). In particular, $\mathcal{M}\zeta = 0$ in (4.40).

We now prove that $\mathbf{w} = 0$. Multiplying the first equation of (4.40) with $\mathbf{w}$ and integrating the resultant formula over B_R and using the divergence theorem of Gauss gives that

$$0 = ik\|\mathbf{w}\|^2_{L_2(B_R)} - \sigma(\mathcal{B}_R\zeta, \mathbf{n}\cdot\mathbf{w})_{S_R} + \frac{\mu}{2}\|\mathbf{D}(\mathbf{w})\|^2_{L_2(B_R)},$$

because $\operatorname{div}\mathbf{w} = 0$ in B_R. By the second equation of (4.40) with $\mathcal{M}\zeta = 0$, we have

$$\sigma(\mathcal{B}_R\zeta, \mathbf{n}\cdot\mathbf{w})_{S_R} = \sigma(\mathcal{B}_R\zeta, ik\zeta)_{S_R} + \sum_{j=1}^N \frac{1}{|B_R|}\int_{B_R} w_j\,dt(\mathcal{B}_R\zeta, R^{-1}x_j)_{S_R},$$

where we have used $\mathbf{n} = R^{-1}x = R^{-1}(x_1, \ldots, x_N)$ for $x \in S_R$. Thus,

$$(\mathcal{B}_R\zeta, x_j)_{S_R} = (\zeta, (\Delta_{S_R} + \frac{N-1}{R^2})x_j)_{S_R} = 0.$$

Moreover, since ζ satisfies (4.37), we know that $-(\mathcal{B}_R\zeta, \zeta)_{S_R} \geq c\|\zeta\|^2_{L_2(S_R)}$ for some positive constant c, and therefore we have $\mathbf{w} = 0$. And then, $\nabla\mathfrak{q} = 0$, which yields that $\mathfrak{q}$ is a constant. Since $\mathcal{B}_R\zeta - \mathfrak{q} = 0$ on S_R, integrating this formula on S_R, we have $\mathfrak{q}|S_R| = 0$, because $(\mathcal{B}_R\zeta, 1)_{S_R} = (N-1)R^{-2}(\zeta, 1)_{S_R} = 0$, and so $\mathfrak{q} = 0$.

Finally, combining $\mathcal{B}_R\zeta = 0$ on S_R and $(\zeta, 1)_{S_R} = (\zeta, x_j)_{S_R} = 0$ gives that $\zeta = 0$. This completes the proof of the uniqueness.

4.11 Stationary solution

Let me consider the following stationary problem:

$$\begin{aligned}
\mathcal{L}\mathbf{v}_S - \operatorname{Div}(\mu\mathbf{D}(\mathbf{v}_S) - \mathfrak{q}_k\mathbf{I}) = \mathbf{f}_S &\quad \text{in } B_R,\\
\operatorname{div}\mathbf{v}_S = g_S = \operatorname{div}\mathbf{g}_S &\quad \text{in } B_R,\\
\mathcal{M}\eta_S - (\mathcal{A}\mathbf{v}_S)\cdot\mathbf{n} = d_S &\quad \text{on } S_R,\\
(\mu\mathbf{D}(\mathbf{v}_S) - \mathfrak{q}_S\mathbf{I})\mathbf{n} - (\mathcal{B}_R\eta_S)\mathbf{n} = \mathbf{h}_S &\quad \text{on } S_R.
\end{aligned} \tag{4.38}$$

The following theorem holds.

Theorem 12. *Let $1 < q < \infty$. Then, for any $\mathbf{f}_S \in L_q(B_R)^N$, $g_S \in H^1_q(B_R)$, $d_S \in W^{2-1/q}_q(S_R)$, $\mathbf{h}_S \in H^1_q(B_R)^N$, and $\mathbf{g}_S \in L_q(B_R)^N$, problem (4.38) admits unique solutions $\mathbf{v}_S \in H^2_q(B_R)^N$, $\mathfrak{q}_S \in H^1_q(B_R)$, and $\rho_S \in W^{3-1/q}_q(S_R)$ possessing the estimate:*

$$\begin{aligned}
&\|\mathbf{v}_S\|_{H^2_q(B_R)} + \|\nabla\mathfrak{q}_S\|_{L_q(B_R)} + \|\rho_S\|_{W^{3-1/q}_q(S_R)}\\
&\quad \leq C(\|\mathbf{f}_S\|_{L_q(B_R)} + \|d_S\|_{W^{2-1/q}_q(S_R)} + \|(g_S, \mathbf{h}_S)\|_{H^1_q(B_R)} + \|\mathbf{g}_S\|_{L_q(B_R)})
\end{aligned} \tag{4.39}$$

for some constant $C > 0$.

To prove Theorem 12, in view of the Riesz-Schauder theorem, Fredholm alternative principle, it is sufficien to prove the uniqueness in the L_2 framework. Let $\mathbf{w} \in H^2_2(B_R)^N$, $\mathfrak{q} \in H^1_2(B_R)$ and $\zeta \in W^{3-1/2}_2(S_R)$ satisfy the homogeneous equations:

$$\begin{aligned}
\mathcal{L}\mathbf{w} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{w}) - \mathfrak{q}\mathbf{I}) = 0, \quad \operatorname{div}\mathbf{w} = 0 &\quad \text{in } B_R,\\
\mathcal{M}\zeta - (\mathcal{A}\mathbf{w})\cdot\mathbf{n} = 0 &\quad \text{on } S_R,\\
(\mu\mathbf{D}(\mathbf{w}) - \mathfrak{q}\mathbf{I})\mathbf{n} - \sigma(\mathcal{B}_R\zeta)\mathbf{n} = 0 &\quad \text{on } S_R.
\end{aligned} \tag{4.40}$$

Employing the same argument as in Subsec.4.10, we have

$$(\zeta, 1)_{S_R} = 0, \quad (\zeta, x_j)_{S_R} = 0 \quad \text{for } j = 1, \ldots, N. \tag{4.41}$$

We now prove that $\mathbf{w} = 0$. Multiplying the first equation of (4.40) with $\mathbf{w}$ and integrating the resultant formula over B_R and using the divergence theorem of Gauss gives that

$$0 = (\mathcal{L}\mathbf{w}, \mathbf{w})_{B_R} - \sigma(\mathcal{B}_R\zeta, \mathbf{n}\cdot\mathbf{w})_{S_R} + \frac{\mu}{2}\|\mathbf{D}(\mathbf{w})\|^2_{L_2(B_R)},$$

because $\operatorname{div}\mathbf{w} = 0$ in B_R. Recalling that $\mathcal{L}\mathbf{v}_S = 2\pi\sum_{k=1}^M(\mathbf{v}_S, \mathbf{p}_k)_{\mathbb{T}}\,\mathbf{p}_k$, we have

$$(\mathcal{L}\mathbf{w}, \mathbf{w})_{B_R} = \sum_{k=1}^M |(\mathbf{w}, \mathbf{p}_k)_{B_R}|^2.$$

Employing the same argument as in Subsec.4.10, we have

$$\sigma(\mathcal{B}_R\zeta, \mathbf{n}\cdot\mathbf{w})_{S_R} = \sum_{k=1}^N \frac{1}{|B_R|}\int_{B_R} w_j\,dt(\mathcal{B}_R\zeta, R^{-1}x_j)_{S_R} = 0.$$

Thus,

$$0 = \sum_{k=1}^M |(\mathbf{w}, \mathbf{p}_k)_{B_R}|^2 + \frac{\mu}{2}\|\mathbf{D}(\mathbf{w})\|^2_{L_2(B_R)},$$

which yields that $\mathbf{w} = 0$. And then, $\nabla\mathfrak{q} = 0$, which shows that $\mathfrak{q}$ is a constant. Thus, $\mathcal{B}_R\zeta - \mathfrak{q} = 0$ on S_R. Integrating this formula on S_R, we have $\mathfrak{q}|S_R| = 0$, and so $\mathfrak{q} = 0$.

Finally, combining $\mathcal{B}_R\zeta = 0$ on S_R and $(\zeta, 1)_{S_R} = (\zeta, x_j)_{S_R} = 0$ gives that $\zeta = 0$. This completes the proof of the uniqueness.

Proof of Theorem 8. Since solutions $\mathbf{v}$, $\mathfrak{q}$ and ρ of equations (4.27) are represented as

$$(\mathbf{v}, \mathfrak{q}, \rho) = (\mathbf{v}_\varphi, \mathfrak{q}_\varphi, \rho_\varphi) + \sum_{1\le|k|\le k_0}(\mathbf{v}_k, \mathfrak{q}_k, \rho_k) + (\mathbf{v}_S, \mathfrak{q}_S, \rho_S),$$

applying estimate (4.33), Theorem 10, and Theorem 12 yields Theorem 8. □

5 Proof of Theorem 7

Theorem 7 is proved by the standard Banch fixed point theorem. Let $\epsilon > 0$ be a small number determined later and let $\mathcal{I}_\epsilon$ be an underlying space defined by setting

$$\begin{aligned}\mathcal{I}_\epsilon = \{(\mathbf{v}, \rho) \mid\ & \mathbf{v} \in L_{p,\mathrm{per}}((0, 2\pi), H^2_q(B_R)^N) \cap H^1_{p,\mathrm{per}}((0, 2\pi), L_q(B_R)^N), \\ & \rho \in L_{p,\mathrm{per}}((0, 2\pi), W^{3-1/q}_q(S_R)) \cap H^1_{p,\mathrm{per}}((0, 2\pi), W^{2-1/q}_q(S_R)) \cap H^1_{\infty,\mathrm{per}}((0, 2\pi), W^{1-1/q}_q(S_R)), \\ & \sup_{t\in(0,2\pi)} \|H_\rho\|_{H^1_\infty(B_R)} \le \delta, \quad E(\mathbf{v}, \rho) \le \epsilon\},\end{aligned} \tag{5.1}$$

where we have set

$$E(\mathbf{v}, \rho) = \|\mathbf{v}\|_{L_p((0,2\pi), H^2_q(B_R)^N)} + \|\mathbf{v}\|_{H^1_p((0,2\pi), L_q(B_R)^N)}$$

$$+ \|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))} + \|\rho\|_{H_p^1((0,2\pi),W_q^{2-1/q}(S_R))} + \|\partial_t\rho\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))}.$$

Let $(\mathbf{v}, \rho) \in \mathcal{I}_\epsilon$ and let $\mathbf{u}$, $\mathfrak{q}$ and η be solutions of linear equations:

$$\begin{cases} \partial_t\mathbf{u} + \mathcal{L}\mathbf{u}_S - \mathrm{Div}\,(\mu(\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mathbf{G} + \mathbf{F}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \mathrm{div}\,\mathbf{u} = g(\mathbf{u}, \rho) = \mathrm{div}\,\mathbf{g}(\mathbf{v}, \rho) & \text{in } B_R \times (0, 2\pi), \\ \partial_t\eta + \mathcal{M}\eta - \mathcal{A}\mathbf{u}\cdot\mathbf{n} = \tilde{d}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q})\mathbf{n} - (\mathcal{B}_R\eta)\mathbf{n} = \mathbf{h}(\mathbf{v}, \rho) & \text{on } S_R \times (0, 2\pi), \end{cases} \tag{5.2}$$

Applying Theorem 8 to equations (5.2) yields that

$$\begin{aligned} &\|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R)^N)} + \|\mathbf{v}\|_{H_p^1((0,2\pi),L_q(B_R)^N)} \\ &\quad + \|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(S_R))} + \|\rho\|_{H_p^1((0,2\pi),W_q^{2-1/q}(S_R))} \\ &\qquad \le C(\|\mathbf{G}\|_{L_p((0,2\pi),L_q(B_R))} + \mathcal{E}(\mathbf{v}, \rho)\} \end{aligned} \tag{5.3}$$

with

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \rho) &= \|\mathbf{F}(\mathbf{v}, \rho)\|_{L_p((0,2\pi),L_q(B_R))} + \|\tilde{d}(\mathbf{v}, \rho)\|_{L_p((0,2\pi),W_q^{2-1/q}(S_R))} + \|\mathbf{g}(\mathbf{v}, \rho)\|_{H_p^1((0,2\pi),L_q(B_R))} \\ &\quad + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{H_p^{1/2}((0,2\pi),L_q(B_R))} + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{L_p((0,2\pi),H_q^1(B_R))}. \end{aligned}$$

To estimate $\|\partial_t\eta\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))}$, the following estimate is used:

$$\|\partial_t\eta\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))} \le C(\|\mathcal{M}\rho\|_{L_\infty,W_q^{1-1/q}(S_R))} + \|\mathbf{v}\|_{L_\infty((0,2\pi),H_q^1(B_R))} + \|\tilde{d}\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))}),$$

which follows from the third equation of equations (5.2). The main task is to prove that

$$\mathcal{E}(\mathbf{v}, \rho) + \|\tilde{d}\|_{L_\infty((0,2\pi),W_q^{1-1/q}(S_R))} \le C\epsilon^2 \tag{5.4}$$

with some constant $C > 0$ independent of ϵ. In the proof, it is assumed that $N < q < \infty$, $2 < p < \infty$ and $2/p + N/q <$. In particular, the first assumption is to use Sobolev's immbedding theorem. In fact, the following inequalities are used:

$$\begin{aligned} \|f\|_{L_\infty(B_R)} &\le C\|f\|_{H_q^1(B_R)}, \\ \|fg\|_{H_q^1(B_R)} &\le C\|f\|_{H_q^1(B_R)}\|g\|_{H_q^1(B_R)}, \\ \|fg\|_{H_q^2(B_R)} &\le C(\|f\|_{H_q^2(B_R)}\|g\|_{H_q^1(B_R)} + \|f\|_{H_q^1(B_R)}\|g\|_{H_q^2(B_R)}) \\ \|fg\|_{W_q^{1-1/q}(S_R)} &\le C\|f\|_{W_q^{1-1/q}(S_R)}\|g\|_{W_q^{1-1/q}(S_R)}, \\ \|fg\|_{W_q^{2-1/q}(S_R)} &\le C(\|f\|_{W_q^{2-1/q}(S_R)}\|g\|_{W_q^{1-1/q}(S_R)} + \|f\|_{W_q^{1-1/q}(S_R)}\|g\|_{W_q^{2-1/q}(S_R)}), \end{aligned}$$

which follows from the Sobolev inequality and the fact that $\|u|_{S_R}\|_{W_q^{1-1/q}(S_R)} \le C\|u\|_{H_q^1(B_R)}$ for $u \in H_q^1(B_R)$. To estimate the lower order derivatives of $\mathbf{v}$ and ρ, the following inequalities are used:

$$\begin{aligned} \|\mathbf{v}\|_{L_\infty((0,2\pi),B_{q,p}^{2(1-1/p)}(B_R))} &\le C(\|\mathbf{v}\|_{L_p((0,2\pi),H_q^2(B_R))} + \|\partial_t\mathbf{v}\|_{L_p((0,2\pi),L_q(B_R))}), \\ \|\rho\|_{L_\infty((0,2\pi),W_{q,p}^{3-1/p-1/q}(B_R))} &\le C(\|\rho\|_{L_p((0,2\pi),W_q^{3-1/q}(B_R))} + \|\partial_t\rho\|_{L_p((0,2\pi),W_q^{2-1/q}(B_R))}), \end{aligned}$$

which follows from real interpolation theorem. In particular, to obtain $\nabla\mathbf{v} \in L_\infty$, it is used the assumption: $2/p + N/q < 1$.

To estimate $\|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{H_p^{1/2}((0,2\pi),L_q(B_R))} + \|(g(\mathbf{v}, \rho), \mathbf{h}(\mathbf{v}, \rho))\|_{L_p((0,2\pi),H_q^1(B_R))}$, the following two lemmas are used:

Lemma 13. *Let* $1 < p < \infty$ *and* $N < q < \infty$. *Let*

$$a \in H^1_{\infty,\mathrm{per}}((0,2\pi), L_q(B_R)) \cap L_{\infty,\mathrm{per}}((0,2\pi), H^1_q(B_R)),$$
$$b \in H^{1/2}_{p,\mathrm{per}}((0,2\pi), L_q(B_R)) \cap L_{p,\mathrm{per}}((0,2\pi), H^1_q(B_R)).$$

Then,

$$\begin{aligned}&\|ab\|_{H^{1/2}_p((0,2\pi),L_q(B_R))} + \|ab\|_{L_p((0,2\pi),H^1_q(B_R))}\\ &\quad\le C(\|a\|_{H^1_\infty((0,2\pi),L_q(B_R))} + \|a\|_{L_\infty((0,2\pi),H^1_q(B_R))})^{1/2}\|a\|^{1/2}_{L_\infty((0,2\pi),H^1_q(B_R))}\\ &\quad\times(\|b\|_{H^{1/2}_p((0,2\pi),L_q(B_R))} + \|b\|_{L_p((0,2\pi),H^1_q(B_R))}).\end{aligned}$$

Remark 14. This lemma holds for more general domains.

Proof. The lemma follows from the following complex interpolation relation of order 1/2:

$$\begin{aligned}&H^{1/2}_{p,\mathrm{per}}((0,2\pi), L_q(B_R)) \cap L_{p,\mathrm{per}}((0,2\pi), H^{1/2}_q(B_R))\\ &\quad= (L_{p,\mathrm{per}}((0,2\pi), L_q(B_R)), H^1_{p,\mathrm{per}}((0,2\pi), L_q(B_R)) \cap L_{p,\mathrm{per}}((0,2\pi), H^1_q(B_R)))_{1/2}.\end{aligned}$$

□

Lemma 15. *Let* $1 < p, q < \infty$. *Then, there exists a constant* C *such that for any* u *with*

$$u \in H^1_{p,\mathrm{per}}((0,2\pi), L_q(B_R)) \cap L_{p,\mathrm{per}}((0,2\pi), H^2_q(B_R)),$$

we have

$$\|u\|_{H^{1/2}_p((0,2\pi),H^1_q(B_R))} \le C(\|u\|_{H^1_p((0,2\pi),L_q(B_R))} + \|u\|_{L_p((0,2\pi),H^2_q(B_R))})$$

for some constant $C > 0$.

Remark 16. This lemma holds for more general domains.

Proof. For a proof, refer to [6]. □

Proof of Theorem 7. Combining (5.3) and (5.4) yields that

$$E(\mathbf{u},\eta) \le C\|\mathbf{G}\|_{L_p((0,\infty),L_q(B_R))} + C\epsilon^2$$

for some constant $C > 0$ independent of ϵ. Thus, choosing $\epsilon > 0$ so small that $C\epsilon < 1/2$ yields that

$$E(\mathbf{u},\eta) \le C\|\mathbf{G}\|_{L_p((0,\infty),L_q(B_R))} + \epsilon/2.$$

Choosing $\mathbf{f}$ so small that $C\|\mathbf{G}\|_{L_p((0,2\pi),L_q(B_R))} \le \epsilon/2$ yields that $E(\mathbf{u},\eta) \le \epsilon$, and so $(\mathbf{u},\eta) \in \mathcal{I}_\epsilon$. Let Ψ be a map acting on $(\mathbf{u},\rho) \in \mathcal{I}_\epsilon$ defined by $\Psi(\mathbf{u},\rho) = (\mathbf{v},\eta)$, and then Ψ is a map from $\mathcal{I}_\epsilon$ into itself. It also can be proved that

$$E(\Psi(\mathbf{v}_1,\rho_1) - \Psi(\mathbf{v}_2,\rho_2)) \le C\epsilon E((\mathbf{v}_1,\rho_1) - (\mathbf{v}_2,\rho_2))$$

for any $(\mathbf{v}_i,\rho_i) \in \mathcal{I}_\epsilon$ $(i = 1,2)$. Choosing $\epsilon > 0$ smaller if necessary, we may assume that $C\epsilon < 1$, and so Ψ is a contraction map from $\mathcal{I}_\epsilon$ into itself. Thus, there exists a unique fixed point $(\mathbf{v},\rho) \in \mathcal{I}_\epsilon$, which is a required unique solution of equations (4.22).

Finally, we define $\xi(t)$ by setting

$$\xi(t) = \int_0^t \xi'(s)\,ds + c = \frac{1}{|B_R|}\int_0^t\int_{B_R} \mathbf{v}(x,s)(1+J_0(x,s))\,dxds + c$$

where c is a constant for which

$$\int_0^{2\pi} \xi(s)\,ds = 0, \text{that is}, c = -\frac{1}{2\pi|B_R|}\int_0^{2\pi}\Big(\int_0^t\int_{B_R}(\mathbf{v}(x,s)(1+J_0(x,s))\,dxds\Big)\,dt.$$

We define Ω_t and Γ_t by the formulas in (4.13). And then, setting $\mathbf{u}(x,t) = \mathbf{v}(\Phi^{-1}(x,t),t)$ and $\mathfrak{p}(x,t) = \mathfrak{q}(\Phi^{-1}(x,t),t)$, we see that Ω_t, Γ_t, $\mathbf{u}(x,t)$ and $\mathfrak{q}(x,t)$ satisfy the equations:

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) + \sum_{k=1}^M \int_0^{2\pi} (\mathbf{u}(\cdot,t),\mathbf{p}_k)_{\Omega_t}\,dt\,\mathbf{p}_k = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega_t,$$
$$(\mu(\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t \qquad \text{on } \Gamma_t,$$

In particular, $\mathrm{div}\,\mathbf{u} = 0$ implies that $|\Omega_t|$ is a constant, and so we set $|\Omega| = |B_R|$. And also, we see that

$$\xi(t) = \int_{\Omega_t} x\,dx,$$

and so by (4.18), (4.20) and (4.21),

$$\partial_t\rho - \mathcal{A}\mathbf{v}\cdot\mathbf{n} = d(\mathbf{v},\rho).$$

Thus, the kinematic condition: $V_{\Gamma_t} = \mathbf{u}\cdot\mathbf{n}_t$ holds on Γ_t. Finally, the assumption on $\mathbf{f}$ implies (4.10), and therefore, $\mathbf{u}$ and $\mathfrak{p}$ satisfy equations (4.1). This completes the proof of Theorem 7. For the detailed proof, see Eiter, Kyed and Shibata [1].

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