On the two–dimensional exterior boundary–value problem for the steady–state Navier–Stokes equations*

Mikhail V. Korobkov[†], Konstantin Pileckas[‡] and Remigio Russo[§]

June 28, 2020

Abstract

We study the boundary value problem for the stationary Navier–Stokes system in two dimensional exterior domains. In particular, we discuss the history of the problem, its linear analogs (the Stokes paradox and Oseen system), some recent results and open questions.

1 Introduction

The stationary motions of an infinite cylinder of simple connected section $\Omega' \subset \mathbb{R}^2$ in a viscous fluid $\mathcal{F}$ are governed by the Navier–Stokes system[1]

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla p = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where

$$\Omega = \mathbb{R}^2 \setminus \overline{\Omega'},$$

$\boldsymbol{u} : \Omega \to \mathbb{R}^2$, $p : \Omega \to \mathbb{R}$ are the unknown velocity and pressure field and $\nu > 0$ is the assigned kinematical viscosity.

*2010 *Mathematical Subject classification*. Primary 76D05, 35Q30; Secondary 31B10, 76D03; *Key words*: stationary Stokes and Navier Stokes equations, two–dimensional exterior domains, asymptotic behavior.

[†]School of Mathematical Sciences, Fudan University, Shanghai 200433, China; and Sobolev Institute of Mathematics, pr-t Ac. Koptyug, 4, Novosibirsk, 630090, Russia; korob@math.nsc.ru

[‡]Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str., 24, Vilnius, 03225 Lithuania; konstantinas.pileckas@mif.vu.lt

[§]Dipartimento di Matematica e Fisica Università degli studi della Campania "Luigi Vanvitelli," viale Lincoln 5, 81100, Caserta, Italy; e-mail: remigio.russo@unicampania.it

[1]We use a standard notation as, *e.g.*, in [7]. In particular, italic light–face letters except o, x, y, ξ, ζ that denote points of $\mathbb{R}^3$, and small upper–case letters indicate scalars and vectors respectively; $o \in \mathbb{R}^3\setminus\overline{\Omega}$ is the origin of the reference frame $(o, \{\boldsymbol{e}_i\}_{i=1,2})$; $\boldsymbol{x} = x - o$, $r = r(x) = |\boldsymbol{x}|$; div $(\boldsymbol{u}\otimes\boldsymbol{u})$ is the vector with components $u_i\partial_i u_j$. C_R is the disk of radius R centered at 0.

To (1) we append the condition at infinity

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0, \tag{2}$$

with $\boldsymbol{u}_0$ assigned constant vector.

The determination of a solution to system (1)–(2) is a far from trivial problem and has attracted the attention of eminent mathematicians.

In this paper we aims at highlight the most important steps in the study of problem (1)–(2), to give the most recent results and to point out the main open problems.

Acknowledgment. The work of M.K. is supported by Mathematical Center in Akademgorodok, the agreement with Ministry of Science and High Education of the Russian Federation number 075-15-2019-1613.

2 The Stokes equations

Our history starts from the first attempt of the great scientist G.G. Stokes to determine the motion of a rigid body through a viscous liquid.

When the inertial effects $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})$ in $(1)_1$ are negligible, one is allowed to linearize $(1)_1$ around the solution $(\boldsymbol{0}, c)$, to get the *Stokes system*

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \nabla p &= \boldsymbol{0} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} \quad \text{on } \partial\Omega, \\ \lim_{r\to+\infty} \boldsymbol{u}(x) &= \boldsymbol{u}_0. \end{aligned} \tag{3}$$

The first rigorous study of existence of a solution to (3) in the exterior of a disk, was performed by G.G. Stokes in 1851 [36], in order to determine the resistence $\boldsymbol{\varrho}$ on the *obstacle* due to a rigid motion of a right circular cylinder in $\mathcal{F}$. While for $\boldsymbol{a} = \boldsymbol{\omega}\times\boldsymbol{\xi}$, he found the solution

$$\boldsymbol{u}(x) = \boldsymbol{\omega}\times\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}, \quad p = p_0,$$

with p_0 arbitrary constant, unlike to the analogous three–dimensional problem of a translational motion of a ball of radius R, where he found the famous formula $\boldsymbol{\varrho} = 6\nu\pi\boldsymbol{u}_0$, his method, based on a suitable use of the stream function, led him to the conclusion that translational motions of a cylinder in $\mathcal{F}$ were impossible: *it appears that the supposition of steady motion is inadmissible* ([36], p. 63). The same Stokes gave the following explanation: *The pressure of the cylinder on the fluid continuously tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of the fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on* ([36], p. 65).

This impossibility of a slow steady–state translational motion of a cylinder in a viscous fluid becomes known, with a bit of emphasis (see Remark 1), as *Stokes paradox*[2].

In 1896 the Nobel price H. Lorentz found the fundamental solution to the equations $(3)_{1,2}$:

$$\begin{aligned} \mathcal{U}_{ij}(x-y) &= \frac{1}{4\pi\nu}\left[\log\frac{\delta_{ij}}{|x-y|}+\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right], \\ Q_i(x-y) &= \frac{1}{2\pi}\frac{x_i-y_i}{|x-y|^2}, \end{aligned} \tag{4}$$

and in 1930, F.K.G. Odqvist [26] by a suitable use of the Green identities introduced the simple and double layer hydrodynamical potentials:

$$\begin{aligned} v_i[\boldsymbol{\psi}](x) &= \int_{\partial\Omega}\mathcal{U}_{ij}(x-\zeta)\psi_j(\zeta)d\sigma_\zeta, \\ P[\boldsymbol{\psi}](x) &= \int_{\partial\Omega}\varpi_i(x-\zeta)\psi_i\zeta)d\sigma_\zeta, \end{aligned} \tag{5}$$

$$\begin{aligned} \boldsymbol{w}[\boldsymbol{\varphi}](x) &= \frac{2}{\pi}\int_{\partial\Omega}\frac{(x-\zeta)[(x-\zeta)\cdot\boldsymbol{n}(\zeta)][(x-\zeta)\cdot\boldsymbol{\varphi}(\zeta)]ds_\zeta}{|x-\zeta|^4}, \\ \varpi[\boldsymbol{\varphi}](x) &= \frac{2\nu}{\pi}\partial_i\int_{\partial\Omega}\frac{(x_j-\zeta_j)n_j(\zeta)\varphi_i(\zeta)ds_\zeta}{|x-\zeta|^2}, \end{aligned} \tag{6}$$

of densities $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, where $\boldsymbol{n}$ is the unit outward (with respect to Ω) normal to $\partial\Omega$. They are analytic solutions to $(3)_{1,2}$ in $\mathbb{R}^2\setminus\partial\Omega$. He observed that the singulary in the integral of $(6)_1$ is the same as the Newtonian double layer potential:

$$w[\varphi](x) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-\zeta_i)n_i(\zeta)\varphi(\zeta)ds_\zeta}{|x-\zeta|^2}. \tag{7}$$

Therefore, at least for regular Ω (say Liapounov), one can apply the Fredholm theory to the integral vector equation

$$\boldsymbol{a}(\xi) = -\frac{\boldsymbol{\varphi}}{2}+\frac{2}{\pi}\int_{\partial\Omega}\frac{(\xi-\zeta)[(\xi-\zeta)\cdot\boldsymbol{n}(\zeta)][(\xi-\zeta)\cdot\boldsymbol{\varphi}(\zeta)]ds_\zeta}{|x-\zeta|^4}, \tag{8}$$

as for (7) to the integral equation

$$a(\xi) = \frac{\varphi(\xi)}{2}+\frac{1}{2\pi}\int_{\partial\Omega}\frac{(\xi_i-\zeta_i)n_i(\zeta)\varphi(\zeta)ds_\zeta}{|x-\zeta|^2}. \tag{9}$$

The kernel of (8) is the space of rigid motions of $\mathbb{R}^2$, so that the homogeneous adjoint equation to (8) has three linearly independent solutions that are the densities $\{\boldsymbol{\psi}_i\}_{i=1,2,3}$,

[2]In his famous monograph [20] H. Lamb did not use this locution to denote the phenomenon found by Stokes.

of the single layer potentials giving the rigid motions of Ω'. Odqvist concluded that a solution to $(3)_{1,2,3}$ in the form of a potential of double layer, and so vanishing at infinity, exists if only if $\boldsymbol{a}$ is orthogonal to every $\boldsymbol{\psi}_i$. On the other hand, translational motions $\boldsymbol{v}[\boldsymbol{\psi}_i]$ grow logaritmically at infinity and this, according to Odqvist, mathematically explains the Stokes paradox ([26], p. 356–357)[3].

Clearly, Odqvist's conclusions hold in the class of solutions expressed by layer potentials. On the other hand, a standard argument shows that his conclusions can be extended to every smooth solution. Indeed, any regular solution ($\boldsymbol{u} = o(r), p = o(1)$) to $(3)_{1,2}$ behaves at infinity according to

$$\begin{aligned} u_i(x) &= u_{0i} + \mathfrak{U}_{ij}(x) \int_{\partial\Omega} s_i(\boldsymbol{u}, p) + \vartheta_i(x), \\ p(x) &= \varpi_i(x) \int_{\partial\Omega} s_i(\boldsymbol{u}, p) + \theta(x), \end{aligned} \tag{10}$$

where

$$\nabla_k \boldsymbol{\vartheta} = O(r^{-1-k}), \quad \nabla_k \theta = O(r^{-2-k}), \quad \nabla_k = \underbrace{\nabla \ldots \nabla}_{k-\text{times}},$$

and

$$s_i(\boldsymbol{u}, p) = -pn_i + \mu(\partial_i u_j + \partial_j u_i)n_i$$

it the traction of $\partial\Omega$. If (3) had a solution, then (10) should imply that

$$\int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{u}, p) = \boldsymbol{0}, \tag{11}$$

and by an integration by parts, one should have

$$\int_{\Omega} |\nabla \boldsymbol{u}|^2 = \int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot s(\boldsymbol{u}, p). \tag{12}$$

Hence it follows that if $\boldsymbol{a} = \boldsymbol{0}$, then (3) is solvable only for $\boldsymbol{u}_0 = \boldsymbol{0}$ (Stokes' paradox).

To better understand Stokes paradox from a mathematical point of view, by allowing also less regular domains and boundary data, one can express the solution to (3) by a simple layer potential as showed in [30].

Let us consider the simple layer potential (5) with density $\boldsymbol{\psi} \in W^{-1/2,2}(\partial\Omega)$ (say), where by abuse of notation the integral could means the value of the functional $\boldsymbol{\psi}$ at $\boldsymbol{\mathfrak{U}}$. With such a density, (5) is a variational solution to $(3)_{1,2}$ in $\mathbb{R}^2 \setminus \partial\Omega$ and it is straightforward to see that $\boldsymbol{v}[\boldsymbol{\psi}] \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$. Then

$$\mathcal{S} : \boldsymbol{\psi} \in W^{-1/2,2}(\partial\Omega) \to \mathcal{S}[\boldsymbol{\psi}] = \text{tr}_{|\partial\Omega} \boldsymbol{v}[\boldsymbol{\psi}] \in W^{1/2,2}(\partial\Omega) \tag{13}$$

is a well defined, self–adjoint, linear and continuous operator, $\boldsymbol{s}(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])^{\pm} \in W^{-1/2,2}(\partial\Omega)$ and $\boldsymbol{s}(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])^{+} - \boldsymbol{s}(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])^{-} = \boldsymbol{\psi}$. Set

$$\mathfrak{M} = \Big\{ \boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] = \text{constant} \Big\}, \qquad \mathfrak{M}_0 = \Big\{ \boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] = \boldsymbol{0} \Big\}. \tag{14}$$

[3]For $n = 3$ Odqvist's approach is explained in detail in Ch. 3 of [19].

Clearly,

$$\dim \mathfrak{M} = 2$$

and

$$\Big\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2\Big\} \text{ basis of } \mathfrak{M} \implies \Big\{\int_{\partial\Omega} \boldsymbol{\psi}_1, \int_{\partial\Omega} \boldsymbol{\psi}_2\Big\} \text{ basis of } \mathbb{R}^2. \tag{15}$$

By using these classical properties, and standard a priori estimates on variational solutions to $(3)_{1,2}$, one easily show that $\mathcal{S}$ is Fredholm with index zero and

$$\operatorname{Kern} \mathcal{S} = \operatorname{spn}\{\boldsymbol{n}\} \cap \mathfrak{M}_0. \tag{16}$$

With this information available, it is routine to get

Proposition 1. *Let Ω be Lipschitz. If $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, then there is $\boldsymbol{\psi} \in W^{-1/2,2}(\partial\Omega)$ such that the pair*

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{v}[\boldsymbol{\psi}](x) + \boldsymbol{\sigma}(x) + \boldsymbol{\kappa}, \\ p(x) &= P[\boldsymbol{\psi}](x), \end{aligned} \tag{17}$$

is a variational solution to $(3)_{1,2,3}$, where

$$\boldsymbol{\sigma}(x) = \frac{x_i}{2\pi|\boldsymbol{x}|^2} \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}$$

and $\boldsymbol{\kappa}$ is defined by

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{\kappa}) \cdot \boldsymbol{\psi}' = 0, \quad \forall \boldsymbol{\psi}' \in \mathfrak{M}_0.$$

It is unique in the class of variational solutions $\{(\boldsymbol{u}, p) : \boldsymbol{u} = o(r)\}$ modulo a pair in the two dimensional linear space $\{(\boldsymbol{v}[\boldsymbol{\psi}'] - \mathcal{S}[\boldsymbol{\psi}'], P[\boldsymbol{\psi}']),\ \boldsymbol{\psi}' \in \mathfrak{M}\}$.

Looking for a solution to $(3)_{1,2,3}$ which converges at infinity, from (15) and (17) it follows that

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{v}[\tilde{\boldsymbol{\psi}}](x) + \boldsymbol{\sigma}(x) + \boldsymbol{u}_0, \\ p(x) &= P[\tilde{\boldsymbol{\psi}}](x), \end{aligned} \tag{18}$$

with

$$\begin{aligned} \tilde{\boldsymbol{\psi}} &= \boldsymbol{\psi} + \boldsymbol{\psi}', \quad \boldsymbol{\psi}' \in \mathfrak{M} : \int_{\partial\Omega} \tilde{\boldsymbol{\psi}} = \boldsymbol{0}, \\ \boldsymbol{u}_0 &= \boldsymbol{\kappa} - \mathcal{S}[\boldsymbol{\psi}'], \end{aligned} \tag{19}$$

takes the value value $\boldsymbol{a}$ on the boundary and $\boldsymbol{v}[\tilde{\boldsymbol{\psi}}]$ tends to zero at infinity. Therefore, it holds

Theorem 1. *If $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, then $(3)_{1,2,3}$ has a solution which converges uniformly at infinity. Moreover, (3) has a solution if and only if*

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{\psi}' = 0, \quad \forall \boldsymbol{\psi}' \in \mathfrak{M}. \tag{20}$$

Remark 1 - In general, the natural behavior at infinity of a solution to an elliptic differential system is that exhibited by the fundamental solution and, in this sense, Proposition 1 does not depart from this "principle". System (3) becomes then ovedetermined and, as a consequence, its solvability requires suitable compatibility conditions, like (20). *The real paradox should be then, not the the lack of solutions to System* (3), *but its solvability!* $\diamond$

Remark 2 - By classical results of R.R. Coifman, A. McIntosh and Y. Meyer [3], the restriction of (13) to $L^2(\partial\Omega)$

$$\mathcal{S} : \boldsymbol{\psi} \in L^2(\partial\Omega) \to \mathcal{S}[\boldsymbol{\psi}] \in W^{1,2}(\partial\Omega), \tag{21}$$

is continuous. Therefore, using a procedure by J. Nečăs based on Rellich's inequalities [25] (see also [5], [14], [37]), one shows that (21) is Fredholm with index zero, as well as its adjoint [30]

$$\mathcal{S}' : \boldsymbol{\psi} \in W^{-1,2}(\partial\Omega) \to \mathcal{S}[\boldsymbol{\psi}] \in L^2(\partial\Omega). \tag{22}$$

Moreover, if Ω and $\boldsymbol{a}$ are more regular, then so does the solution (17). In particular, [4], [24], [35]

(*i*) by well—known stability results (see, *e.g.*, [13]), there is $\epsilon > 0$ depending on $\partial\Omega$, such that

$$\mathcal{S}' : \boldsymbol{\psi} \in W^{s,q}(\partial\Omega) \to \mathcal{S}[\boldsymbol{\psi}] \in W^{s+1,q}(\partial\Omega), \tag{23}$$

is Fredholm with index zero for all $s \in [-1, 0]$ and $q \in (2-\epsilon, 2+\epsilon)$:

(*ii*) there is $\mu_0 \in [0, 1])$ such that if $\boldsymbol{a} \in C^{0,\mu}(\partial\Omega)$, $\mu \in [0, \mu_0)$, then $\boldsymbol{u} \in C^{0,\mu}_{\mathrm{loc}}(\overline{\Omega})$;

(*iii*) If Ω is of class C^1, then in (*i*), (*ii*) we can take $q \in (1, +\infty)$ and $\mu_0 = 1$.

For more regular domains (of class $C^{k,\mu}$, say) we quote the classical monograph of C. Miranda [23]. $\diamond$

Remark 3 - The results quikly recalled above seem to be quite complete from a mathematical point of view. Nevertheless, there is a point which deserves attention. To be not merely formal, the compatibility condition (20) requires an analytic expression of the fields in $\mathfrak{M}$. In our opinion this is a very interesting problem in both pure and applied mathematics. As far as we know, it is has been solved only the ellipse. Indeed, if $f(\xi) = 1$ is its equation, then (see Sect. 4 of [22] for a simple proof), then

$$\mathfrak{M} = \mathrm{spn}\left\{\frac{\boldsymbol{e}_1}{|\nabla f|}, \frac{\boldsymbol{e}_2}{|\nabla f|}\right\}$$

Hence, if Ω is an ellipse, than (3) has a solution if and only if

$$\int_{\partial\Omega} \frac{\boldsymbol{a} - \boldsymbol{u}_0}{|\nabla f|} = \mathbf{0}.$$

In particular, if Ω is a disk and $\boldsymbol{u}_0 = \mathbf{0}$, then (3) has a solution if and only if

$$\int_{\partial\Omega} \boldsymbol{a} = \mathbf{0} \tag{24}$$

and we rediscover Stokes' paradox. ◇

Remark 4 - Theorem 1 and the results in remark 2 can be stated also for the more general exterior domain

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} \overline{\Omega}_i, \tag{25}$$

where Ω_i are N simply connected bounded domains such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing$, $i \neq j$. The only difference is that now the field $\boldsymbol{\sigma}$ from Proposition 1 writes

$$\boldsymbol{\sigma}(x) = \sum_{i=1}^{k} \frac{(x - \bar{x}_i)}{2\pi |x - \bar{x}_i|^2} \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n},$$

with $\bar{x}_i$ in Ω_i. ◇

3 The Oseen equations

At the beginning of the twentieth century the belief that Stokes' approximation $(3)_{1,2}$ was valid not far from the obstacle and that in order to determine the slow motion of a cylinder in a viscous fluid (and so to find a definite value of the resistance) the effects of inertia, expressed by the non-linear term $\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})$, had to be taken into account, although without renouncing the benefits of linearity of the system of differential equations (see [20] Sections 340–343) . To take then partially into account the inertial effects, in 1910 C.W. Oseen [27] proposed to linearize the Navier Stokes equations around the solution $(\boldsymbol{u}_0, c)$ and to replace system (3) by

$$\begin{aligned} \nu\Delta \boldsymbol{u} - \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} - \nabla p = \mathbf{0} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) = \mathbf{0}, & \end{aligned} \tag{26}$$

today known as *Oseen equations*. He was able to find a solution to (26) for the disk and to determine the resistance (see [20] Section 343). In 1927 [28] the same Oseen found the fundamental solution to $(26)_{1,2}$ $(\mathcal{E}_{ij}(x-y), \varpi_i(x-y))$. It enjoy the following properties (see [7], Ch. VII)

(*i*) $\mathcal{E}_{ij}(x-y) = \mathcal{U}_{ij}(x-y) + o(1)$ as $|\boldsymbol{u}_0||x-y| \to 0$;

(*ii*) $\nabla_k \mathcal{E}_{ij}(\boldsymbol{t}) = O(|\boldsymbol{t}|^{-(k+1)/2})$, $\forall k \in \mathbb{N}_0$;

(*i*) implies that the trace operator

$$\mathcal{S}_o : W^{-1/2,2}(\partial\Omega) \to W^{1/2,2}(\partial\Omega), \tag{27}$$

associated with the Oseen simple layer potential

$$
\begin{aligned}
v_{oi}[\boldsymbol{\psi}](x) &= \int\limits_{\partial\Omega} \mathcal{E}_{ij}(x-\xi)\psi_j(\xi)\mathrm{ds}_\xi, \\
P[\boldsymbol{\psi}](x) &= \int\limits_{\partial\Omega} \varpi_i(x-\xi)\cdot\psi_i(\xi)\mathrm{ds}_\xi,
\end{aligned} \tag{28}
$$

is Fredholm with index zero and [32]

$$
\mathrm{Kern}\,\mathcal{S}_o = \mathrm{Kern}\,\mathcal{S}'_o = \mathrm{spn}\{\boldsymbol{n}\}.
$$

Hence, taking also into account Remark 2, it follows

Proposition 2. *Let Ω be Lipschitz. There is $\epsilon > 0$ such that if $\boldsymbol{a} \in W^{s,q}(\partial\Omega)$, $s \in [0,1]$, $q \in (2-\epsilon, 2+\epsilon)$, then* (26) *has a solution expressed by*

$$
\begin{aligned}
\boldsymbol{u}_o(x) &= \boldsymbol{v}_o[\boldsymbol{\psi}](x) + \boldsymbol{\sigma}(x), \\
p_o(x) &= P[\boldsymbol{\psi}](x) - \boldsymbol{u}_0\cdot\boldsymbol{\sigma},
\end{aligned} \tag{29}
$$

for some $\boldsymbol{\psi} \in W^{-1,q}(\partial\Omega)$. If Ω is of class C^1, we can take $q \in (1,+\infty)$.

4 Navier Stokes equations

From here and henceforth let Ω be an exterior domain in $\mathbb{R}^2$, i.e.,

$$
\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} \overline{\Omega}_i, \tag{30}
$$

where Ω_i are N pairwise disjoint bounded domains with connected Lipschitz boundaries $\Gamma_i = \partial\Omega_i$.

The first existence theorem for equations (1) was established by J. Leray in his celebrated PhD thesis [21] (1933). He consider the sequence of boundary value problems

$$
\begin{aligned}
\nu\Delta\boldsymbol{u}_k - \mathrm{div}\,(\boldsymbol{u}_k\otimes\boldsymbol{u}_k) - \nabla p_k &= \boldsymbol{0} && \text{in } \Omega_k, \\
\mathrm{div}\,\boldsymbol{u}_k &= 0 && \text{in } \Omega_k, \\
\boldsymbol{u}_k &= \boldsymbol{a} && \text{on } \partial\Omega, \\
\boldsymbol{u}_k &= \boldsymbol{u}_0 && \text{on } \partial C_k,
\end{aligned} \tag{31}
$$

for Ω and $\boldsymbol{a}$ sufficiently regular, where $\Omega_k = \Omega \cap C_k$ with $C_k = \{x \in \mathbb{R}^2 : |x| < R_k\}$, $R_k < R_{k+1} \to +\infty$. Under the condition

$$
\int\limits_{\Gamma_i} \boldsymbol{a}\cdot\boldsymbol{n} = 0, \qquad i = 1,\ldots,N, \tag{32}
$$

using Odqvist 's results [26] and a fixed point argument, he proved that (31) has a regular solution $(\boldsymbol{u}_k, p_k)$ that satisfies the estimates

$$\int_{\Omega} |\nabla \boldsymbol{u}_k|^2 \leq c, \tag{33}$$

for some positive constant c independent of k. Thanks to (33), Leray was able to show that the sequence $(\boldsymbol{u}_k, p_k)$ converges to a regular solution $(\boldsymbol{u}, p)$ to (1), having finite Dirichlet integral

$$\int_{\Omega} |\nabla \boldsymbol{u}|^2 \leq c, \tag{34}$$

and known as *Leray solution*. Today, a solution to the Navier Stokes equations (1) satisfying (34) in a neighborhood of the infinity is called *D–solution*. The Leray argument can be easily repeated to prove existence of variational solutions (see [19], Ch. 5).

Let us observe that the only hypothesis required by Leray consists in excluding by (32) the presence of source or sink in the fluid. This assumption has been partially removed in [31] (2009), and using also the results stated in Remark 2, one can claim the following Theorem.

Theorem 2. *If $\partial\Omega$ is Lipschitz and $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfies*

$$\sum_{i=1}^{N} \left| \int_{\Gamma_i} \boldsymbol{a} \cdot \boldsymbol{n} \right| < 2\pi\nu, \tag{35}$$

then (1) *has a D–solution, which is analytic in* Ω.

Despite this great achievement, Leary left open a problem of undoubted interest. Since every $\boldsymbol{u}_k$ takes the value $\boldsymbol{u}_0$ on ∂C_k, one should expect that the limit $\boldsymbol{u}$ of $\boldsymbol{u}_k$ "remember" (at least in weak form) the value at infinity, as happens in the three dimensional case. But the only information available on the behavior at infinity of $\boldsymbol{u}$ is given by (34), and it is well–known that a function having finite Dirichlet integral in Ω can grows at infinity as $\log^{\alpha} r$ for $\alpha < 1/2$. Nevertheless, under suitable assumptions of symmetry (say) on data and solutions, on can say that the Leray solution vanishes in a weak sense at infinity. Assuming that Ω is polar symmetric with respect to o, *i.e.*, $x \in \Omega \Rightarrow -x \in \Omega$ and $\boldsymbol{a}(\xi) = -\boldsymbol{a}(-\xi)$, for every $\xi \in \partial\Omega$, one shows that the Leray procedure yields a polar symmetric solution $\boldsymbol{u}(x) = -\boldsymbol{u}(-x)$, for every $x \in \Omega$[4], so that for large R

$$\int_0^{2\pi} \boldsymbol{u}(R,\theta) d\theta = \boldsymbol{0}. \tag{36}$$

By the trace theorem, (36) and Poincaré's inequality

$$\int_0^{2\pi} |\boldsymbol{u}(R,\theta)|^2 d\theta \leq \frac{c}{R^2} \int_{C_{2R}\setminus C_R} |\boldsymbol{u}|^2 + \int_{C_{2R}\setminus C_R} |\nabla \boldsymbol{u}|^2 \leq c \int_{C_{2R}\setminus C_R} |\nabla \boldsymbol{u}|^2$$

[4] In this case we have to set $\boldsymbol{u}_0 = \boldsymbol{0}$ on ∂C_k.

Hence it follows

$$\lim_{R\to+\infty}\int_0^{2\pi}|\boldsymbol{u}(R,\theta)|^2 d\theta=\boldsymbol{0}. \tag{37}$$

In 1961 H. Fujita [12] and, independently Vorovich & Yudovich [38], using a method different from that of Leray, based on a Galerkin scheme[5], were able to prove existence of a D–solution to (1).

A first important, deep result (and, as far as we know, unique until today) in the existence problem for system (1)–(2) was given by R. Finn and D.R. Smith in 1967 [6]. By a technique based on a fixed point argument and on the existence of solutions to Oseen system (26), they proved that if Ω and $\boldsymbol{a}$ are sufficiently regular and

$$\|\boldsymbol{a}-\boldsymbol{u}_0\|_{C^2(\partial\Omega)}=o(\lambda\log\lambda^{-1})\quad\text{as }\lambda=\frac{|\boldsymbol{u}_0|}{\nu}\to 0,$$

then (1)–(2) has a regular D–solution.

It is clear that, at least for small data and $\boldsymbol{u}_0\neq\boldsymbol{0}$, Finn and Smith's theorem ruled out Stokes' paradox from the nonlinear theory of viscous fluid.

Finn and Smith results were rediscovered by G.P. Galdi [8] (see also [34]) by a different method, under less restrictive assumptions on Ω and $\boldsymbol{a}$ (see also [7], Ch. XII).

Due to the lack of a uniqueness theorem the three solutions we discussed above are not comparable. Therefore, results holding for every D–solution are of great interest.

5 Asymptotic behavior of D solutions

The problem of the asymptotic behavior at infinity of Leray's solution $(\boldsymbol{u}_k,p_k)$ was tacked by D. Gilbarg & H. Weinberger in 1974 [10]. They proved that $\boldsymbol{u}_k$ is bounded, there are a scalar p_0 and a constant vector $\boldsymbol{u}_\infty$ such that

$$\lim_{r\to+\infty}p_k(x)=p_0 \tag{38}$$

(one can choose, say, $p_0=0$),

$$\lim_{r\to+\infty}\int_0^{2\pi}|\boldsymbol{u}_k(r,\theta)-\boldsymbol{u}_\infty|^2 d\theta=0, \tag{39}$$

and

$$\begin{aligned}\omega(x)&=o(r^{-3/4}),\\ \nabla\boldsymbol{u}(x)&=o(r^{-3/4}\log r),\end{aligned} \tag{40}$$

where

$$\omega=\partial_2 u_1-\partial_1 u_2$$

[5]This technique is clearly described in [7], Ch IX.

is the vorticity. Two years later the same authors [11] showed any D–solution $(\boldsymbol{u}, p)$ satisfies (38) and

$$\begin{aligned}
&\boldsymbol{u}(z) = o(\log^{1/2} r),\\
&\omega(z) = o(r^{-3/4}\log^{1/8} r),\\
&\nabla\boldsymbol{u}(z) = o(r^{-3/4}\log^{9/8} r),\\
&\nabla\omega \in L^2(\Omega).
\end{aligned}$$

If $\boldsymbol{u}$ is bounded, then it satisfies the same properties as the Leray solution and if $\boldsymbol{u}_\infty = \boldsymbol{0}$, then

$$\boldsymbol{u}(z) \to 0 \qquad \text{uniformly as } \ |z| \to \infty. \tag{41}$$

Moreover, if $\boldsymbol{u}_\infty \neq 0$, then there exists a sequence of radii $\rho_n \in (2^n, 2^{n+1})$, $n \geq n_0$, such that

$$\sup_{\theta\in[0,2\pi]} |\boldsymbol{u}(\rho_n, \theta) - \boldsymbol{u}_\infty| \to 0 \qquad \text{as } \ n \to \infty. \tag{42}$$

Some years later C.J. Amick [1] proved that a D–solution to the problem of a flow around an obstacle ($\boldsymbol{a} = \boldsymbol{0}$) has the following asymptotic properties:

(i) $\boldsymbol{u}$ is bounded and, as a consequence, it satisfies (39), (40);

(ii) the total head pressure $\Phi = p + \frac{1}{2}|\boldsymbol{u}|^2$ and the absolute value of the velocity $|\boldsymbol{u}|$ have the uniform limit at infinity, i.e.,

$$|\boldsymbol{u}(r, \theta)| \to |\boldsymbol{u}_\infty| \qquad \text{as } \ r \to \infty, \tag{43}$$

where $\boldsymbol{u}_\infty$ is the constant vector from the condition (39);

(iii) if $\partial\Omega$ is symmetric with respect to the x_1–axis (say), and $(\boldsymbol{u}, p)$ is symmetri, *i.e.* u_1 is even and u_2 is odd with repect to x_2, then $\boldsymbol{u}$ converges uniformly at infinity to a constant vector $\mu\boldsymbol{e}_1$, for some scalar μ. Moreover, the Leary procedure yields to a nontrivial solution.

In [2] the same author proved that if $\mu \neq 0$, then the solution in (iii) behaves at infinity as that of the linear Oseen equation. These result was extended by L.I. Sazonov [33] to an arbitrary D–solution converging uniformly at large distance to a nonzero constant vector (see also [9] and Ch. XII of [7]).

6 Recent results and open problems

In recent papers [15], [16], [17], [18], [29] we were able to give positive answers to some problems outlined above for general exterior domains defined by (25). In particular, the following theorems has been established.

Theorem 3. [29] – *Let Ω be a Lipschitz exterior domain of $\mathbb{R}^2$ symmetric with respect to the coordinate axes and let $\boldsymbol{a} = (a_1, a_2) \in W^{1/2,2}(\partial\Omega)$. If*

$$\left.\begin{aligned}
a_1(\xi_1,\xi_2) = a_1(\xi_1,-\xi_2), \quad a_2(\xi_1,\xi_2) = -a_2(\xi_1,-\xi_2)\\
a_1(\xi_1,\xi_2) = -a_1(-\xi_1,\xi_2), \quad a_2(\xi_1,\xi_2) = a_2(-\xi_1,\xi_2)
\end{aligned}\right\}, \quad \forall(\xi_1,\xi_2) \in \partial\Omega,$$

then (1) *has a D–solution vanishing uniformly at infinity.*

Theorem 4. [15] – *Let Ω be a Lipschitz exterior domain of $\mathbb{R}^2$ symmetric with respect to the axis x_1 and let $\boldsymbol{a} = (a_1, a_2) \in W^{1/2,2}(\partial\Omega)$. If*

$$a_1(\xi_1, \xi_2) = a_1(\xi_1, -\xi_2), \quad a_2(\xi_1, \xi_2) = -a_2(\xi_1, -\xi_2), \quad \forall(\xi_1, \xi_2) \in \partial\Omega,$$

then (1) *has a D–solution.*

Theorem 5. [16] – *If $\Omega \subset \mathbb{R}^2$ is an exterior domain of class C^2 and $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ satisfies*

$$\int\limits_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \tag{44}$$

then (1) *has a D–solution.*

So in the last theorem we replace the assumption (32) (that flux through every boundary component is zero) by weaker assumption that total flux is zero. Unfortunately, this last assumption is also far from being a necessary condition for the solvability of the problem (cf. with Theorem 2) .

Theorem 6. [16] – *Let $(\boldsymbol{u}, p)$ be a D–solution to the Navier–Stokes equations*

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \nabla p = \boldsymbol{0} \quad &\text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad &\text{in } \Omega, \end{aligned} \tag{45}$$

in the exterior domain $\Omega \subset \mathbb{R}^2$. Then $\boldsymbol{u}$ is uniformly bounded in $\Omega_0 = \mathbb{R}^2 \setminus B$, i.e.,

$$\sup_{z \in \Omega_0} |\boldsymbol{u}(z)| < \infty,$$

where $B = B_{R_0}$ is an open disk with sufficiently large radius: $B \Supset \partial\Omega$.

Theorem 7. [17] – *Let $(\boldsymbol{u}, p)$ be a D–solution to the Navier Stokes equations (45) in the exterior domain $\Omega \subset \mathbb{R}^2$. Then $\boldsymbol{u}$ converges uniformly at infinity, i.e., there exists a vector $\boldsymbol{u}_\infty \in \mathbb{R}^2$ such that*

$$\boldsymbol{u}(z) \to \boldsymbol{u}_\infty \qquad \text{uniformly as } |z| \to \boldsymbol{u}_\infty.$$

Theorem 8. [18] – *Let Ω be an exterior domain in $\mathbb{R}^2$ with C^2-smooth compact boundary, and $\boldsymbol{0} \neq \boldsymbol{u}_0 \in \mathbb{R}^2$. Take a sequence $\boldsymbol{u}_k$ of solutions to system (31) with boundary data $\boldsymbol{a} = 0$, and take further arbitrary weakly convergent subsequence $\boldsymbol{u}_{k_n} \rightharpoonup \boldsymbol{u}$. Then the limiting solution $\boldsymbol{u}$ to* (1) *is nontrivial (i.e., $\boldsymbol{u}$ is not identically zero). In particular, the Leray solution is nontrivial.*

Theorem 9. [18] – *Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth compact boundary, and let $\boldsymbol{a} \in \mathbb{R}^2$ be a nonzero constant vector. Take a sequence $\boldsymbol{u}_k$ of solutions to the system*

$$\begin{cases} -\nu\Delta\boldsymbol{u}_k + (\boldsymbol{u}_k \cdot \nabla)\boldsymbol{u}_k + \nabla p_k = \boldsymbol{0} & \text{in } \Omega_k, \\ \operatorname{div} \boldsymbol{u}_k = 0 & \text{in } \Omega_k, \\ \boldsymbol{u}_k = \boldsymbol{a} & \text{on } \partial\Omega, \\ \boldsymbol{u}_k = \boldsymbol{0} & \text{for } |z| = R_k, \end{cases} \tag{46}$$

and take further an arbitrary weakly convergent subsequence $\boldsymbol{u}_{k_n} \rightharpoonup \boldsymbol{u}$. Then $\boldsymbol{u}$ is a nontrivial solution to the system (1), *i.e., $\boldsymbol{u} \neq \boldsymbol{a}$.*

Together with Finn and Smith theorem [6], recalled at the end of section 4, Theorem 8–9 show that Stokes' paradox is typical of the Stokes equations (3).

Note that in all the above theorem there is no restriction on the size of the data $\boldsymbol{a}$ and ν. The proof of many results here based on some real analysis tools and fine properties of functions, such as Coarea formula, etc.

Clearly, many important problems remain open. Let us point out what we deem most significant:

(i) the relation between the constant vectors $\boldsymbol{u}_0$ in $(31)_4$ and $\boldsymbol{u}_\infty$ in Theorem 7;

(ii) the existence of a D–solution without assumption (44);

(iii) the validity of Finn- and Smith theorem for every data ν, $\boldsymbol{a}$ and $\boldsymbol{u}_0$;

(iv) uniqueness of a D–solution;

(v) the rate of decay of a D–solution vanishing at infinity, of course depending on ν in view of Hamel counter–examples (see [7] p. 805 and [19] p. xi, xii).

References

[1] C.J. Amick: On Leray's problem of steady Navier-Stokes flow past a body in the plane, *Acta Math* **161** (1988), 71–130. 11

[2] C.J. Amick: On the Asymptotic Form of Navier-Stokes Flow Past a Body in the Plane,*J. Differential Equations*, **91** (1991), 149–167. 11

[3] R.R. Coifman, A. McIntosh and Y. Meyer: L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les curves Lipschitziennes, *Ann. Math.* **116** (1982), 361–388. 6

[4] E.B. Fabes, M. Jodeit Jr. and N.M. Rivière: Potential techniques for boundary value problems on C^1 domains, *Acta Math.* **141** (1978), 165–185. 6

[5] E.B. Fabes, C.E. Kenig and G.C. Verchota: The Dirichlet problem for the Stokes system on Lipschitz domains, *Duke Math. J.* **57** (1988), 769–793. 6

[6] R. Finn and D.R. Smith, On the Stationary Solution of the Navier- Stokes Equations in Two Dimensions, *Arch. Rational Mech. Anal.*, **25** (1967), 26–39 10, 13

[7] G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady–state problems*, Springer (2011). 1, 7, 10, 11, 13

[8] G.P. Galdi: Existence and uniqueness at low Reynolds number of stationary plane flow of a vscous fluid in exterior domains, *Recent Developments in Theoretical Fluid Mechanics* , Galdi, G.P., and Nečas, J., Eds., Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, **291** (1993), 1–33. 10

[9] G.P. Galdi and H. Sohr: On the Asymptotic Structure of Plane Steady Flow of a Viscous Fluid in Exterior Domains, *Arch. Rational Mech. Anal.* **131** (1995), 101–119. 11

[10] D. Gilbarg and H.F. Weinberger: Asymptotic properties of Leray's solution of the stationary two–dimensional Navier–Stokes equations, *Russian Math. Surveys* **29** (1974), 109–123. 10

[11] D. Gilbarg and H.F. Weinberger: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral, *Ann. Scuola Norm. Pisa (4)* **5** (1978), 381–404. 11

[12] H. Fujita: On the existence and regularity of the steady-state solutions of the Navier-Stokes squation, *J. Fac. Sci. Univ. Tokyo*, (1A) **9** (1961), 59–102. 10

[13] N. Kalton and M. Mitrea: Stability results on interpolation scales of quasi-Banach spaces and applications, *Trans. Amer. Math. Soc* **350** (1988), 3903–3922. 6

[14] C. Kenig: *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, Regional Conference Series in Mathematics **83**, AMS (1994). 6

[15] M.V. Korobkov, K. Pileckas and R. Russo: The existence of a solution with finite Dirichlet integral for the steady Navier–Stokes equations in a plane exterior symmetric domain, *J. Math. Pures. Appl.* **101** (2014), 257–274. 11, 12

[16] M.V. Korobkov, K. Pileckas and R. Russo: On the steady Navier–Stokes equations in $2D$ exterior domains, *J. Differential Equations*, **269** (2020), 1796–1828. 11, 12

[17] M.V. Korobkov, K. Pileckas and R. Russo: On Convergence of Arbitrary D-Solution of Steady Navier–Stokes System in 2D Exterior Domains, *Arch. Rational Mech. Anal.*, **233** (2019), 385–407 11, 12

[18] M.V. Korobkov, K. Pileckas and R. Russo: Leray's plane steady state solutions are nontrivial, ArXiv: 1911.11546v1 [math.AP] November 26, 2019. 11, 12

[19] O.A. Ladyzhenskaia: *The Mathematical theory of viscous incompressible fluid*, Gordon and Breach (1969). 4, 9, 13

[20] H. Lamb: *Hydrodynamics*, Sixth Edition, Cambridge University Press (1932). 3, 7

[21] J. Leray: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82. 8

[22] P. Maremonti, R. Russo and G. Starita: On the Stokes equations: the boundary value problem. *Advances in fluid dynamics*, P. Maremonti ed., *Quad. Mat.* **4** (1999), 69–140. 6

[23] C. Miranda: *Partial Differential Equations of Elliptic Type*, Springer (1970). 6

[24] M. Mitrea and M. Taylor: Navier–Stokes equations on Lipschitz domains in Riemannian manifolds, *Math. Ann.* **321** (2001), 955–987. 6

[25] J. Nečas: *Les méthodes directes en théorie des équations élliptiques*, Masson-Paris and Academie-Prague (1967); English translation: *Direct Methods in the Theory of elliptic equations*, Springer (2012). 6

[26] F.K.G. Odqvist: Über die randwertaufgaben der hydrodynamik zäher flflüssigkeiten, *Math. Z.* **32** (1930). 329–375. 3, 4, 9

[27] C.W. Oseen: Über die Stokessche formel und über eine Verwandte aufgabe in derhydrodynamik, *Ark. Mat. Astron. Fys.*, **6** (1910) ,1–20 7

[28] C.W. Oseen: Neueremethoden und ergebnisse in der hydrodynamik, Leipzig, *Akad. Verlagsgesellschaft M.B.H.* (1927). 7

[29] K. Pileckas and R. Russo: On the existence of vanishing at infinity symmetric solutions to the plane stationary exterior Navier–Stokes problem, *Math. Ann.* **352** (2012), 643, 658. 11

[30] R. Russo: On Stokes' problem, in *Advances in Mathematica Fluid Mechanics*, Eds. R. Rannacher and A. Sequeira, p. 473–511, Springer–Verlag Berlin Heidelberg (2010). 4, 6

[31] A. Russo: A Note on the Exterior Two-Dimensional Steady-State Navier–Stokes Problem, *J. Math. Fluid Mech.* **11** (2009), 407–414. 9

[32] R. Russo and C.G. Simader: A Note on the Existence of Solutions to the Oseen System in Lipschitz Domains, *J. Math. Fluid Mech.* **8** (2006), 64–76. 8

[33] L.I. Sazonov: On the asymptotic behavior of the solution of the two-dimensional stationary problem of the flow past a body far from it, (Russian) *Mat. Zametki* **65** (1999), 246–253; translation in *Math. Notes* **65** (1999), 202–207. 11

[34] L.I Sazonov: Justification of the asymptotic expansion of the solution of a two-dimensional flow problem for small Reynolds numbers. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **67** (2003), 125–154; translation in *Izv. Math.* **67** (2003), 975–1006. 10

[35] Z. Shen: A note on the Dirichlet problem for the Stokes system in Lipschitz domains, *Proc. Amer. Math. Soc.* **123** (1995), 801–811. 6

[36] G.G. Stokes: On the Effect of the Internal Friction of Fluids on the Motion of Pendulums, *Trans. Cambridge Phil. Soc.* **9** (1851), 8–106.

[37] G. Verchota: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains, *J. Funct. Anal.* 2

[38] I.I.Vorovich and V.I. Youdovich: Stationary Flow of a Viscous Incompressible Fluid, *Mat. Sb.*, **53** (1961), 393-428 (in Russian). 6