A Remark on the Solvability of Plane Steady-State Exterior Navier-Stokes Problem for Arbitrarily Large Data

Giovanni P. Galdi

1 Introduction

In his celebrated paper of 1933, J.Leray studied the existence of solutions to the boundary-value problem of Navier-Stokes equations in the complement, $\Omega \subset \mathbb{R}^2$, of a smooth two-dimensional compact set (the "obstacle"). In a suitable dimensionless form, the problem is formulated as follows: Given a vector $v_\infty \in \mathbb{R}^2$, find a pair (v, p) –representing velocity and pressure fields, respectively– satisfying the following set of equations

$$\left.\begin{array}{l}\Delta v = v \cdot \nabla v + \nabla p \\ \nabla \cdot v = 0\end{array}\right\} \text{ in } \Omega \qquad v = 0 \text{ at } \partial\Omega \tag{1.1}$$

along with the condition at infinity

$$\lim_{|x|\to\infty} v(x) = v_\infty. \tag{1.2}$$

The most significant contribution of Leray to the resolution of problem (1.1), (1.2) consisted in proving that, for *any* prescribed nonzero v_∞ there is at least one solution to (1.1), which is also as smooth as allowed by the smoothness of Ω. However, by his arguments, he was not able to infer that these solutions verify also the fundamental condition (1.2). Actually, the only "asymptotic property" he was able to show was that the velocity field v of his solution possesses a finite Dirichlet integral, namely,

$$\nabla v \in L^2(\Omega)\,. \tag{1.3}$$

The question of whether solutions constructed by Leray or, more generally, solutions in the class (1.3), satisfy (1.2) has become the focus of deep researches by outstanding mathematicians. In particular, D.Gilbarg and H.Weinberger [10], [11] were the first to show that the solution constructed by Leray is bounded and that it converges at large distances, in the mean square over the angle, to a certain vector v_0. More detailed information about convergence was provided later on by C.Amick [1], if the solution is symmetric. Specifically, a pair $(w(x) \equiv (w_1(x), w_2(x)), \mathsf{p}(x))$, $x = (x_1, x_2)$, is said *symmetric* if

$$w_1(x_1, x_2) = w_1(x_1, -x_2)\,,\ \ w_2(x_1, x_2) = -w_2(x_1, -x_2)\,;\ \ \mathsf{p}(x_1, x_2) = p(x_1, -x_2). \tag{1.4}$$

If Ω is symmetric around the x_1-axis, namely, $(x_1, x_2) \in \partial\Omega$ implies $(x_1, -x_2) \in \partial\Omega$, and $v_\infty = \lambda e$, with e unit vector along x_1, Leray's construction leads to a symmetric solution. In such a case, in [1] it is shown that v tends to v_0 uniformly pointwise. This result has been recently improved by M.Korobkov, K.Pileckas and R.Russo [12], who relaxed the symmetry assumption. However, it is *not* known whether $v_\infty = v_0$ (v_0 may even be zero!) and, consequently, the question of whether Leray's solution satisfies (1.2) remains open.

It must be observed, however, that existence of solutions to (1.1), (1.2), for *small* v_∞ and by methods completely different than Leray's, was shown by R.Finn and D.R.Smith [4], [5], [15], and, successively, by me [6], [**?**]. Moreover, these solutions are *physically reasonable* in the sense of Finn [15], and are locally unique.

In view of all the above considerations, the fundamental question that remains still open is whether (1.1), (1.2) is solvable for arbitrary *large* v_∞.

Denote by $\mathcal{C}$ the class of pairs constituted by a vector field $w = (w_1, w_2)$ and scalar field p satisfying (1.4) and having a finite Dirichlet integral. More than 20 years ago, in [7] (see also [8, §4.3]), I proved the following result.

Theorem 1.1 *Let Ω be symmetric around the x_1-axis. Assume that the following problem*

$$\left.\begin{aligned}&\Delta u = u \cdot \nabla u + \nabla \phi\\&\nabla \cdot u = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$u = 0 \ \ \text{at } \partial\Omega \tag{1.5}$$
$$\lim_{|x|\to\infty} u(x) = 0, \quad \text{uniformly}$$

has only the zero solution in the class $\mathcal{C}$. Then, there is a set M with the following properties:

(i) $M \subset [0,\infty)$;

(ii) $M \supset [0,c)$ *for some* $c = c(\Omega) > 0$;

(iii) M *is unbounded;*

(iv) *For any $\mu \in M$, the problem*

$$\left.\begin{aligned}&\Delta v = v \cdot \nabla v + \nabla p\\&\nabla \cdot v = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$v = 0 \ \ \text{at } \partial\Omega, \qquad \lim_{|x|\to\infty} v(x) = \mu e \tag{1.6}$$

has at least one solution in the class $\mathcal{C}$.

The importance of this result resides in the fact that it assures existence of solutions to (1.1)–(1.2) for all v_∞ in an *unbounded* set of $\mathbb{R}^2$. The difficult part, however, is to show that its assumption is indeed satisfied, namely, that the homogeneous problem (1.5) has only the zero solution in the class $\mathcal{C}$. Even though plausible, to date, to prove (or disprove!) such a property has remained an open question.

Objective of the present note is to give a contribution toward answering this question. Precisely, we shall show that if, in addition to u satisfying (1.3), we assume

$$\int_{\partial\Omega} \mathbb{T}(u,\phi) \cdot n = 0 \tag{1.7}$$

with $\mathbb{T}(u,\phi) := \nabla u + (\nabla u)^\top - \phi\,\mathbb{I}$ Cauchy stress tensor and n unit normal at $\partial\Omega$, then $u \equiv \nabla\phi \equiv 0$. From the physical viewpoint, the left-hand side of (1.7) represents the total net force exerted by the liquid on the obstacle. Therefore, our result can be equivalently reformulated as follows: *if in the class $\mathcal{C}$ there is a nontrivial solution to* (1.5)*, then such a solution must produce a non-zero net force on the obstacle.* A conclusion, the latter, that appears to be quite paradoxical, given the absence of any driving mechanism.

The above statement is proved in the following section by means of a simple argument used also by H.Kozono and H.Sohr [13] in a different context.

2 On the Existence of Solutions for Arbitrarily Large v_∞

The general idea behind our argument is rather basic. It consists in finding a suitable extension of the solution u to a field $\widetilde{u}$ that, on one hand, possesses a finite Dirichlet integral over $\mathbb{R}^2$ and, on the other hand, satisfies equations $(1.5)_{1,2}$ *in the whole of* $\mathbb{R}^2$, in a distributional sense. In fact, whenever such an extension is found, then one may apply a well-known Liouville-type theorem ensuring that $\widetilde{u}$ must vanish identically in $\mathbb{R}^2$ and, as a consequence, thus vanishes the original solution u in Ω. One way of constructing the desired extension is to assume the validity of (1.7) (see also Remark 2.1). Precisely, we can show the following result.

Theorem 2.1 *Let* (u,ϕ) *be a distributional solution to* (1.5) *with* $\nabla u \in L^2(\Omega)$*. Then, if* (1.7) *holds (in a trace sense), necessarily* $u \equiv \nabla\phi \equiv 0$*. As a result, problem* (1.6) *has at least one solution for all* μ *in the unbounded set* M *specified in* Theorem 1.1.

Proof. First of all, we observe that, under the stated assumption, well–known results on the regularity of distributional solutions ensure (at least) that, in fact, $(u,\phi) \in [C^\infty(\Omega)\times C^\infty(\Omega)] \cap [W^{2,q}_{\rm loc}(\overline{\Omega}) \times W^{1,q}_{\rm loc}(\overline{\Omega})]$, all $q \in [1,\infty)$; see [9, Section X.1].[(1)] This implies, in particular, $\mathbb{T}(u,\phi) \in W^{1-\frac{1}{q},q}(\partial\Omega)$, so that (1.7) is meaningful in the classical trace sense. Next, we perform a suitable extension of problem (1.5) to the whole plane $\mathbb{R}^2$ as follows. Let Ω_0 denote the complement of Ω in $\mathbb{R}^2$, and extend (u,ϕ) to the whole of $\mathbb{R}^2$ by setting

$$\widetilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega_0 \end{cases} \qquad \widetilde{\phi} = \begin{cases} \phi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega_0 \end{cases} .$$

We shall now show that $\widetilde{u}, \widetilde{\phi}$ is a solution to the following problem

$$\left.\begin{array}{l} \Delta\widetilde{u} - \widetilde{u}\cdot\nabla\widetilde{u} - \nabla\widetilde{\phi} = 0 \\ \operatorname{div}\widetilde{u} = 0 \end{array}\right\} \text{ in } \mathbb{R}^2 . \tag{2.1}$$

in the distributional sense. In fact, observing that $(1.5)_1$ is equivalent to

$$\operatorname{div}(\mathbb{T}(u,\phi) + u\otimes u) = 0 \ \text{ in } \Omega ,$$

for all $\psi \in C_0^\infty(\mathbb{R}^2)$, we infer, by integration by parts, $(1.5)_3$, and assumption (1.7),

$$\begin{aligned} \int_{\mathbb{R}^2} [\mathbb{T}(\widetilde{u},\widetilde{\phi}) + \widetilde{u}\otimes\widetilde{u}]\cdot\nabla\psi &= \int_\Omega [\mathbb{T}(u,\phi) + u\otimes u]\cdot\nabla\psi \\ &= -\int_\Omega \psi \operatorname{div}(\mathbb{T}(u,\phi) + u\otimes u) + \int_{\partial\Omega} [\mathbb{T}(u,\phi) - u\otimes u]\cdot n\psi \\ &= \int_{\partial\Omega} \mathbb{T}(u,\phi)\cdot n\psi = 0 , \end{aligned}$$

which shows the desired property for $(\widetilde{u},\widetilde{\phi})$. Now, by hypothesis, $\nabla\widetilde{u} \in L^2(\mathbb{R}^2)$. As a consequence, thanks to a classical regularity result [9, Theorem IX.5.1], we obtain $(\widetilde{u},\widetilde{\phi}) \in C^\infty(\mathbb{R}^2)$. Thus, $\widetilde{u}$ is a smooth vector function satisfying (2.1) and having finite Dirichlet integral. By a well-known Liouville-type result [11] (see also [9, Theorem XII.3.1]), it then follows that $\widetilde{u}$ must be necessarily constant in the whole of $\mathbb{R}^2$. This, in turn, implies $u \equiv \nabla\phi \equiv 0$, which completes the proof of the theorem. □

Remark 2.1 It is worth noticing that a sufficient condition to satisfy (1.7), is to require that (u,ϕ) possesses suitable symmetry properties. For example, suppose that Ω is symmetric also around the

[(1)] Actually, the *interior* regularity property can be proved under the weaker assumption that u *only* belongs to $L^r_{\rm loc}(\Omega)$, for some $r > 2$; see [9, Theorem IX.5.1].

x_2-axis, that is, $(x_1, x_2) \in \Omega$ implies $(-x_1, x_2) \in \Omega$, and, besides being in the class defined by (1.4), the solution $(u = (u_1, u_2), \phi)$ to (1.5) meets the further symmetry conditions around x_2:

$$u_1(x_1, x_2) = -u_1(-x_1, x_2)\,, \quad u_2(x_1, x_2) = u_2(-x_1, x_2)\,; \quad \phi(x_1, x_2) = \phi(-x_1, x_2)\,. \tag{2.2}$$

Then, it is easily checked that (1.7) holds.

Remark 2.2 Employing $(1.5)_1$ and the assumption that u possesses a finite Dirichlet integral, it is easy to show that a necessary and sufficient condition for the validity of (1.7) is that

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|=R} (p\, x + u\, u \cdot x) = 0\,.$$

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University of Pittsburgh
Department of Mechanical Engineering
and Materials Science
Pittsburgh, PA 15260 U.S.A
E-mail address: galdi@pitt.edu