The Stokes operator in exterior Lipschitz domains

Keiichi Watanabe[1]

Global Center for Science and Engineering, Waseda University

1 Introduction

This article gives a summary of [17]. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be an exterior Lipschitz domain in $\mathbb{R}^n$ $(n \geq 3)$, *i.e.*, the complement of a bounded Lipschitz domain. We consider the Stokes resolvent problem in an exterior Lipschitz domain with homogeneous Dirichlet boundary conditions

$$\begin{cases} \lambda u - \Delta u + \nabla \pi = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u = {}^{\top}(u_1, \ldots, u_n) \colon \Omega \to \mathbb{C}^n$ and $\pi \colon \Omega \to \mathbb{C}$ are the unknown velocity field and the pressure, respectively. Here, the right-hand side f is assumed to be divergence free and L^p-integrable for an appropriate number $1 < p < \infty$ and the resolvent parameter λ is assumed to be contained in a sector $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \theta\}$ with $\theta \in (0, \pi)$.

It is known that the results of the Helmholtz projection and the Stokes operator in bounded Lipschitz domains are available only in a restricted set of exponents p, which is quite different from the case of bounded smooth domains. Indeed, Fabes, Méndez, and Mitrea [2] showed that the Helmholtz decomposition of $\mathrm{L}^p(D; \mathbb{C}^n)$ exists whenever $(3/2) - \varepsilon < p < 3 + \varepsilon$, where D is a bounded Lipschitz domain and ε is a positive number depending on the Lipschitz character of D. They also showed that the range of p is sharp, see [2, Thm. 12.2]. This result leaded to the following conjecture posed by Taylor [14, Sec. 4].

Conjecture 1.1. *For a given Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the negative of the Stokes operator generates an analytic semigroup, provided $(3/2) - \varepsilon < p < 3 + \varepsilon$.*

*1 keiichi-watanabe@akane.waseda.jp

Using the potential theory and a weak reversed Hölder estimate, Shen [13] obtained resolvent bounds for numbers p satisfying the condition

$$\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2n}+\varepsilon,$$

where ε is a number depending only on the dimension n, the opening angle θ, and the Lipschitz character of Ω. Notice that when Ω is a bounded smooth domain, such a result was established for all $p \in (1,\infty)$ in [5]. A corollary of Shen's result is that the negative of the Stokes operator generates a bounded analytic semigroup, which gave an affirmative answer to Conjecture 1.1 in the case of bounded Lipschitz domains. Recently, this result was extended by Kustmann and Weis [9] and Tolksdorf [15]. In [9], a new criteria of the H^∞-calculus was given and the boundedness of H^∞-calculus of the Stokes Stokes operator was proved, which yields an investigation of the domain of the square root of the Stokes operator as $\mathrm{W}^{1,p}_{0,\sigma}(\Omega)$, see [15]. This characterization provides L^p-L^q mapping properties of the Stokes semigroup and its gradient with optimal decay. As an application of his result, the existence of solutions to the three-dimensional Navier–Stokes equations in the critical space $\mathrm{L}^\infty(0,\infty;\mathrm{L}^3_\sigma(\Omega))$ was shown in terms of a maximal L^q-regularity approach.

Our interest is to consider the Stokes operator in the case of exterior Lipschitz domains Ω. If the boundary $\partial\Omega$ is connected, it was shown by Lang and Mendéz that the Helmholtz decomposition of $\mathrm{L}^p(\Omega;\mathbb{C}^n)$ exists. More precisely, they proved that the Helmholtz projection $\mathbb{P}$ defines a bounded projection from $\mathrm{L}^p(\Omega;\mathbb{C}^n)$ onto $\mathrm{L}^p_\sigma(\Omega)$ for $3/2 < p < 3$. Here and in the following, $\mathrm{L}^p_\sigma(\Omega)$ denotes the closure of

$$\mathrm{C}^\infty_{c,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^n) \mid \operatorname{div}(\varphi)=0\}$$

in $\mathrm{L}^p(\Omega;\mathbb{C}^n)$. We will investigate the Stokes operator defined on $\mathrm{L}^2_\sigma(\Omega)$ by using a sesquilinear form, see [12, Ch. 4]. On $\mathrm{L}^p_\sigma(\Omega)$ for $1 < p < \infty$, the Stokes operator A_p is defined in two steps. First, we take the part of A_2 in $\mathrm{L}^p_\sigma(\Omega)$, *i.e.*,

$$\mathcal{D}(A_2|_{\mathrm{L}^p_\sigma(\Omega)}) := \{u \in \mathcal{D}(A_2)\cap \mathrm{L}^p_\sigma(\Omega) \mid A_2 u \in \mathrm{L}^p_\sigma(\Omega)\},$$

where $A_2|_{\mathrm{L}^p(\Omega)}u$ is given by $A_2 u$ for u in its domain. Notice that $A_2|_{\mathrm{L}^p_\sigma(\Omega)}$ is densely defined because $\mathrm{C}^\infty_{c,\sigma}(\Omega) \subset \mathcal{D}(A_2|_{\mathrm{L}^p_\sigma(\Omega)})$ and that it is closable. Secondary, we define A_p to be the closure of $A_2|_{\mathrm{L}^p_\sigma(\Omega)}$ in $\mathrm{L}^p_\sigma(\Omega)$.

The purpose of this article is to study a family of operators $\{\lambda(\lambda + A_p)^{-1}\}_{\lambda\in\Sigma_\theta}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\mathrm{L}^p_\sigma(\Omega))$. Since $\mathcal{R}$-boundedness implies uniform boundedness of the family of operators, this result yields that the negative of the Stokes operator generates a bounded analytic semigroup on $L^p_\sigma(\Omega)$ for *certain* p.

2 Preliminaries

Let us first introduce the notion of exterior Lipschitz domains that is considered in this article.

Definition 2.1. Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain, *i.e.*, D is a bounded, open, and connected set satisfying the following: For each $x_0 \in \partial\Omega$, there exists a Lipschitz function $\zeta\colon \mathbb{R}^{n-1} \to \mathbb{R}$, a coordinate system (x', x_n), and a radius $r > 0$ such that

$$B_r(x_0) \cap D = \{(x', x_n) \in \mathbb{R}^n \mid x_n > \zeta(x')\} \cap B_r(x_0),$$
$$B_r(x_0) \cap \partial D = \{(x', x_n) \in \mathbb{R}^n \mid x_n = \zeta(x')\} \cap B_r(x_0),$$

where $B_r(x_0)$ denotes the ball with radius r centered at x_0 and $x' := (x_1, \dots, x_{n-1})$. Then an exterior Lipschitz domain $\Omega \subset \mathbb{R}^n$ is the complement of a bounded Lipschitz domain, *i.e.*, $\Omega := \mathbb{R}^n \setminus D$.

Remark 2.2. The definition of exterior Lipschitz domains stated above excludes the case of multi-connected domains, *i.e.*, the presence of holes inside the exterior domain. However, it should be possible to add holes by employing the argument used in [11]. Notice that the rest of this article works with holes appearing in the exterior domain.

2.1 Remark on the Helmholtz projection

Let $\Xi \subset \mathbb{R}^n$ be a domain and $\mathbb{P}_{2,\Xi}$ be the Helmholtz projection on $L^2(\Xi; \mathbb{C}^n)$ that is the orthogonal projection onto $L^2_\sigma(\Xi)$. It is wildly known that the Helmholtz projection implies the orthogonal decomposition

$$L^2(\Xi; \mathbb{C}^n) = L^2_\sigma(\Xi) \oplus G_2(\Xi).$$

Here, for $1 < p < \infty$ we write

$$G_p(\Xi) := \{\nabla g \in L^p(\Xi; \mathbb{C}^n) \mid g \in L^p_{\mathrm{loc}}(\Xi)\}.$$

For $1 < p < \infty$, we say that the Helmholtz decomposition of $L^p(\Xi; \mathbb{C}^n)$ exists if an algebraic and topological decomposition of the form

$$L^p(\Xi; \mathbb{C}^n) = L^p_\sigma(\Xi) \oplus G_p(\Xi) \tag{2.1}$$

exists. Then, the Helmholtz projection $\mathbb{P}_{p,\Xi}$ on $\mathrm{L}^p(\Xi;\mathbb{C}^n)$ is defined as the projection of $\mathrm{L}^p(\Xi;\mathbb{C}^n)$ onto $\mathrm{L}^p_\sigma(\Xi)$. If $\Xi = \mathbb{R}^n$, the Helmholtz decomposition of $\mathrm{L}^p(\mathbb{R}^n;\mathbb{C}^n)$ exists for all $1 < p < \infty$, and the projection can be represented by

$$\mathbb{P}_{p,\mathbb{R}^n} := \mathcal{F}^{-1}\left[1 - \frac{\xi \otimes \xi}{|\xi|^2}\right],$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

In the case $\Xi = D$, where $D \subset \mathbb{R}^n$ is a bounded Lipschitz domain, we know the result by Fabes, Méndez, and Mitrea [2] who showed that there exists $\varepsilon = \varepsilon(D) > 0$ such that the Helmholtz decomposition of $\mathrm{L}^p(D;\mathbb{C}^n)$ exists if p satisfy

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6} + \varepsilon. \tag{2.2}$$

Notice that the range of p is optimal, see [2, Thm. 12.2]. On the other hand, if $\Omega \subset \mathbb{R}^n$ is an exterior Lipschitz domain with connected boundary, there is the result by Lang and Méndez [10, Thm. 6.1] who showed that the Helmholtz decomposition of $\mathrm{L}^p(\Omega;\mathbb{C}^n)$ exists for all p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{6} + \varepsilon, \frac{1}{2} - \frac{1}{n}\right\}. \tag{2.3}$$

We easily observe that the condition (2.3) is slightly stronger than (2.2). For example, when $n = 3$, we deduce that (2.2) implies $(3/2) - \varepsilon < p < 3 + \varepsilon$ and that (2.3) implies $3/2 < p < 3$. Here, it seems to be difficult to show the existence of mild solutions to the three-dimensional Navier–Stokes equations in the critical space $\mathrm{L}^\infty(0,T;\mathrm{L}^3_\sigma(\Omega))$ based on the result due to Lang and Méndez because we need information for p in the interval $[3, 3+\varepsilon)$ to prove the existence of mild solutions, *cf.*, [6, 15, 16] for the cases of the whole space and bounded/smooth Lipschitz domains. However, by a slight modification of the proof of Lang and Méndez, we can recover the interval (2.2) for exterior domains Ω with connected boundary. In fact, Lang and Méndez proved that the Helmholtz decomposition of $\mathrm{L}^p(\Omega;\mathbb{C}^n)$ exists if and only if there exists a function ψ that solves the following Neumann problem:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega, \\ \nu \cdot \nabla\psi = h \in \mathrm{B}^{-1/p}_{p,p}(\partial\Omega) & \text{on } \partial\Omega, \\ \nabla\psi \in \mathrm{L}^p(\Omega;\mathbb{C}^n) \end{cases} \tag{2.4}$$

with $h := \nu \cdot (u - \nabla\Pi_\Omega(\mathrm{div}\,(u)))$, where $\Pi_\Omega(\mathrm{div}\,(u))$ is the Newton potential of $\mathrm{div}\,(u)$ extended by zero to $\mathbb{R}^n$. According to [10, Thm. 5.8], it holds

$$\|\nabla\psi|_{\mathrm{L}^p(\Omega;\mathbb{C}^n)} \le C\|h\|_{\mathrm{B}^{-1/p}_{p,p}(\partial\Omega)}$$

for all p that satisfy (2.2) but from [10, Cor. 2.3] it holds

$$\|\nabla\Pi_\Omega(\mathrm{div}\,(u))\|_{\mathrm{L}^p(\Omega;\mathbb{C}^n)} \leq C\|u\|_{\mathrm{L}^p(\Omega;\mathbb{C}^n)}$$

for all p that satisfy $n/(n-1) < p < n$, where these conditions induce (2.3). To get rid of the condition $n/(n-1) < p < n$, we replace in the definition of h the term $\nabla\Pi_\Omega \mathrm{div}\,(u)$ by $\mathcal{F}^{-1}[(\xi\otimes\xi)|^{-2}\mathcal{F}[U]]$, where U is the extension of u to $\mathbb{R}^n$ by zero. Hence, the term h is given by $h := \nu\cdot(\mathbb{P}_{p,\mathbb{R}^n}U)|_{\partial\Omega}$ and we obtain the estimate

$$\|h\|_{\mathrm{B}^{-1/p}_{p,p}(\partial\Omega)} \leq C\|u\|_{\mathrm{L}^p(\Omega;\mathbb{C}^n)}$$

for all $1 < p < \infty$. Summing up, the Neumann problem (2.4) is uniquely solvable for all p that satisfy (2.2) but only if $\partial\Omega$ is connected. To generalize this result into the case when $\partial\Omega$ is not connected, we decompose Ω into its connected components

$$\Omega = \Omega_0 \cup \left(\bigcup_{k=1}^{N} \Omega_k\right),$$

where Ω_0 is the unbounded connected component and Ω_k $(k = 1,\ldots,N)$ are bounded Lipschitz domains. From [2, Thm. 11.1], there exists $\varepsilon > 0$ such that the Helmholtz projection $\mathbb{P}_{p,\Omega_k}$ is bounded on $\mathrm{L}^p(\Omega_k;\mathbb{C}^n)$ for all p satisfying (2.2) and $k = 1,\ldots,N$. Then the Helmholtz projection on Ω can be defined by

$$[\mathbb{P}_{p,\Omega}f](x) := [\mathbb{P}_{p,\Omega_k}R_k f](x) \qquad (x\in\Omega_k, k = 0,\ldots,N, f\in\mathrm{L}^p(\Omega;\mathbb{C}^n)),$$

where R_k is the restriction operator of functions on Ω to Ω_k. Therefore, we now arrive at the following proposition on he existence result of the Helmholtz decomposition on exterior Lipschitz domains.

Proposition 2.3. *Let $\Omega\subset\mathbb{R}^n$ be an exterior domain and p enjoy* (2.2). *Then the Helmholtz decomposition* (2.1) *exists.*

The details of the discussion in this section can be found in [17, Sec. 2.1].

2.2 Maximal regularity

Consider the abstract Cauchy problem:

$$\begin{cases} \partial_t u + \mathcal{A}u = f & \text{on } (0,\infty), \\ \qquad u(0) = u_0, \end{cases} \tag{2.5}$$

where $-\mathcal{A}$ generates a bounded analytic semigroup on a Banach space X. Here, f and u_0 are appropriate given data. The definition of the maximal regularity can be read as follows.

Definition 2.4. Let $1 < s < \infty$, $f \in \mathrm{L}^s(0,\infty;X)$, and $u_0 \in (X, \mathcal{D}(\mathcal{A}))_{1-1/s,s}$. An operator $\mathcal{A}$ is said to admit a maximal regularity if the system (2.5) has a unique solution u, which is differential for almost every $t > 0$, satisfies $u(t) \in \mathcal{D}(\mathcal{A})$ for almost every $t > 0$, and

$$\|\partial_t u\|_{\mathrm{L}^s(0,\infty;X)} + \|\mathcal{A}u\|_{\mathrm{L}^s(0,\infty;X)} \leq C\Big(\|f\|_{\mathrm{L}^s(0,\infty;X)} + \|u_0\|_{(X,\mathcal{D}(\mathcal{A}))_{1-1/s,s}}\Big).$$

Here, $\mathcal{D}(\mathcal{A})$ and $(X, \mathcal{D}(\mathcal{A}))_{1-1/s,s}$ denote the domain $\mathcal{A}$ and the real interpolation space, respectively.

According to Weis [19, Thm. 4.2], we know the characterization of the maximal regularity of a closed operator $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset X \to X$ on a Banach space X. To this end, we introduce the concept of $\mathcal{R}$-boundedness.

Definition 2.5. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded if there exists a positive constant C such that for any $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ $(j = 1, \dots, N)$ it holds

$$\Big\| \sum_{j=1}^{N} r_j(\cdot) T_j x_j \Big\|_{\mathrm{L}^2(0,1;Y)} \leq C \Big\| \sum_{j=1}^{N} r_j(\cdot) x_j \Big\|_{\mathrm{L}^2(0,1;X)}. \tag{2.6}$$

Here, $r_j(t) := \mathrm{sgn}(\sin(2^j \pi t))$ are the Rademacher-functions. Besides, the infimum over all $C > 0$ such that the inequality holds is said to be the $\mathcal{R}$-bound of $\mathcal{T}$ and will be denoted $\mathcal{R}_{X\to Y}\{\mathcal{T}\}$. Especially, if $X = Y$, we simply write $\mathcal{R}_X\{\mathcal{T}\}$.

Remark 2.6. It is known that $\mathcal{R}$-boundedness of a family of operators yields its uniform boundedness (take $N = 1$ in the above definition). If X and Y are Hilbert spaces, then $\mathcal{R}$-boundedness is *equivalent* to uniform boundedness.

The following proposition was shown in [19, Thm. 4.2].

Proposition 2.7. *Let X be a space of type* UMD *and let $-\mathcal{A}$ be the generator of a bounded analytic semigroup on X. The operator $\mathcal{A}$ has maximal regularity if and only if there exists $\theta \in (\pi/2, \pi)$ such that a family of operators $\{\lambda(\lambda + \mathcal{A})^{-1}\}_{\lambda \in \Sigma_\theta}$ is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.*

Remark 2.8. It is known that L^p-spaces for $1 < p < \infty$ are type UMD. Besides, all closed subspaces of UMD-spaces are type UMD. See, *e.g.*, Amman [1, Thm. 4.5.2] for details.

3 Main results

The following two theorems are main results in this article, see also [17].

Theorem 3.1. *There exists a positive constant $\varepsilon > 0$ such that for all p satisfying*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2n} + \varepsilon \tag{3.1}$$

the Stokes operator A_p is closed and densely defined and admits maximal regularity. Furthermore, $-A_p$ generates a bounded analytic semigroup $\{T(t)\}_{t\geq 0}$ on $\mathrm{L}^p_\sigma(\Omega)$. Here, ε only depends only on the dimension n, the opening angle θ, and quantities describing the Lipschitz geometry.

Theorem 3.2. *For all $1 < p \leq q < \infty$ that both satisfy (3.1), there exists a constant $C > 0$ such that*

$$\|T(t)f\|_{\mathrm{L}^q_\sigma(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{\mathrm{L}^p_\sigma(\Omega)} \qquad (t > 0, \ \ f \in \mathrm{L}^p_\sigma(\Omega)).$$

If additionally $p \leq 2$ and $q < n$, there exists a constant C such that

$$\|\nabla T(t)f\|_{\mathrm{L}^q(\Omega)} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{\mathrm{L}^p_\sigma(\Omega)} \qquad (t > 0, \ \ f \in \mathrm{L}^p_\sigma(\Omega)).$$

Remark 3.3. As an application of Theorems 3.1 and 3.2, we can construct a mild solution to the three-dimensional Navier-Stokes equations in the critical space $\mathrm{L}^\infty(0, T; \mathrm{L}^3_\sigma(\Omega))$ via an iteration scheme due to Giga [6]. Namely, the three-dimensional Navier–Stokes equations admits local-in-time mild solutions $u \in \mathrm{BC}([0, T_0); \mathrm{L}^p_\sigma(\Omega))$, and if the initial value is sufficiently small, then the solution is global in time, *i.e.*, $T_0 = +\infty$. If one have an interest in control of the gradient of the solution, one have to show gradient estimate for the Stokes semigroup in L^3, see, *e.g.*, Kato [8]. Further details can be found in [17, Thm. 1.3], see also Tolksdorf [16, Sec. 6.3].

4 Analysis on bounded Lipschitz domains

The proofs of Theorems 3.1 and 3.2 rely on the study of the Stokes resolvent problem (1.1). More precisely, our task is to show that there exists a positive constant C such that

$$\mathcal{R}_{\mathrm{L}^p(\Omega;\mathbb{C}^n)}\{\lambda(\lambda + A_p)^{-1}\mathbb{P}_{p,\Omega} \mid \lambda \in \Sigma_\theta\} \leq C$$

for some $\theta \in (\pi/2, \pi)$. To show this estimate for large values of λ, we follow the cut-off technique due to Geissert *et al.* [4] and construct a parametrix of the resolvent problem in an exterior domain by using the solutions to a problem on the whole space and to a problem on a bounded Lipschitz domain. On the other hand, to deal with small values of λ, we employ the compactness argument (essentially based on Fredholm theory), where Iwashita's proof [7] is extended. When we consider the resolvent bounds for small values of λ, we emphasize that the standard contradiction argument will fail to use because the maximal regularity property requires a randomized version of the resolvent estimate due to the lack of compact embeddings for vector-valued Lebesgue spaces.

Let us recall the results for the Stokes operator in bounded Lipschitz domains. According to Shen [13], Kunstmann and Weis [9], and Tolksdorf [15], we know the following result.

Proposition 4.1. *Let $D \subset \mathbb{R}^n$, $n \geq 3$, be a bounded Lipschitz domain and $\theta \in (0, \pi)$. Then there exists a constant $\varepsilon > 0$ depending only on n, θ, and the Lipschitz character of D such that for all $p \in (1, \infty)$ satisfying*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2n} + \varepsilon \tag{4.1}$$

it holds $\Sigma_\theta \subset \rho(-A_{p,D})$ and there exists a constant $C > 0$ such that

$$\mathcal{R}_{\mathrm{L}^p(D;\mathbb{C}^n) \to \mathrm{L}^p_\sigma(D)} \{\lambda(\lambda + A_{p,D})^{-1} \mathbb{P}_{p,D} \mid \lambda \in \Sigma_\theta\} \leq C.$$

Besides, for all such p it holds $\mathcal{D}(A^{1/2}_{p,D}) = \mathrm{W}^{1,p}_{0,\sigma}(D)$ and there exists a constant $C > 0$ such that

$$\|\nabla u\|_{\mathrm{L}^p(D;\mathbb{C}^{n^2})} \leq C \|A^{1/2}_{p,D} u\|_{\mathrm{L}^p_\sigma(D)}$$

for $u \in \mathcal{D}(A^{1/2}_{p,D})$.

When we construct a parametrix of the resolvent problem in an exterior domain, we use the following Bogovskii lemma to keep the divergence free conditions.

Proposition 4.2. *Let $D \subset \mathbb{R}^n$ $(n \geq 2)$, be a bounded Lipschitz domain, $1 < p < \infty$, and $k \in \mathbb{N}$. Let $\mathrm{L}^p_0(D) := \{F \in \mathrm{L}^p(D) \mid \int_D F \, \mathrm{d}x = 0\}$. Then there exists a continuous operator $\mathcal{B} \colon \mathrm{L}^p(D) \to \mathrm{W}^{1,p}_0(D;\mathbb{C}^n)$ with $\mathcal{B} \in \mathcal{L}(\mathrm{W}^{k,p}_0(D), \mathrm{W}^{k+1,p}_0(D;\mathbb{C}^n))$ such that $\mathrm{div}\,(\mathcal{B}g) = g$ for $g \in \mathrm{L}^p_0(D)$. Furthermore, the operator $\mathcal{B}$ extends to a bounded operator from $\mathrm{W}^{-1,p}_0(D)$ to $\mathrm{L}^p(D;\mathbb{C}^n)$. Here, the space $\mathrm{W}^{-1,p}_0(D)$ stands the dual space of $\mathrm{W}^{1,p'}(D)$ with $(1/p) + (1/p') = 1$. The operator $\mathcal{B}$ is called the Bogovskii operator defined on D.*

To employ the cut-off technique due to Geissert *et al.* [4], it is crucial to prove a decay estimate in λ for the pressure term π_λ. The proof of the following proposition can be found in [17, Prop. 4.3].

Proposition 4.3. *Define the operator* $P_\lambda\colon \mathrm{L}^p_\sigma(D) \to \mathrm{L}^p_0(D)$ *by* $P_\lambda f := \pi_\lambda$. *There exist positive constants* $\varepsilon, C > 0$ *and* $\delta \in (0,1)$ *such that for all* p *satisfying* (4.1) *and all numbers* α *satisfying*

$$\begin{cases} 0 \leq 2\alpha < 1 - \dfrac{1}{p} & \text{if } p \geq \dfrac{2}{1+\delta}, \\ 0 \leq 2\alpha < 2 - \dfrac{3}{p} + \delta & \text{if } p < \dfrac{2}{1+\delta} \end{cases}$$

it holds

$$\mathcal{R}_{\mathrm{L}^p_\sigma(D)\to\mathrm{L}^p_0(D)}\{|\lambda|^\alpha P_\lambda \mid \lambda \in \Sigma_\theta\} \leq C.$$

The proof of Proposition 4.3 relies on mapping properties of the Helmholtz projection on D and the following lemma, see [17, Lem. 3.3].

Lemma 4.4. *Let* $D \subset \mathbb{R}^n$, $n \geq 3$, *be a bounded Lipschitz domain and let* $p \in (1,\infty)$ *satisfy* (4.1). *For all* $\theta \in (0,\pi)$, $\alpha \in (0,1)$, *and* $\beta \in [0,1/2]$ *there exists* $C > 0$ *such that*

$$\mathcal{R}_{\mathrm{L}^p(D;\mathbb{C}^n)\to\mathrm{L}^p_\sigma(D)}\{|\lambda|^\alpha A^{1-\alpha}_{p,D}(\lambda + A_{p,D})^{-1}\mathbb{P}_{p,D} \mid \lambda \in \Sigma_\theta\} \leq C,$$
$$\mathcal{R}_{\mathrm{L}^p(D;\mathbb{C}^n)\to\mathrm{L}^p_\sigma(D)}\{|\lambda|^\beta \nabla(\lambda + A_{p,D})^{-1}\mathbb{P}_{p,D} \mid \lambda \in \Sigma_\theta\} \leq C.$$

5 Outline of the proofs of Theorems 3.1 and 3.2

In this last section, we give outline of the proofs of Theorems 3.1 and 3.2. Before explaining the proofs, we first introduce the following convention for ε and p.

Convention 5.1. Let $\varepsilon > 0$ be such that the assertions of Propositions 2.3 and 4.1 are satisfied for all p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2n} + \varepsilon.$$

Let us take $R > 1$ large such that $\Omega^c \subset B_R(0) = \{x \in \mathbb{R}^n \mid |x| < R\}$. Define

$$\begin{aligned} D &:= \Omega \cap B_{R+5}(0), \\ K_1 &:= \{x \in \Omega \mid R < |x| < R+3\}, \\ K_2 &:= \{x \in \Omega \mid R+2 < |x| < R+5\}. \end{aligned}$$

Besides, we define cut-off functions $\varphi, \eta \in \mathrm{C}^\infty(\mathbb{R}^n; [0,1])$ by

$$\varphi(x) = \begin{cases} 0 & \text{for } |x| \le R+1, \\ 1 & \text{for } |x| \ge R+2, \end{cases}$$

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \le R+3, \\ 0 & \text{for } |x| \ge R+4. \end{cases}$$

For $f \in \mathrm{L}^p_\sigma(\Omega)$ let f^R be the zero extension of f to $\mathbb{R}^n$, while set $f^D = \eta f - \mathcal{B}_2((\nabla \eta) \cdot f)$. Here, for $\ell = 1, 2$, the symbol $\mathcal{B}_\ell$ denotes the Bogovskii operator defined on K_ℓ. Let (u^R_λ, g) and $(u^D_\lambda, \pi^D_\lambda)$ satisfy

$$\begin{cases} \lambda u^R_\lambda - \Delta u^R_\lambda + \nabla g = f^R & \text{in } \mathbb{R}^n, \\ \operatorname{div}(u^R_\lambda) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

and

$$\begin{cases} \lambda u^D_\lambda - \Delta u^D_\lambda + \nabla \pi^D_\lambda = f^D & \text{in } D, \\ \operatorname{div}(u^D_\lambda) = 0 & \text{in } D, \\ u^D_\lambda = 0 & \text{on } \partial D \end{cases}$$

respectively. Here, the pressure term g is given by $\nabla g = (\mathrm{Id} - \mathbb{P}_{p,\mathbb{R}^n}) f^R$ and we normalize it to satisfy $\int_D g \, \mathrm{d}x = 0$. Define the operators U_λ and Π_λ by

$$\begin{aligned} U_\lambda f &:= \varphi u^R_\lambda + (1-\varphi) u^D_\lambda - \mathcal{B}_1((\nabla \varphi) \cdot (u^R_\lambda - u^D_\lambda)), \\ \Pi_\lambda f &:= (1-\varphi) \pi^D_\lambda + \varphi g, \end{aligned}$$

respectively. Then we see that the pair $(U_\lambda f, \Pi_\lambda f)$ enjoys the system

$$\begin{cases} (\lambda - \Delta) U_\lambda f + \nabla \Pi_\lambda f = (\mathrm{Id} + T_\lambda) f & \text{in } \Omega, \\ \operatorname{div}(U_\lambda f) = 0 & \text{in } \Omega, \\ U_\lambda f = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of distributions. Here, the remainder term T_λ is given by

$$\begin{aligned} T_\lambda f := &-2[(\nabla \varphi) \cdot \nabla](u^R_\lambda - u^D_\lambda) - (\Delta \varphi)(u^R_\lambda - u^D_\lambda) \\ &+ (\nabla \varphi)(g - \pi^D_\lambda) - (\lambda - \Delta) \mathcal{B}_1((\nabla \varphi) \cdot (u^R_\lambda - u^D_\lambda)). \end{aligned}$$

Notice that it is clear that for each $f \in \mathrm{L}^p(\Omega; \mathbb{C}^n)$ it holds $\operatorname{supp}(T_\lambda f) \subset \overline{K_1}$ and T_λ is a compact operator on $\mathrm{L}^p(\Omega; \mathbb{C}^n)$. Concerning for the resolvent bounds, we have the following lemmas.

Lemma 5.2. *Let $\theta \in (0,\pi)$. Let ε and $p \in (1,\infty)$ be subject to Convention 5.1. There exists $\lambda_* \geq 1$ such that for all $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \lambda_*$ the operator*

$$\mathrm{Id} + \mathbb{P}_{p,\Omega} T_\lambda \colon \mathrm{L}^p_\sigma(\Omega) \to \mathrm{L}^p_\sigma(\Omega)$$

is invertible. Besides, λ_* can be taken such that

$$\mathcal{R}_{\mathrm{L}^p_\sigma(D)}\{(\mathrm{Id} + \mathbb{P}_{p,\Omega} T_\lambda)^{-1} \mid \lambda \in \Sigma_\theta, |\lambda| \geq \lambda_*\} \geq 2.$$

Lemma 5.3. *Let ε and p be subject to Convention 5.1 with $p < n/2$, $\theta \in (0,\pi)$, and $\lambda_* > 0$. For all $\lambda \in \overline{\Sigma_\theta \cap B_{\lambda_*}(0)}$ the operator $\mathrm{Id}+T_\lambda \colon \mathrm{L}^p(\Omega;\mathbb{C}^n) \to \mathrm{L}^p(\Omega;\mathbb{C}^n)$ is invertible and there exists a constant $C > 0$ such that*

$$\mathcal{R}_{\mathrm{L}^p(\Omega;\mathbb{C}^n)}\{(\mathrm{Id} + T_\lambda)^{-1} \mid \lambda \in \Sigma_\theta \cap B_{\lambda_*}(0)\} \geq C.$$

We first assume ε and p are subject to Convention 5.1 with $p < n/2$. Thanks to the Helmholtz decomposition, we write

$$f + T_\lambda f = f + \mathbb{P}_{p,\Omega} T_\lambda f + (\mathrm{Id} - \mathbb{P}_{p,\Omega}) T_\lambda f =: f + \mathbb{P}_{p,\Omega} T_\lambda f + \nabla \Phi_\lambda f.$$

Then, $U_\lambda f$ and $\Pi_\lambda f - \Phi_\lambda f$ solve the Stoles resolvent problem with right-hand side $f + \mathbb{P}_{p,\Omega} T_\lambda f$. Thus, if $\lambda \geq \lambda_*$, we see that the functions

$$\begin{aligned} u &:= U_\lambda (\mathrm{Id} + \mathbb{P}_{p,\Omega} T_\lambda)^{-1}, \\ \pi &:= (\Pi_\lambda - \Phi_\lambda)(\mathrm{Id} + \mathbb{P}_{p,\Omega})^{-1} f \end{aligned}$$

are solutions to the Stokes resolvent problem with right-hand side f. On the other hand, if $|\lambda| < \lambda_*$, we observe that

$$\begin{aligned} u &:= U_\lambda (\mathrm{Id} + T_\lambda)^{-1} f, \\ \pi &:= \Pi_\lambda (\mathrm{Id} + T_\lambda)^{-1} f \end{aligned}$$

are the solutions to the Stokes resolvent problem with right-hand side f. Hence, there exists $C > 0$ such that it holds

$$\mathcal{R}_{\mathrm{L}^p(\Omega;\mathbb{C}^n)}\{\lambda(\lambda + A_p)^{-1} \mathbb{P}_{p,\Omega} \mid \lambda \in \Sigma_\theta\} \leq C \tag{5.1}$$

with $p < n/2$.

According to [16, Prop. 5.2.5], the resolvent estimate for the case $p = 2$ has been already known. Since uniform boundedness is equivalent to $\mathcal{R}$-boundedness if $p = 2$, we have (5.1) with $p = 2$. By the complex interpolation, the estimate (5.1) is valid for all $p \leq 2$ subject to Convention 5.1. Therefore, from the duality result due to Weis [19, Lem. 3.1], we obtain (5.1) for all p with Convention 5.1. This yields Theorem 3.1. Finally, Theorem 3.2 follows from the interpolation theorem due to Voigt [18] and the Cauchy formula. For further (rigorous) discussion, we refer to [17, Sec. 5].

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