On the $\mathscr{R}$-boundedness for the generalized Stokes resolvent problem in an infinite layer with Neumann boundary condition

名古屋大学・多元数理科学研究科　大石　健太
Kenta Oishi
Graduate School of Mathematics,
Nagoya University

1 Introduction

Consider the generalized Stokes resolvent problem and the nonstationary Stokes problem with Neumann boundary conditions

$$\begin{cases} \lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u},\theta) = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = g & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$$\begin{cases} \partial_t\mathbf{U} - \operatorname{Div}\mathbf{S}(\mathbf{U},\Theta) = \mathbf{F}, \quad \operatorname{div}\mathbf{U} = G & \text{in } \Omega\times(0,\infty), \\ \mathbf{S}(\mathbf{U},\Theta)\nu = \mathbf{H} & \text{on } \partial\Omega\times(0,\infty), \\ \mathbf{U}|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{1.2}$$

in an infinite layer

$$\Omega = \{x=(x',x_N)\in\mathbb{R}^N \mid x'=(x_1,\cdots,x_{N-1})\in\mathbb{R}^{N-1},\ 0<x_N<\delta\} \quad (\delta>0,\ N\geq 2).$$

Here, the unknowns $\mathbf{u}=(u_1(x),\cdots,u_N(x))^\top$ and $\theta=\theta(x)$ are N-component velocity vector and scalar pressure, respectively, while known functions are scalar function $g=g(x)$ and N-vector functions $\mathbf{f}=(f_1(x),\cdots,f_N(x))^\top$ and $\mathbf{h}=(h_1(x),\cdots,h_N(x))^\top$. By $\mathbf{U}=\mathbf{U}(x,t)$, $\Theta=\Theta(x,t)$, $\mathbf{F}=\mathbf{F}(x,t)$, $G=G(x,t)$ and $\mathbf{H}=\mathbf{H}(x,t)$, we denote the counterparts of them for (1.2). The symbol $\mathbf{S}(\mathbf{u},\theta)$ is the stress tensor given by $\mu\mathbf{D}(\mathbf{u})-\theta\mathbf{I}$, where μ is a positive constant which denotes the viscosity coefficient, and $\mathbf{I}$ is the $N\times N$ identity matrix. Also, $\mathbf{D}(\mathbf{u})$ stands for the doubled deformation tensor whose (j,k) component is $\mathbf{D}_{jk}(\mathbf{u})=\partial_k u_j+\partial_j u_k$ with $\partial_j=\partial/\partial x_j$. We denote by $\nu=(\nu_1(x),\cdots,\nu_N(x))^\top$ the unit outer normal vector to $\partial\Omega$. For an $N\times N$ matrix-valued function $\mathbf{M}=(M_{ij})_{ij}$, the i-th component of $\operatorname{Div}\mathbf{M}$ is defined by $\sum_{j=1}^N\partial_k M_{ij}$.

Several mathematicians have been studied these problems for the Neumann-Dirichlet boundary condition, namely, the boundary condition on the lower boundary is replaced by Dirichlet one:

$$\mathbf{u}=0 \text{ on } \Gamma_0=\{x=(x',x_N)\in\mathbb{R}^N \mid x_N=0\}.$$

Abe [1] proved the resolvent estimates with $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$ where

$$\Sigma_{\varepsilon,\gamma_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \varepsilon,\ |\lambda| > \gamma_0\}. \tag{1.3}$$

Abels obtained them for $\gamma_0 = 0$ in [5] and, then, he extended the result to those estimates for asymptotically flat layers in [3]. He also showed that the Stokes operator admits a bounded H_∞-calculus in [4] and, as a consequence, the maximal L_q regularity for (1.2) with $3/2 < q < \infty$. Finally, Saito [10] established the $\mathscr{R}$-boundedness of the solution operator families with resolvent parameter $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for any $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$ and, as a corollary, he obtained the maximal L_p-L_q regularity for $1 < p, q < \infty$. Moreover, Shibata [12, 13] developed a theory for general domains, with Dirichlet boundary condition on $\Gamma_b \subset \partial\Omega$ and Neumann boundary condition on $\Gamma = \partial\Omega \setminus \Gamma_b$, under the assumption: the unique existence of solution $\theta \in \mathcal{W}^1_q(\Omega)$ to the weak Dirichlet-Neumann problem

$$(\nabla\theta, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega \text{ for any } \varphi \in \mathcal{W}^1_q(\Omega) \tag{1.4}$$

for any $\mathbf{f} \in L_q(\Omega)^N$. Here, $\mathcal{W}^1_q(\Omega)$ is any closed subspace of $\widehat{W}^1_{q,\Gamma}(\Omega)$ containing $W^1_{q,\Gamma}(\Omega)$, where $\widehat{W}^1_{q,\Gamma}(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla\theta \in L_q(\Omega)^N,\ \theta|_\Gamma = 0\}$ and $W^1_{q,\Gamma}(\Omega) = \{\theta \in W^1_q(\Omega) \mid \theta|_\Gamma = 0\}$. To be precise, he showed the resolvent estimate with $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for some $\gamma_0 > 0$ in [12]. In [13], he extended this estimate to the $\mathscr{R}$-boundedness for (1.1) and developed the maximal L_p-L_q regularity for (1.2) with the help of it. As for the case of the Neumann boundary conditions on both sides of the boundary, however, our knowledge of the $\mathscr{R}$-boundedness as well as the maximal regularity is much less, and even the unique solvability of (1.4) has not been proved as far as we know. We note that the unique solvability of (1.4) with $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,0}(\Omega)$ follows from the $\mathscr{R}$-boundedness thanks to observation by [12, Remark 1.7].

In this paper, we establish the $\mathscr{R}$-boundedness of solution operator families of (1.1) with the resolvent parameter λ in a sector $\Sigma_{\varepsilon,\gamma_0}$ for arbitrary $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$, and it implies the resolvent estimates with λ in the same sector. And then we prove the maximal L_p-L_q regularity for (1.2) with $1 < p, q < \infty$ from the $\mathscr{R}$-boundedness combined with the operator-valued Fourier multiplier theorem due to Weis [22, Theorem 3.4]. It is worth pointing out that we gain an exact solution formula to (1.1) by applying the partial Fourier transform with respect to tangential variable $x' \in \mathbb{R}^{N-1}$. And also, the formula enables us to take any $\gamma_0 > 0$ in (1.3) although it was taken large enough in the study of general domain, see [13]. We wish to obtain the resolvent estimates with $\lambda \in \Sigma_{\varepsilon,0}$, that would be the first step toward decay properties of solutions to (1.2). However, the assumption $\gamma_0 > 0$ seemed to be needed essentially for the estimate of the determinant $\det \mathbf{L}$ in the solution formula. In fact, $|\det \mathbf{L}|^{-1}$ is too singular at the origin in Fourier side when $\lambda = 0$ although the solution formula is also available for $\lambda = 0$. Our approach follows [18] and [10]; we regard the solution formula as a singular integral and, then, estimate it by the fact that the kernel is estimated by $|x|^{-N}$, see the proof of Lemma 3.1. However, the formula involves a symbol which possesses higher singularity at the origin than that for Neumann-Dirichlet boundary condition. The reason is that the determinant degenerates for $\xi' \to 0$ since it has two similar rows caused by the same boundary conditions on both sides of $\partial\Omega$, see Remark 3.2. We get around this difficulty by using the idea of Saito [10, Lemma 5.5]. The point is to estimate the solution formula in the tangential direction uniformly with respect to the normal variable by regarding the formula as the singular integral on $\mathbb{R}^{N-1}$ with a kernel decaying like $|x'|^{-(N-1)}$. We then find the desired $\mathscr{R}$-boundedness since the layer is bounded in the normal direction, see the proof of Lemma 3.2. As another difficulty, the estimate of $|\det \mathbf{L}|^{-1}$ is inhomogeneous in the sense that it is bounded for $|\xi'| \to \infty$ but it diverges for $|\xi'| \to 0$. We resolve it by a cut-off procedure.

Problems (1.1) and (1.2) arise from a free boundary problem of Navier-Stokes equations describing the motion of incompressible and viscous fluid flow without surface tension. The problem is to find a time-dependent domain $\Omega(t)$, a velocity field $\mathbf{v} = (v_1(x,t), \cdots, v_N(x,t))^\mathsf{T}$ and a pressure $\pi = \pi(x,t)$ satisfying the following equation for given initial velocity $\mathbf{v}_0 = (v_{01}(x), \cdots, v_{0N}(x))^\mathsf{T}$:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{Div}\mathbf{S}(\mathbf{v},\pi) = 0, \quad \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ 0<t<T, \\ \mathbf{S}(\mathbf{v},\pi)\nu_t = 0, \quad \mathbf{v}\cdot\nu_t = V_n & \text{on } \partial\Omega(t),\ 0<t<T, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

Here, V_n and ν_t in the boundary condition stand for the velocity of the evolution of $\partial\Omega(t)$ and the unit outer normal to $\partial\Omega(t)$, respectively. The novelty of the problem (1.5) is that we consider free boundary conditions to be determined on both upper and lower ones.

Problem (1.5) has been studied for the case that the lower boundary is fixed while the upper surface is still free in an asymptotic layer

$$\Omega(t) = \{x = (x', x_N) \in \mathbb{R}^N \mid -b(x') < x_N < \eta(t,x')\}.$$

In the L_2-framework, Beale [8] established the existence of solutions locally in time without surface tension. Allain [6, 7] and Tani [19] proved it with surface tension, that is, for the following boundary condition:

$$\mathbf{S}(\mathbf{v},\pi)\nu_t = \sigma\mathcal{H}\nu_t, \quad \mathbf{v}\cdot\nu_t = V_n, \quad 0<t<T,$$

where $\mathcal{H}$ and $\sigma > 0$ stand for doubled mean curvature of $\partial\Omega(t)$ and the coefficient of surface tension. Beale [9] obtained the global well-posedness with surface tension, and Tani and Tanaka [20] proved it with and without surface tension. Furthermore, Abels [2] obtained the local well-posedness in L_q-framework, and Saito [11] developed the global well-posedness in the L_p-L_q setting, without surface tension. In general domains, Shibata proved the local well-posedness theorem without and with surface tension in [14] and [16, 15], respectively. For our boundary conditions, however, results seem to be less developed. In this paper, we establish the local well-posedness in L_p-in-time and L_q-in-space setting for $2 < p < \infty$ and $N < q < \infty$ by means of the fix-point arguments with the help of maximal L_p-L_q regularity obtained above, in the similar way to [11].

As mentioned above, the uniqueness of the weak Dirichlet problem (1.4) with $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,0}(\Omega)$ follows from the $\mathscr{R}$-boundedness. Then the theory for general domains due to Shibata

[13, 14] leads to the local well-posedness of (1.5), in addition to the maximal regularity (but only for $\gamma_0 \gg 1$). Nevertheless, we develop the theory in the layer (for arbitrary $\gamma_0 > 0$) since it would be better not to rely on the general theory in [13, 14].

The next section is devoted to stating the main theorem and corollaries. In Section 3, we present the sketch of the proof of the $\mathscr{R}$-boundedness for (1.1).

2 Main Results

In this section, we provide our main theorem and corollaries. First, we introduce some notation and, then, review the definition of the $\mathscr{R}$-boundedness. we often write $\gamma = \operatorname{Re}\lambda$ and $\tau = \operatorname{Im}\lambda$ for $\lambda \in \mathbb{C}$. Let $D = \mathbb{R}^N$, $\mathbb{R}^N_+$ or the layer Ω. We set

$$\begin{aligned} &\widehat{W}^{-1}_q(D) = \text{ the dual space of } \widehat{W}^1_{q',0}(D), \\ &\text{where } \ \widehat{W}^1_{q',0}(D) = \{\varphi \in \widehat{W}^1_{q'}(D) \mid \varphi|_{\partial D} = 0\}. \end{aligned} \tag{2.1}$$

for $1 < q < \infty$, where q' denotes the dual exponent given by $1/q + 1/q' = 1$. We define

$$[\Lambda^{1/2}_\gamma f](t) = [\mathscr{L}^{-1}_\gamma |\lambda|^{1/2} \mathscr{L}[f]](t) \tag{2.2}$$

with $\mathscr{L}$ and $\mathscr{L}^{-1}_\gamma$ being the Laplace transform and its inverse transform, respectively, which are given by

$$\mathscr{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-i\tau t} e^{-\gamma t} f(t)\, dt, \qquad \mathscr{L}^{-1}_\gamma[g](t) = \frac{1}{2\pi} e^{\gamma t} \int_{-\infty}^{\infty} e^{i\tau t} g(\lambda)\, d\tau$$

for functions f vanishing on $(-\infty, 0)$ and g. We note that we have, as in [13, Appendix],

$$\|e^{-\gamma t}\Lambda^{1/2}_\gamma G\|_{L_p(\mathbb{R}, L_q(\Omega))} \le C(\|e^{-\gamma t}\partial_t G\|_{L_p(\mathbb{R}, \widehat{W}^{-1}_q(\Omega))} + \|e^{-\gamma t}\nabla G\|_{L_p(\mathbb{R}, L_q(\Omega))}) \tag{2.3}$$

for any function G with $e^{-\gamma t}\partial_t G \in L_p(\mathbb{R}, \widehat{W}^{-1}_q(\Omega))$, $e^{-\gamma t}\nabla G \in L_p(\mathbb{R}, L_q(\Omega))$ and $G(t) = 0$ $(t < 0)$. The definition of the $\mathscr{R}$-boundedness is given by the following.

Definition 2.1. *Let X and Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is said to be $\mathscr{R}$-bounded if there exist $1 \le p < \infty$ and $C > 0$ such that for $m \in \mathbb{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{x_j\}_{j=1}^m \subset X$, and sequences $\{r_j\}_{j=1}^m$ of independent, symmetric and $\{\pm 1\}$-valued random variables on $(0, 1)$, the following estimate holds:*

$$\int_0^1 \Big\| \sum_{j=1}^m r_j(u) T_j x_j \Big\|_Y^p du \le C \int_0^1 \Big\| \sum_{j=1}^m r_j(u) x_j \Big\|_X^p du.$$

The infimum of such C is called $\mathscr{R}$-bound and denoted by $\mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$, or $\mathscr{R}_{\mathcal{L}(X)}(\mathcal{T})$ if $X = Y$.

The main result on the $\mathscr{R}$-boundedness for (1.1) is stated as follows.

Theorem 2.1. *Set*

$$X_q(\Omega) = \{(F_1, F_2, F_3, F_4, F_5, F_6) \mid \\ F_1, F_4, F_5 \in L_q(\Omega)^N, F_2 \in \widehat{W}_q^{-1}(\Omega), F_3 \in L_q(\Omega), F_6 \in L_q(\Omega)^{N^2}\}.$$

For all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, there exist operators $\mathcal{U}(\lambda) = (\mathcal{U}_1(\lambda), \cdots, \mathcal{U}_N(\lambda))$ and $\mathcal{P}(\lambda)$ satisfying $\mathcal{U}(\lambda) \in \mathcal{L}(X_q(\Omega), W_q^2(\Omega)^N)$ and $\mathcal{P}(\lambda) \in \mathcal{L}(X_q(\Omega), \widehat{W}_q^1(\Omega))$ for $1 < q < \infty$ such that the following assertions hold:

a) For all $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and the data

$$(\mathbf{f}, g, \mathbf{h}) \in L_q(\Omega)^N \times (\widehat{W}_q^{-1}(\Omega) \cap W_q^1(\Omega)) \times W_q^1(\Omega)^N,$$

the couple $(\mathbf{u}, \theta) \in W_q^2(\Omega) \times \widehat{W}_q^1(\Omega)$ given by

$$(\mathbf{u}, \theta) = (\mathcal{U}(\lambda), \mathcal{P}(\lambda))(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})$$

is a unique solution of (1.1).

b) For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\ell = 0, 1$ and $1 \le m, n, J \le N$, there hold

$$\begin{aligned}
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},\\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},\\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},\\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},\\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m \mathcal{P}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},
\end{aligned}$$

where $\lambda = \gamma + i\tau$ and $\Sigma_{\varepsilon,\gamma_0}$ is given by (1.3).

We obtain the maximal L_p-L_q regularity as a corollary of Theorem 2.1 combined with the operator-valued Fourier multiplier theorem due to Weis [22, Theorem 3.4]. We may skip the proof since it is similar to the proof in [10, Theorem 2.1].

Theorem 2.2. *Recall that $\widehat{W}_q^{-1}(\Omega)$ and $\Lambda_\gamma^{1/2}$ are given by (2.1) and (2.2). Let $1 < p, q < \infty$, $\gamma_0 > 0$. Then, for any data $\mathbf{F}$, G, $\mathbf{H}$ such that*

$$e^{-\gamma_0 t}\mathbf{F} \in L_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{-\gamma_0 t} G \in W_p^1(\mathbb{R}, \widehat{W}_q^{-1}(\Omega)) \cap L_p(\mathbb{R}, W_q^1(\Omega)),$$
$$e^{-\gamma t}\Lambda_\gamma^{1/2}\mathbf{H} \in L_p(\mathbb{R}, L_q(\Omega)^N) \ (\gamma \ge \gamma_0), \quad e^{-\gamma_0 t}\mathbf{H} \in L_p(\mathbb{R}, W_q^1(\Omega)^N),$$

with $(\mathbf{F}(t), G(t), \mathbf{H}(t)) = (0, 0, 0)$ $(t < 0)$, problem (1.2) admits a unique solution $(\mathbf{U}, \Theta)$ with

$$e^{-\gamma_0 t}\mathbf{U} \in W_p^1(0, \infty, L_q(\Omega)^N) \cap L_p(0, \infty, W_q^2(\Omega)), \quad e^{-\gamma_0 t}\Theta \ \in L_p(0, \infty, \widehat{W}_q^1(\Omega)).$$

Moreover, it satisfies the estimate (note that the right-hand side is finite by (2.3)):

$$\begin{aligned}
&\|e^{-\gamma t}(\partial_t \mathbf{U}, \gamma \mathbf{U}, \Lambda_\gamma^{1/2}\nabla \mathbf{U}, \nabla^2 \mathbf{U}, \nabla\Theta)\|_{L_p(0,\infty, L_q(\Omega))}\\
&\le C_{\gamma_0}\{\|e^{-\gamma t}(\mathbf{F}, \Lambda_\gamma^{1/2} G, \nabla G, \Lambda_\gamma^{1/2}\mathbf{H}, \nabla \mathbf{H})\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}\partial_t G\|_{L_p(\mathbb{R}, \widehat{W}_q^{-1}(\Omega))}\}
\end{aligned}$$

for any $\gamma \ge \gamma_0$ with some constant C_{γ_0} independent of γ.

Finally, we develop the local well-posedness of the nonlinear free boundary problem (1.5). To this end, we first formulate this problem in Lagrange coordinates by using the relation between Euler coordinates $x \in \Omega(t)$ and Lagrange ones $y \in \Omega$:

$$x = y + \int_0^t \mathbf{u}(y,s)\,ds \equiv \mathbf{X}_{\mathbf{u}}(y,t), \tag{2.4}$$
$$\mathbf{u} = (u_1(y,t), \cdots, u_N(y,t)) = \mathbf{v}(\mathbf{X}_{\mathbf{u}}(y,t),t), \quad \theta(y,t) = \pi(\mathbf{X}_{\mathbf{u}}(y,t),t).$$

And then we get the following quasilinear problem (cf. [17, Appendix A]):

$$\begin{cases} \partial_t\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u},\theta) = \mathbf{f}(\mathbf{u}), \quad \operatorname{div}\mathbf{u} = g(\mathbf{u}) = \operatorname{div}\mathbf{g}(\mathbf{u}) & \text{in } \Omega\times(0,T), \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h}(\mathbf{u}) & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega, \end{cases} \tag{2.5}$$

where nonlinear terms $\mathbf{f}(\mathbf{u})$, $g(\mathbf{u})$, $\mathbf{g}(\mathbf{u})$ and $\mathbf{h}(\mathbf{u})$ are given by

$$\mathbf{f}(\mathbf{u}) = \mathbf{V}_1\left(\int_0^t \nabla\mathbf{u}\,ds\right)\partial_t\mathbf{u} + \mathbf{V}_2\left(\int_0^t \nabla\mathbf{u}\,ds\right)\nabla^2\mathbf{u} + \mathbf{V}_3\left(\int_0^t \nabla\mathbf{u}\,ds\right)\int_0^t \nabla^2\mathbf{u}\,ds\cdot\nabla\mathbf{u},$$
$$g(\mathbf{u}) = \mathbf{V}_4\left(\int_0^t \nabla\mathbf{u}\,ds\right)\nabla\mathbf{u}, \quad \mathbf{g}(\mathbf{u}) = \mathbf{V}_5\left(\int_0^t \nabla\mathbf{u}\,ds\right)\mathbf{u}, \quad \mathbf{h}(\mathbf{u}) = \mathbf{V}_6\left(\int_0^t \nabla\mathbf{u}\,ds\right)\nabla\mathbf{u}$$

with some polynomials satisfying $\mathbf{V}_i(0) = 0$ for $i = 1, \cdots, 6$. As the linearlized problem, we consider the nonstationary Stokes equation (1.2) with the initial velocity $\mathbf{v}_0$:

$$\begin{cases} \partial_t\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u},\theta) = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = g & \text{in } \Omega\times(0,\infty), \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h} & \text{on } \partial\Omega\times(0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases} \tag{2.6}$$

By setting $\mathbf{u} = e^{-A_q t}\mathbf{v}_0 + \mathbf{U}$ and $\theta = \mathcal{K}(e^{-A_q t}\mathbf{v}_0) + \Theta$, we get (1.2). Here, $e^{-A_q t}\mathbf{v}_0$ is the Stokes analytic semigroup, whose generator A_q is defined by

$$D(A_q) = \{\mathbf{u} \in J_q(\Omega) \cap W_q^2(\Omega)^N \mid \mathbf{S}(\mathbf{u},\mathcal{K}(\mathbf{u})) = 0 \text{ on } \partial\Omega\}, \quad A_q\mathbf{u} = \operatorname{Div}\mathbf{S}(\mathbf{u},\mathcal{K}(\mathbf{u})),$$

where

$$J_q(\Omega) = \{\mathbf{u} \in L_q(\Omega)^N \mid \operatorname{div}\mathbf{u} = 0\},$$

and $\mathcal{K}$ is the solution operator which gives θ from $\mathbf{u}$ in the equation (2.6) with $(\mathbf{f}, g, \mathbf{h}) = (0,0,0)$ or in the equation

$$\begin{cases} \Delta\theta = 0 & \text{in } \Omega, \\ \theta = 2\mu\partial_N u_N & \text{on } \partial\Omega. \end{cases}$$

We are on the point of stating the local well-posedness of (2.5) instead of (1.5). We obtain the solution of (1.5) by applying the Lagrange transformation (2.4) again. The proof may be omitted since it is the same way as in [11, Theorem 2.2]

Theorem 2.3. *Let $2 < p < \infty$ and $N < q < \infty$. For all $R > 0$, there exists $T = T(R) > 0$ satisfying the following: for any initial data $\mathbf{v}_0 \in (J_q(\Omega), D(A_q))_{1-1/p,p} \subset B^{2(1-1/p)}_{q,p}$ with $\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}} \le R$, the system* (2.5) *admits a unique solution*

$$\mathbf{u} \in W^1_p(0,T,L_q(\Omega)^N) \cap L_p(0,T,W^2_q(\Omega)^N)$$

with some pressure term $\theta \in L_p(0,T,\widehat{W}^1_q(\Omega))$, satisfying the following estimate:

$$\|\partial_t \mathbf{u}\|_{L_p(0,T,L_q(\Omega))} + \|\mathbf{u}\|_{L_p(0,T,W^2_q(\Omega))} \le M_0 R$$

with some constant M_0 independent of T and R. Here, $(\cdot,\cdot)_{1-1/p,p}$ is the real interpolation functor.

3 Sketch of proof of Theorem 2.1

In this section, we present the sketch of the proof of our main theorem. First, by solving the divergence equation and the generalized Stokes resolvent problem (1.1) on whole space, we can reduce the problem (1.1) to the case where the data are only on boundary:

$$\begin{cases} \lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u},\theta) = 0, \quad \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h} & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

and, then, we show the $\mathscr{R}$-boundedness for the preceding problem. Theorem 2.1, the full data case, is deduced by combining the $\mathscr{R}$-boundedness for the divergence equation, the problem (1.1) on $\mathbb{R}^N$ and the problem (3.1). The $\mathscr{R}$-boundedness for the problem (3.1) is stated as follows.

Theorem 3.1. *For all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, there exist the operators $\mathcal{S}(\lambda) = (\mathcal{S}_1(\lambda), \cdots, \mathcal{S}_N(\lambda))$ and $\mathcal{T}(\lambda)$ satisfying $\mathcal{S}(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+N^2}, W^2_q(\Omega)^N)$ and $\mathcal{T}(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+N^2}, \widehat{W}^1_q(\Omega))$ for $1 < q < \infty$ such that the following assertions hold:*

a) For all $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\mathbf{h} = (\mathbf{h}', h_N) = (h_1, \cdots, h_N) \in W^1_q(\Omega)^N$, $(\mathbf{u},\theta) \in W^2_q(\Omega)^N \times \widehat{W}^1_q(\Omega)$ given by below solves (3.1)*:*

$$\mathbf{u} = \mathcal{S}(\lambda)(\lambda^{1/2}\mathbf{h}', h_N, \nabla\mathbf{h}), \quad \theta = \mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}', h_N, \nabla\mathbf{h}). \tag{3.2}$$

b) For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\ell = 0, 1$ and $1 \le m, n, J \le N$, there hold

$$\begin{aligned}
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},
\end{aligned}$$

where $\lambda = \gamma + i\tau$ and $\Sigma_{\varepsilon,\gamma_0}$ is given by (1.3).

Remark 3.1. *It is reasonable (and possible by $\gamma_0 > 0$) to show the $\mathscr{R}$-boundedness for*

$$\mathbf{u} = \mathcal{S}(\lambda)(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}), \quad \theta = \mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$$

instead of (3.2)*, but if we do so, we have difficulty estimating pressure term when we prove Theorem 2.1 from Theorem 3.1. This is the reason why we consider the solution* (3.2)*.*

In the sketch of the proof, we focus on the assertions for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ since those for $\mathcal{U}_j(\lambda)$ $(j = 1, \cdots, N-1)$ are obtained as follows. The solution u_j obeys the following system, which consists of the j-th component of the first equation and j-th component of the boundary condition in (3.1):

$$\begin{cases} \lambda u_j - \mu\Delta u_j = -\partial_j\theta & \text{in } \Omega, \\ \partial_N u_j = \mu^{-1}\nu_N h_j - \partial_j u_N & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Once we get the $\mathscr{R}$-boundedness for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$, combining that with the $\mathscr{R}$-boundedness for (3.3) leads to the $\mathscr{R}$-boundedness for $\mathcal{S}_j(\lambda)$.

First, we derive solution formula. We multiply the partial Fourier transform to (1.1) and find the fundamental solution to the resultant ordinary differential equation with respect to ∂_N. Then the partial Fourier transform of u_N and θ are given by

$$\widehat{u}_N(\xi', x_N) = \sum_{\ell=1,2} \mu_{\ell N}\mathcal{M}(d_\ell(x_N)) + \beta_{\ell N}e^{-Bd_\ell(x_N)}, \quad \widehat{\theta}(\xi', x_N) = \sum_{\ell=1,2} \frac{\mu(B+A)}{A}\mu_{\ell N}e^{-Ad_\ell(x_N)}$$

with some constants $\mu_{\ell N}$ and $\beta_{\ell N}$ depending on λ, ξ' and data $\mathbf{h}$, where

$$A = |\xi'|, \quad B = \sqrt{\mu^{-1}\lambda + A^2} \ (\operatorname{Re} B > 0), \tag{3.4}$$

$$d_\ell(x) = \begin{cases} \delta - x_N & \ell = 1, \\ x_N & \ell = 2, \end{cases} \qquad \mathcal{M}(x_N) = \frac{e^{-Bx_N} - e^{-Ax_N}}{B - A}.$$

The constants satisfy the following equation:

$$\mathbf{L}\mathbf{x} = \mathbf{r}$$

where $\mathbf{x} = (x^1, \cdots, x^4)$ and $\mathbf{r} = (r_1, \cdots, r_4)$ are given by

$$x^1 = \mu_{1N}, \quad x^2 = \beta_{1N}, \quad x^3 = \mu_{2N}, \quad x^4 = \beta_{2N},$$

$$r_1 = \mu^{-1}\widehat{h}_d(\xi', \delta), \quad r_2 = -\mu^{-1}\widehat{h}_d(\xi', 0), \quad r_3 = \mu^{-1}A\widehat{h}_N(\xi', \delta), \quad r_4 = -\mu^{-1}A\widehat{h}_N(\xi', 0),$$

and $\mathbf{L} = (L_{ij})_{1\leq i,j\leq 4}$ is defined by

$$\begin{aligned}
&L_{11} = -(B+A), && L_{12} = -(B^2+A^2), \\
&L_{13} = -(B+A)e^{-A\delta} - (B^2+A^2)\mathcal{M}(\delta), && L_{14} = -(B^2+A^2)e^{-A\delta} - (B^2+A^2)(B-A)\mathcal{M}(\delta), \\
&L_{21} = -(B+A)e^{-A\delta} - (B^2+A^2)\mathcal{M}(\delta), && L_{22} = -(B^2+A^2)e^{-A\delta} - (B^2+A^2)(B-A)\mathcal{M}(\delta), \\
&L_{23} = -(B+A), && L_{24} = -(B^2+A^2), \\
&L_{31} = -(B-A), && L_{32} = 2AB, \\
&L_{33} = (B-A)e^{-A\delta} - 2AB\mathcal{M}(\delta), && L_{34} = -2ABe^{-A\delta} - 2AB(B-A)\mathcal{M}(\delta), \\
&L_{41} = -(B-A)e^{-A\delta} + 2AB\mathcal{M}(\delta), && L_{42} = 2ABe^{-A\delta} + 2AB(B-A)\mathcal{M}(\delta), \\
&L_{43} = -(B-A), && L_{44} = 2AB.
\end{aligned}$$

By solving this, we obtain the solution formula of (1.1):

$$u_N(x) = \sum_{k=1}^{4}\sum_{\ell=1}^{2} \mathscr{F}_{\xi'}^{-1}\left[\left\{\frac{L_{k,2\ell-1}}{\det \mathbf{L}}\mathcal{M}(d_\ell(x_N)) + \frac{L_{k,2\ell}}{\det \mathbf{L}} e^{-Bd_\ell(x_N)}\right\} r_k\right](x'),$$
$$\theta(x) = \sum_{k=1}^{4}\sum_{\ell=1}^{2} \mathscr{F}_{\xi'}^{-1}\left[\left\{\frac{\mu(B+A)}{A}\frac{L_{k,2\ell-1}}{\det \mathbf{L}} e^{-Ad_\ell(x_N)}\right\} r_k\right](x'),$$

where $L_{i,j}$ is (i,j) cofactor of $\mathbf{L}$, and the determinant of $\mathbf{L}$ is

$$\det \mathbf{L} = \frac{1}{(B-A)^2}\prod_{+,-}\{(B^2+A^2)^2(1\pm e^{-A\delta})(1\mp e^{-B\delta}) - 4A^3B(1\mp e^{-A\delta})(1\pm e^{-B\delta})\}. \tag{3.5}$$

Remark 3.2. *The determinant causes higher singularity at* $\xi' = 0$ *in the symbol of the solution formula than that for Neumann-Dirichlet boundary condition. Indeed, the third and fourth rows are similar each other for small* ξ' *in the sense that they coincide when* $\xi' = 0$*, and so* $\det \mathbf{L} \to 0$ *as* $\xi' \to 0$*. On the other hand,* $\det \mathbf{L} \not\to 0$ *as* $\xi' \to 0$ *for Neumann-Dirichlet boundary condition.*

In what follows, we focus on one term in the solution formula of λu_N. The other terms can be estimated in the same way. Let φ_δ and φ_0 be cut-off functions such that

$$\varphi_0 \in C_0^\infty(\mathbb{R};[0,1]),\ \varphi_0(x_N) = \begin{cases} 1 & (|x_N| \le 1/3) \\ 0 & (|x_N| \ge 2/3), \end{cases} \quad \varphi_\delta(x_N) = 1 - \varphi_0(x_N) \quad (x_N \in \mathbb{R}). \tag{3.6}$$

We use a trick in Volevich [21] (see (5.24)); by the fundamental theorem of calculus,

$$\begin{aligned} \lambda u_N &= -\mathscr{F}_{\xi'}^{-1}\left[\frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N}\widehat{h_N}(\xi',0)\right](x') + \cdots \qquad (3.7)\\ &= \int_0^\delta \partial_{y_N}\mathscr{F}_{\xi'}^{-1}\left[\varphi_0(y_N)\frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N} e^{-Ay_N}\widehat{h_N}(\xi',y_N)\right](x')\,dy_N + \cdots \\ &= \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[{\varphi_0}'(y_N)\frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N} e^{-Ay_N}\widehat{h_N}(\xi',y_N)\right](x')\,dy_N \\ &\quad - \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[\varphi_0(y_N)\frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A^2 e^{-Bx_N} e^{-Ay_N}\widehat{h_N}(\xi',y_N)\right](x')\,dy_N \\ &\quad + \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[\varphi_0(y_N)\frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N} e^{-Ay_N}\widehat{\partial_N h_N}(\xi',y_N)\right](x')\,dy_N + \cdots . \end{aligned}$$

Then we introduce technical lemmas playing a crucial role in this paper. Since the symbol possesses higher singularity at $\xi' = 0$ as compared to that for Neumann-Dirichlet boundary condition, we employ different lemma depending on the part: the part with same singularity or higher singularity. As a notation, remember that A, B, $\mathcal{M}$, d_ℓ, φ_δ and φ_0 are defined in (3.4) and (3.6), and set

$$k_i(x_N) = \begin{cases} e^{-Bx_N} & i = 1, \\ e^{-Ax_N} & i = 2, \\ B\mathcal{M}(x_N) & i = 3, \end{cases} \qquad \Phi_i(y_N) = \begin{cases} \varphi_\delta(y_N) & i = 1, \\ \varphi_0(y_N) & i = 2, \\ \varphi_0'(y_N) = -\varphi_\delta'(y_N) & i = 3. \end{cases}$$

The following lemma, which is obtained in the same way as in [10, Lemma 5.3], is concerned with the $\mathscr{R}$-boundedness for the part with same singularity as that for Neumann-Dirichlet boundary condition.

Lemma 3.1. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 \geq 0$ and let $m(\lambda, \xi') \in C^\infty(\Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\}))$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} (\tau\partial_\tau)^\ell m(\lambda, \xi')\right| \leq MA^{-|\alpha'|}$$

with some constant $M = M(\varepsilon, \gamma_0, \alpha') > 0$, for $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$, $\ell = 0, 1$ and multi-index α'. Here, $\lambda = \gamma + i\tau$and $\Sigma_{\varepsilon,\gamma_0}$ is given by (1.3). *We define the operator*

$$[K_1(\lambda)h](x) = \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[\Phi_i(y_N)m(\lambda, \xi')Ak_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\widehat{h}(\xi', y_N)\right](x')\, dy_N \quad (3.8)$$

for all $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $i, i_1, i_2 = 1, 2, 3$ and $\ell_1, \ell_2 = 1, 2$. Then, for $1 < q < \infty$ and $\ell = 0, 1$, there holds

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell K_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C_{N,q,\varepsilon,\gamma_0,M}.$$

The $\mathscr{R}$-boundedness for the part with higher singularity is guaranteed by the following lemma, which is proved by means of the same method as in [10, Lemma 5.5].

Lemma 3.2. *Lemma 3.1 holds even if $K_1(\lambda)$ is replaced by the operator $K_2(\lambda)$ defined by*

$$[K_2(\lambda)h](x) = \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[\Phi_i(y_N)m(\lambda, \xi')k_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\widehat{h}(\xi', y_N)\right](x')\, dy_N.$$

The sketches of the proofs of these lemmas are given at the end of this section. From now on, we first prove the $\mathscr{R}$-boundedness for the one term in (3.7) of λu_N from the lemmas above and the following lemma, which is concerned with the estimates of the determinant and the cofactors of $\mathbf{L}$.

Lemma 3.3. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$.*

a) For any $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$, $\ell = 0, 1$ and multi-index α', the estimate

$$\left|\partial_{\xi'}^{\alpha'} (\tau\partial_\tau)^\ell \frac{1}{\det \mathbf{L}}\right| \leq C_{\varepsilon,\gamma_0,\alpha'}(|\lambda|^{1/2} + A)^{-6}\left(1 + \frac{1}{A}\right) \quad (3.9)$$

holds with some constants $C_{\varepsilon,\gamma_0,\alpha'}$. Here, $\tau = Im\,\lambda$, and also, $\Sigma_{\varepsilon,\gamma_0}$, $\det \mathbf{L}$ and A are given in (1.3), (3.5) *and* (3.4).

b) For $k = 1, \cdots, 4$ and $\ell = 1, 2$, we have

$$L_{k,2\ell-1} \in \mathbb{M}_{5,2,\varepsilon,\gamma_0}, \quad L_{k,2\ell} \in \mathbb{M}_{4,2,\varepsilon,\gamma_0}.$$

Sketch of the proof of the assertions for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ in Theorem 2.1. We give rather details of the proof of the $\mathscr{R}$-boundedness for a typical term in (3.7) of λu_N. In view of (3.9), we divide each term of (3.7) into two parts: one possesses the same singularity as in the case of the Neumann-Dirichlet boundary condition, the other does higher singularity. Let ζ_0 and ζ_1 be cut-off functions satisfying

$$\zeta_0 \in C^\infty(\mathbb{R}; [0,1]), \ \zeta_0(\xi') = \begin{cases} 1 & |\xi'| \geq 2, \\ 0 & |\xi'| \leq 1, \end{cases} \qquad \zeta_1(\xi') = A(1 - \zeta_0(\xi'))$$

so that $1 = \zeta_0(\xi') + \zeta_1(\xi')/A$. Then, by using the formula

$$A = \frac{A^2}{A} = \sum_{j'=1}^{N-1} \frac{(-i\xi_{j'})}{A} i\xi_{j'},$$

we rewrite (3.7) as follows:

$$\begin{aligned}
\lambda u_N = &\int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0{}'(y_N)\zeta_0(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N} e^{-Ay_N} \widehat{h_N}(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0{}'(y_N)\zeta_1(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} e^{-Bx_N} e^{-Ay_N} \widehat{h_N}(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{j'=1}^{N-1} \int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0(y_N)\zeta_0(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} \frac{i\xi_{j'}}{A} A e^{-Bx_N} e^{-Ay_N} \widehat{\partial_{j'} h_N}(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{j'=1}^{N-1} \int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0(y_N)\zeta_1(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} \frac{i\xi_{j'}}{A} e^{-Bx_N} e^{-Ay_N} \widehat{\partial_{j'} h_N}(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0(y_N)\zeta_0(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} A e^{-Bx_N} e^{-Ay_N} \widehat{\partial_N h_N}(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\delta \mathscr{F}^{-1}_{\xi'} \left[\varphi_0(y_N)\zeta_1(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} e^{-Bx_N} e^{-Ay_N} \widehat{\partial_N h_N}(\xi', y_N) \right] (x')\, dy_N + \cdots .
\end{aligned}$$

From Lemma 3.3, we can get

$$\zeta_j(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}}, \ \zeta_0(\xi') \frac{\lambda L_{4,4}}{\mu \det \mathbf{L}} \frac{i\xi_{j'}}{A} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$$

for $j = 0, 1$ and $j' = 1, \cdots, N-1$, and so, the $\mathscr{R}$-boundedness for each term is proved by Lemma 3.1 for odd-numbered one and by Lemma 3.2 for even-numbered one. $\square$

Now, we give the sketches of the proofs of Lemma 3.1 and Lemma 3.2.

Sketch of the proof of Lemma 3.1. The proof is done exactly by the same method in [10, Lemma 5.3]. See also [18], which studies half space. Here, we consider only uniform boundedness, since the $\mathscr{R}$-boundedness also can be obtained similarly. For the simplicity, let $\Phi_i = \varphi_0$ in (3.8), that is, $i = 2$. We rewrite $K_1(\lambda)$ in the form of convolution with kernel and extend the domain of integral to $\mathbb{R}^N_+$:

$$K_1(\lambda)h = \int_\Omega k^1_\lambda(x' - y', x_N, y_N)h(y', y_N)\, dy = \int_{\mathbb{R}^N_+} k^1_\lambda(x' - y', x_N, y_N)h(y', y_N)\, dy$$

where the integrand is extended to $\mathbb{R}^N_+$ by setting 0, and k^1_λ is given by

$$k^1_\lambda(z', x_N, y_N) = \mathscr{F}^{-1}_{\xi'}\left[\varphi_0(y_N)m(\lambda, \xi')Ak_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\right](z').$$

Then the kernel satisfies the estimate

$$|k^1_\lambda(z', x_N, y_N)| \le C|(z', x_N + y_N)|^{-N}, \tag{3.10}$$

and thus, we can estimate the integral as follows.

$$\begin{aligned}
\|K_1(\lambda)h\|_{L_q(\Omega)} &= \|\|[K_1(\lambda)h](\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})}\|_{L_q(0,\delta)} \\
&\le \left\|\int_0^\infty \|k^1_\lambda(\cdot, x_N, y_N) * h(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\right\|_{L_q(0,\delta)} \\
&\le \left\|\int_0^\infty \int_{\mathbb{R}^{N-1}} |k^1_\lambda(\cdot, x_N, y_N)|\, dz' \|h(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\right\|_{L_q(0,\delta)} \\
&\le C\left\|\int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{1}{|(z', x_N + y_N)|^N}\, dz' \|h(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\right\|_{L_q(0,\delta)} \\
&= C\left\|\int_0^\infty \frac{1}{x_N + y_N}\int_{\mathbb{R}^{N-1}} \frac{1}{|(z', 1)|^N}\, dz' \|h(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\right\|_{L_q(0,\delta)} \\
&= C\left\|\int_0^\infty \frac{1}{1 + y_N}\int_{\mathbb{R}^{N-1}} \frac{1}{|(z', 1)|^N}\, dz' \|h(\cdot, x_N y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\right\|_{L_q(0,\delta)} \\
&\le C\int_0^\infty \frac{1}{1 + y_N}\left\|\|h(\cdot, x_N y_N)\|_{L_q(\mathbb{R}^{N-1})}\right\|_{L_q(0,\delta)}\, dy_N \\
&= C\int_0^\infty \frac{1}{y_N^{1/q}(1 + y_N)}\left\|\|h(\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})}\right\|_{L_q(0,\delta)}\, dy_N \\
&= C\|h\|_{L_q(\Omega)}.
\end{aligned}$$

Here, we used the Hölder's inequality in the third line, (3.10) in fourth line, change of variables in fifth, sixth and eighth lines, and the Minkowski's inequality for integrals in seventh line. And then the proof is complete. □

As for the operator $K_2(\lambda)$, unfortunately we only have the following decay of the kernel:

$$|k^2_\lambda(z', x_N, y_N)| \le C|(z', x_N, y_N)|^{-N+1}$$

(if we define $k^2_\lambda(z', x_N, y_N)$ similarly). However, we can estimate the integral by fixing x_N and applying the singular integral theory only to the tangential direction. Thanks to uniformity of the estimate with respect to x_N and boundedness of domain in the normal direction, we can also deal with the integral in the normal direction. Namely, we can prove Lemma 3.2 as follows.

Sketch of the proof of Lemma 3.2. This proof is done exactly by the same method in [10, Lemma 5.5]. We only consider uniform boundedness for the same reason in the preceding proof. Since we can get

$$|(\tau\partial_\tau)^\ell \partial^{\alpha'}_{\xi'}(\Phi_i(y_N)m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N)))| \le C_{N,q,\varepsilon,\mu}A^{-|\alpha'|},$$

by the Fourier multiplier theorem on $\mathbb{R}^{N-1}$ and the Holder's inequality, we have

$$\begin{aligned}
&\|K_2(\lambda)h(\cdot,x_N)\|_{L_q(\mathbb{R}^{N-1})}\\
&\le C\int_0^\delta \left\|\mathscr{F}^{-1}_{\xi'}[\Phi_i(y_N)m(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\widehat{h}(\xi',y_N)](x')\right\|_{L_q(\mathbb{R}^{N-1})} dy_N\\
&\le C\int_0^\delta \|h(\cdot,y_N)\|_{L_q(\mathbb{R}^{N-1})}\, dy_N\\
&\le C\delta^{1-1/q}\left\|\|h(\cdot,y_N)\|_{L_q(\mathbb{R}^{N-1})}\right\|_{L_q(0,\delta)} = C\delta^{1-1/q}\|h\|_{L_q(\Omega)}
\end{aligned}$$

for $0 < x_N < \delta$. Since this estimate is uniform for x_N, we finally obtain

$$\|K_3(\lambda)h\|_{L_q(\Omega)} = \|\|K_3(\lambda)h(\cdot,x_N)\|_{L_q(\mathbb{R}^{N-1})}\|_{L_q(0,\delta)} \le C\delta\|h\|_{L_q(\Omega)},$$

which shows the lemma. □

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