Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law

Mitsuo Higaki
Department of Mathematics,
Kobe University

1 Introduction

This note is a summary of the preprint [19]. The paper [19] is concerned with the local regularity of viscous incompressible fluid flows above rough bumpy boundaries $x_3 > \varepsilon\gamma(x'/\varepsilon)$ with γ Lipschitz and the no-slip boundary condition. Although bumpy boundaries have a complicated geometry and low regularity, the flow may paradoxically be better behaved than for smooth or flat boundaries. It is well documented in the physical [22, 33] and the mathematical [29, 12, 18] literature that roughness favors slip of the fluid on the boundary in certain regimes. In the striking paper [8] it is even showed experimentally that roughness may delay the transition to turbulence. This also supports the idea that the vanishing viscosity limit from Navier-Stokes to Euler may be less singular above highly oscillating boundaries than above flat ones [20, 11, 31].

Our goal is to investigate these effects, such as the enhanced slip, or the delay of the transition to turbulence, from the point of view of the regularity theory. Due in particular to vorticity creation at the boundary, the boundary regularity of fluid flows with the no-slip boundary conditions is delicate. In the nonstationary case, it is for instance not known whether there is an analogue of Constantin and Fefferman's [7] celebrated geometric regularity criteria for supercritical blow-up scenarios. For perfect slip or Navier-slip boundary conditions on the contrary, the situation is brighter. In particular an extension of the criteria of [7] is known in this case; see the work [6] by Beirão da Veiga and Berselli. We expect that fluids over bumpy boundaries have an intermediate behavior between these two extreme no-slip and (full-)slip situations, especially as far as the mesoscopic regularity properties are concerned.

Our approach grounds on the use of asymptotic analysis to prove regularity estimates. The success of such methods to prove the regularity to certain Partial Differential Equations is spectacular. One of the striking examples is that of homogenization. The basic idea is that the large-scale regularity is determined by the macroscopic properties of the systems, i.e. in the homogenization limit, while the small-scale regularity is determined by the regularity of the data (coefficients, boundary). Two approaches were developed: (a) blow-up and compactness arguments in periodic homogenization in the wake of the pioneering works [4, 5], (b) quantitative arguments based on suboptimal local error estimates as developed for periodic homogenization [34, 9, 32], almost periodic homogenization [3], and stochastic homogenization [15, 1].

In this work, we focus on the regularity for stationary problems. We consider the three-dimensional stationary Navier-Stokes equations

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon & \text{in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1(0)\,. \end{cases} \tag{NS$^\varepsilon$}$$

The functions $u^\varepsilon = u^\varepsilon(x)$, $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3)^\top \in \mathbb{R}^3$, and $p^\varepsilon = p^\varepsilon(x) \in \mathbb{R}$ denote respectively the velocity field and the pressure field of the fluid. We have set for $\varepsilon \in (0,1]$ and $r \in (0,1]$,

$$
\begin{aligned}
B^{\varepsilon}_{r,+}(0) &= \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,, \quad \varepsilon\gamma(\frac{x'}{\varepsilon}) < x_3 < \varepsilon\gamma(\frac{x'}{\varepsilon}) + r\}\,, \\
\Gamma^{\varepsilon}_{r}(0) &= \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,, \quad x_3 = \varepsilon\gamma(\frac{x'}{\varepsilon})\}\,.
\end{aligned}
\tag{1}
$$

The boundary function $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is assumed to satisfy $\gamma(x') \in (-1,0)$ for all $x' \in \mathbb{R}^2$.

Our use of compactness arguments to tackle the regularity for solutions of (NS$^\varepsilon$) is reminiscent of the pioneering work of Avellaneda and Lin [4, 5] in homogenization, and of the works by Gérard-Varet [10], Gu and Shen [16], and Kenig and Prange [23, 24]. We separate the small-scale regularity, i.e. at scales $\lesssim \varepsilon$, from the mescopic- or large-scale regularity, i.e. at scales $\varepsilon \lesssim r \leq 1$. Concerning the small scales, the classical Schauder regularity theory for the Stokes and the Navier-Stokes equations was started by Ladyženskaja [27] using potential theory and by Giaquinta and Modica [14] using Campanato spaces. These classical estimates require some smoothness of the boundary and typically depend on the modulus of continuity of $\nabla\gamma$ when the boundary is given by $x_3 = \gamma(x')$. Therefore, these estimates degenerate for highly oscillating boundaries $x_3 = \varepsilon\gamma(x'/\varepsilon)$ with sufficiently small $\varepsilon \in (0,1)$. As for the large scales, on the contrary, the regularity is inherited from the limit system when $\varepsilon \to 0$ posed in a domain with a flat boundary. Here no regularity is needed for the original boundary, beyond the boundedness of γ and of its gradient. The mechanism for the regularity at small scales and at large scales is hence completely different. Moreover, it is possible to prove, at the large scales, improved estimates that are known to be false at the small scales. An example of this is our large-scale Lipschitz estimate of Theorem 1.1 below that is known to be false over a Lipschitz graph at the small scales even in the case of a linear elliptic operator [25, 26, 34].

Beyond improved regularity estimates, our objective is to develop local error estimates for the homogenization of viscous incompressible fluids over bumpy boundaries and derive local wall laws. The wall law catches an averaged effect from the $O(\varepsilon)$-scale on large scale flows of order $O(1)$ through homogenization. In the wall law, a rough boundary is modeled as a smooth one and an appropriate condition is imposed on it reflecting the roughness of the original boundary. In typical situations, this process gives a Navier-type condition with slip length of $O(\varepsilon)$, the so-called Navier wall law. This effective boundary condition reads for instance in two dimensions

$$
u_1 = \varepsilon\alpha\partial_2 u_1\,, \quad u_2 = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+ \tag{2}
$$

with a constant α depending only on the boundary function γ. We now briefly review the literature concerned with the derivation of wall laws such as (2) and the proof of error estimates in the global setting. The literature is vast and it is impossible to be exhaustive here. The wall law for simple stationary shear flows is analyzed in the pioneering work Jäger and Mikelić [21] when the boundary is periodic. This result is extended to a random setting by Gérard-Varet [10] and to the almost periodic setting by Gérard-Varet and Masmoudi [12]. Nonstationary cases are studied in Mikelić, Nečasová, and Neuss-Radu [30] under the assumption that the limit flows are space-time C^2 functions. The strong regularity condition in [30] implies that a careful analysis is needed when we study Initial Boundary Value Problems (IBVPs). Indeed, for these cases, no matter how regular the initial data are, there is the loss of regularity of solutions due to the boundary compatibility condition. Higaki [18] considers an IBVP in a bumpy half-space and verifies the Navier wall law for C^1 initial data under natural compatibility conditions. A key ingredient is to make use of the L^∞-regularity theory of the Navier-Stokes equations in the half-spaces. Theorem 1.3 below provides a local counterpart of these global error estimates in the case of the stationary Navier-Stokes equations.

Outline and novelty of our results. Our main results are given in the two theorems below. In Theorem 1.1 we state a uniform Lipschitz estimate. In Theorem 1.3 we give a local error estimate and identify the building blocks of the regularity theory. Both results hold for weak solutions of the nonlinear equations (NS^ε) and hold without any smallness assumption on the size of the solutions.

Theorem 1.1 (mesoscopic Lipschitz estimate)**.** *For all $M \in (0,\infty)$, there exists a constant $\varepsilon^{(1)} \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and M such that the following statement holds. For all $\varepsilon \in (0,\varepsilon^{(1)}]$ and $r \in [\varepsilon/\varepsilon^{(1)}, 1]$, any weak solution $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ to (NS^ε) with*

$$\left(\fint_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2\right)^{\frac{1}{2}} \leq M \tag{3}$$

satisfies

$$\left(\fint_{B^\varepsilon_{r,+}(0)} |u^\varepsilon|^2\right)^{\frac{1}{2}} \leq C^{(1)}_M r\,, \tag{4}$$

where the constant $C^{(1)}_M$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and M. Moreover, $C^{(1)}_M$ is a monotone increasing function of M and converges to zero as M goes to zero.

Remark 1.2. (i) By using the Caccioppoli inequality in Appendix, one can easily prove

$$\left(\fint_{B^\varepsilon_{r,+}(0)} |\nabla u^\varepsilon|^2\right)^{\frac{1}{2}} \leq \widetilde{C^{(1)}_M}$$

for $r \in [\varepsilon/\varepsilon^{(1)}, \frac{1}{2}]$. Here the constant $\widetilde{C^{(1)}_M}$ satisfies the same property as $C^{(1)}_M$.

(ii) In the paper [10], Gérard-Varet obtains a uniform Hölder estimate for weak solutions of the Stokes equations when $\gamma \in C^{1,\omega}(\mathbb{R}^2)$ for a fixed modulus of continuity ω. Let us emphasize that there is a gap in difficulty between the uniform Hölder estimate (right-hand side of (4) replaced by Cr^μ with $\mu \in (0,1)$) and the uniform Lipschitz estimate (4). Indeed the Lipschitz estimate requires the analysis of the boundary layer corrector. Moreover, let us emphasize that the Lipschitz estimate is the best that can be proved for u^ε uniformly in ε. This comment does not contradict the uniform $C^{1,\mu}$ estimate below. Indeed the estimate in Theorem 1.3 is a measure of the oscillation between u^ε and affine functions, and is not an estimate for u^ε directly.

(iii) As in the works [4, 10, 23] one can combine the mesoscopic regularity estimate with the classical regularity, provided the boundary is regular enough, i.e. when $\nabla\gamma$ is Hölder continuous. In that case, we can prove the full Lipschitz estimate $\|\nabla u^\varepsilon\|_{L^\infty(B^\varepsilon_{1,+}(0))}$ for (NS^ε). However, one cannot expect such an estimate to hold in Lipschitz domains even for the Laplace equation with the Dirichlet boundary condition.

(iv) There is a version of Theorem 1.1 for the linear Stokes equations; see Theorem 3.1 in Section 3 below. An important application of such uniform Lipschitz estimates is for estimating the Green and Poisson kernels associated to the Stokes equations in the Lipschitz half-space $\{y_3 > \gamma(y')\}$. Following [4], such estimates were proved for elliptic systems in bumpy domains in [23], or the Stokes equations with periodic coefficients [17]. Such estimates play a crucial role for the homogenization of boundary layer correctors, in particular in the works [13, 2, 35]

Next let us state the result which gives a local justification of the Navier wall law. The following theorem is concerned with the polynomial approximation of weak solutions to (NS^ε) at mesoscopic scales. Remark 1.4 below states consequences of the theorem and Remark 1.5 establishes the connection between our theorem and the Navier wall law.

Theorem 1.3 (polynomial approximation)**.** *Fix $M \in (0,\infty)$ and $\mu \in (0,1)$. Then there exists a constant $\varepsilon^{(2)} \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ such that for all weak solutions $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ to* (NS$^\varepsilon$) *satisfying the bound* (3)*, the following statements hold.*

(i) *For all $\varepsilon \in (0,\varepsilon^{(2)}]$ and $r \in [\varepsilon/\varepsilon^{(2)},1]$, we have*

$$\left(\fint_{B^\varepsilon_{r,+}(0)} \Big|u^\varepsilon(x) - \sum_{j=1}^{2} c^\varepsilon_{r,j} x_3 \mathbf{e}_j\Big|^2 \,\mathrm{d}x\right)^{\frac12} \le C^{(2)}_M (r^{1+\mu} + \varepsilon^{\frac12} r^{\frac12})\,, \tag{5}$$

where the coefficient $c^\varepsilon_{r,j}$, $j \in \{1,2\}$, is a functional of u^ε depending on ε, r, $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ, while the constant $C^{(2)}_M$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ.

(ii) *We assume in addition that $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is 2π-periodic in each variable. Then there exists a constant vector field $\alpha^{(j)} = (\alpha^{(j)}_1, \alpha^{(j)}_2, 0)^\top \in \mathbb{R}^3$, $j \in \{1,2\}$, depending only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ such that for all $\varepsilon \in (0,\varepsilon^{(2)}]$ and $r \in [\varepsilon/\varepsilon^{(2)},1]$, we have*

$$\left(\fint_{B^\varepsilon_{r,+}(0)} \Big|u^\varepsilon(x) - \sum_{j=1}^{2} c^\varepsilon_{r,j} (x_3 \mathbf{e}_j + \varepsilon\alpha^{(j)})\Big|^2 \,\mathrm{d}x\right)^{\frac12} \le \widetilde{C^{(2)}_M} (r^{1+\mu} + \varepsilon^{\frac32} r^{-\frac12})\,, \tag{6}$$

where the coefficient $c^\varepsilon_{r,j}$, $j \in \{1,2\}$, is same as in the estimate (5)*, while the constant $\widetilde{C^{(2)}_M}$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ.*

Remark 1.4. (i) Each of the constants $C^{(2)}_M$ and $\widetilde{C^{(2)}_M}$ satisfies the same property as $C^{(1)}_M$ in Theorem 1.1 as functions of M.

(ii) Note that at the small scale, namely when $r = O(\varepsilon)$, the right-hand side in the estimate (5) is no better than the right-hand side of (4) in Theorem 1.1. Hence there is no improvement at this scale. On the other hand, if we consider the case $r \in [(\varepsilon/\varepsilon^{(2)})^\delta, 1]$ with $\delta \in (0,1)$, then we see that

$$r^{1+\mu} + \varepsilon^{\frac12} r^{\frac12} \le (1 + (\varepsilon^{(2)})^{\frac12} r^{\frac{1-\delta}{2\delta} - \mu}) r^{1+\mu}\,.$$

Therefore, we call the estimate (5) a mesoscopic $C^{1,\mu}$ estimate at the scales $r \in [(\varepsilon/\varepsilon^{(2)})^\delta, 1]$ with $\delta \in (0, (2\mu+1)^{-1}]$.

(iii) A comparison between the estimates (5) and (6) highlights the regularity improvement coming from the boundary periodicity. Indeed, the estimate (6) is sharper than (5) at mesoscopic scales because $\varepsilon^{\frac32} r^{-\frac12} \le \varepsilon^{\frac12} r^{\frac12}$ holds whenever $r \in [\varepsilon, 1]$.

Remark 1.5. Let us denote the polynomial in (6) by $P^\varepsilon_{N,j}$, $j \in \{1,2\}$:

$$P^\varepsilon_{N,j}(x) = x_3 \mathbf{e}_j + \varepsilon \alpha^{(j)}\,. \tag{7}$$

Then each $P^\varepsilon_{N,j}$ is a shear flow in the half-space $\mathbb{R}^3_+$ and is an explicit solution to the Navier-Stokes equations with a Navier-slip boundary condition

$$\begin{cases} -\Delta u^\varepsilon_N + \nabla p^\varepsilon_N = -u^\varepsilon_N \cdot \nabla u^\varepsilon_N & \text{in } \mathbb{R}^3_+ \\ \nabla \cdot u^\varepsilon_N = 0 & \text{in } \mathbb{R}^3_+ \\ u^\varepsilon_{N,3} = 0 & \text{on } \partial\mathbb{R}^3_+ \\ (u^\varepsilon_{N,1}, u^\varepsilon_{N,2})^\top = \varepsilon \overline{M} (\partial_3 u^\varepsilon_{N,1}, \partial_3 u^\varepsilon_{N,2})^\top & \text{on } \partial\mathbb{R}^3_+ \end{cases} \tag{NS$^\varepsilon_N$}$$

with a trivial pressure $p^\varepsilon_N = 0$. Here the 2×2 matrix $\overline{M} = (\alpha^{(j)}_i)_{1\le i,j\le 2}$ can be proved to be positive definite; see Proposition 2.2 (ii). Thus the estimate (6) in Theorem 1.3 reads as follows: any weak solution u^ε to (NS^ε) can be approximated at any mesoscopic scale by a linear combination of the Navier polynomials $P^\varepsilon_{N,1}$ and $P^\varepsilon_{N,2}$ multiplied by constants depending on u^ε. This is a local version of the Navier wall law at the $O(\varepsilon^\delta)$-scales, which has been widely studied in the global framework.

The novelty of our results can be summarized as follows:
(I) Singular boundary: it is just Lipschitz and has no structure (except in Theorem 1.3 (ii)).
(II) No smallness assumption on the size of solutions.
(III) Derivation of a local wall law and local error estimates.

As is stated in (I), one of the originalities of Theorem 1.1 is that it does not rely on the smoothness of the boundary such as, the Hölder continuity of $\nabla\gamma$. Moreover, one cannot use any Fourier methods due to the lack of structure of the boundary. In fact, when working with Lipschitz boundaries, the classical Schauder theory is not applicable directly since there is no improvement of flatness coming from zooming on the boundary as is explained in [24]. The smoothing happens at scales larger than that of the boundary layer thickness.

Concerning point (II), we are able to remove any smallness assumption on the size of the solutions in Theorem 1.1 and Theorem 1.3. This is in stark contrast with previous works concerned with the regularity of elliptic or Stokes systems [4, 10, 16, 23, 24]. Moreover, as far as we know the error estimates in the stationary global setting are all in the perturbative regime; see for instance [12].

Point (III) is concerned with Theorem 1.3. It is important physically as well as mathematically since we are interested in the effects of rough boundaries on viscous fluids. Our result is a first-step toward understanding roughness effects on the Navier-Stokes flows in view of regularity improvement. As far as we know, estimate (6) is the first justification of a local wall law.

These three aspects are further discussed in connection with our strategy in the paragraph below.

Difficulties and strategy. The proof of Theorem 1.1 and Theorem 1.3 is based on a compactness argument as in [23, 24] originating from the works [4, 5] on uniform estimates in homogenization. In principle, we follow the strategy of [24] concerned with the regularity theory of elliptic systems in bumpy domains. The main points in [24] are: (1) construction of a boundary layer corrector in the Lipschitz half-space, (2) proof of the mesoscopic regularity by compactness and iteration. This strategy entails difficulties related to the lack of structure of the boundary which implies a lack of compactness of the solution to the boundary layer problem, and to the unavailability of Fourier methods up to the boundary. In addition to these difficulties, our proof is more involved due to: (i) the vectoriality of the equations (NS^ε) and the divergence-free condition, (ii) the nonlocal pressure, (iii) the nonlinearity of the Navier-Stokes equations and the lack of smallness of the solutions.

Concerning the first point, the (vectorial) divergence-free condition $\nabla\cdot u^\varepsilon = 0$ causes a difficulty in the compactness argument even for the Stokes equations; see Section 3, especially Lemma 3.2 and its proof. A key idea is that no boundary layer is needed on the vertical component of the velocity. Therefore the boundary layer corrector is naturally constructed as a divergence-free function.

Concerning the second point, let us stress a key difference between the stationary Navier-Stokes equations and the nonstationary ones. For the stationary Stokes equations imposed in a ball $B_1(0)$, one can estimate the pressure directly in terms of the velocity as

$$\|p - (\overline{p})_{B_1(0)}\|_{L^2(B_1(0))} \leq C\|\nabla p\|_{H^{-1}(B_1(0))} \leq C\|\nabla u\|_{L^2(B_1(0))}. \tag{8}$$

Similar estimates in balls intersecting the boundary and for the Navier-Stokes equations are intensively used in our note. This is in strong contrast with the nonstationary Navier-Stokes equations where the pressure interacts with the time derivative of the velocity.

The third aspect is partly related to (ii). In typical statements of the partial regularity theory for the nonstationary Navier-Stokes equations, one assumes smallness of certain scale-critical quantities in ε and hence one obtains linear equations in the limit $\varepsilon \to 0$. Then the regularity theory for the linear equations yields a space-time Hölder regularity improvement for the original solution; see Lin [28] for example. However, for the stationary Navier-Stokes equations discussed in our note, we do not need such a smallness condition; see Theorem 1.1. The limit equations when $\varepsilon \to 0$ are not linear, but we can prove the smoothness of weak solutions because H^1 bounds are enough to control both the nonlinear term and pressure term in L^2 space (see Appendix for details). Then bootstrapping using the standard elliptic regularity in a smooth domain leads to the (spatial) C^∞-regularity for the limit equations. Estimate (8) is the reason why one can bootstrap the regularity. Once the regularity is inherited at a fixed scale $\theta \in (0,1)$, a serious difficulty arises in the iteration of such an estimate. At each step in the induction, we need to use the Caccioppoli inequality from Appendix to control the norm $\|u^\varepsilon\|_{L^2}$. A naive approach yields an estimate that depends algebraically on the size M of u^ε as in (3). Hence the naive estimate becomes unbounded in M as the iteration proceeds. This prevents one from closing the induction due to the lack of uniformity. We overcome this difficulty by choosing the free parameter θ in the compactness lemma in terms of the data γ and M. This is done in the spirit of the Newton shooting method. We will make this idea precise in Section 4. It should finally be emphasized that the boundary layer corrector, entering the scheme for the nonlinear Navier-Stokes equations (NS^ε), solves the linear Stokes equations. This is expected from the following formal heuristics. Indeed, in the boundary layer $u^\varepsilon \simeq \varepsilon v(x/\varepsilon)$, so that v solves $-\frac{1}{\varepsilon}\Delta v + \varepsilon v \cdot \nabla v + \nabla q = 0$, $\nabla \cdot v = 0$.

Outline of the note. In Section 2 we summarize the results concerning the boundary layer equations. In Section 3 we prove the linear version of Theorem 1.1 in order to show how the compactness method works in the regularity argument. In Section 4 we prove the main results namely Theorem 1.1 and Theorem 1.3. The regularity theory in a domain with a flat boundary and the Caccioppoli inequality are stated in Appendix.

Notations. Let us summarize the notations in this note for easy reference. For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, we denote by x' its tangential part $(x_1, x_2)^\top$. For $r \in (0,1]$ and $\varepsilon \in (0,1]$, we define $B^\varepsilon_{r,+}(0)$ and $\Gamma^\varepsilon_r(0)$ as is done in (1) and set

$$\begin{aligned}
B_r(0) &= \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,, \quad x_3 \in (-r,r)\} = (-r,r)^3\,,\\
B_{r,+}(0) &= \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,, \quad x_3 \in (0,r)\}\,,\\
\Gamma_r(0) &= \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,, \quad x_3 = 0\}\,.
\end{aligned}$$

Note that formally we have $B_{r,+}(0) = B^0_{r,+}(0)$ and $\Gamma_r(0) = \Gamma^0_r(0)$. For an open set $\Omega \subset \mathbb{R}^3$ and a Lebesgue measurable function f on Ω, we set

$$⨍_\Omega |f| = \frac{1}{|\Omega|}\int_\Omega |f|\,, \qquad (\overline{f})_\Omega = \frac{1}{|\Omega|}\int_\Omega f\,, \tag{9}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Note that, since our interest is in the local boundary regularity of (NS^ε), the boundary condition is prescribed only on the lower part of $\partial B^\varepsilon_{r,+}(0)$. We work in the framework of weak solutions of (NS^ε). A vector function $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ is said to be a weak solution to (NS^ε) if u^ε satisfies $\nabla \cdot u^\varepsilon = 0$ in the sense of distributions, $u^\varepsilon|_{\Gamma^\varepsilon_1(0)} = 0$ in the trace sense, and

$$\int_{B^\varepsilon_{1,+}(0)} \nabla u^\varepsilon \cdot \nabla \varphi = -\int_{B^\varepsilon_{1,+}(0)} (u^\varepsilon \cdot \nabla u^\varepsilon) \cdot \varphi \tag{10}$$

for any $\varphi \in C^\infty_{0,\sigma}(B^\varepsilon_{1,+}(0))$. Here $C^\infty_{0,\sigma}(\Omega)$ denotes the space of test functions $\{f \in C^\infty_0(\Omega)^3 \mid \nabla \cdot f = 0\}$ when Ω is an open set in $\mathbb{R}^3$. For the pressure p^ε, we emphasize that the unique existence in $L^2(B^\varepsilon_{1,+}(0))$ up to an additive constant can be proved in a functional analytic way using the weak formulation (10).

2 The boundary layer corrector

In this section we summarize the results concerning the boundary layer problem in Lipschitz half-spaces without proof.

General case. The boundary layer equations for $j \in \{1, 2\}$ are written as

$$\begin{cases} -\Delta v + \nabla q = 0\,, & y \in \Omega^{\mathrm{bl}} \\ \nabla \cdot v = 0\,, & y \in \Omega^{\mathrm{bl}} \\ v(y', \gamma(y')) = -\gamma(y')\mathbf{e}_j\,, & \end{cases} \tag{$\mathrm{BL}^{(j)}$}$$

where $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ and Ω^{bl} denotes the Lipschitz half-space $\Omega^{\mathrm{bl}} = \{y \in \mathbb{R}^3 \mid \gamma(y') < y_3 < \infty\}$. The unique existence of weak solutions to ($\mathrm{BL}^{(j)}$) is stated as follows.

Proposition 2.1. *Fix $j \in \{1, 2\}$ and let $\gamma \in W^{1,\infty}(\mathbb{R}^2)$. Then there exists a unique weak solution $(v, q) = (v^{(j)}, q^{(j)}) \in H^1_{loc}(\overline{\Omega^{\mathrm{bl}}})^3 \times L^2_{loc}(\overline{\Omega^{\mathrm{bl}}})$ to ($\mathrm{BL}^{(j)}$) satisfying*

$$\sup_{\eta \in \mathbb{Z}^2} \int_{\eta + (0,1)^2} \int_{\gamma(y')}^{\infty} |\nabla v^{(j)}(y', y_3)|^2 \, \mathrm{d}y_3 \, \mathrm{d}y' \le C\,, \tag{11}$$

where the constant C depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$.

The basic idea of the proof is to decompose the domain Ω^{bl} into $\overline{\mathbb{R}^3_+} \cup (\Omega^{\mathrm{bl}} \setminus \overline{\mathbb{R}^3_+})$ and to derive an equivalent equations to ($\mathrm{BL}^{(j)}$) on the infinite channel $\Omega^{\mathrm{bl}} \setminus \overline{\mathbb{R}^3_+}$ but involving the Dirichlet-to-Neumann operator. This formulation allows us to apply the Poincaré inequality in the vertical direction when estimating the local energy. The reader is referred to [19, Section 3] for the details.

Periodic case. We note the asymptotic behavior of the boundary layer corrector at spatial infinity when the boundary is periodic; see [19, Proposition 11].

Proposition 2.2. *Fix $j \in \{1, 2\}$ and let $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ be 2π-periodic in each variable. Then the weak solution $(v^{(j)}, q^{(j)})$ to ($\mathrm{BL}^{(j)}$) provided by Proposition 2.1 satisfies the following properties.*

(i) *There exists a constant vector field $\alpha^{(j)} = (\alpha^{(j)}_1, \alpha^{(j)}_2, 0)^\top \in \mathbb{R}^3$ such that*

$$|v^{(j)}(y) - \alpha^{(j)}| + y_3 |q^{(j)}(y)| \le C \|v^{(j)}(\cdot, 0)\|_{L^2((0,2\pi)^2)} \, e^{-\frac{y_3}{2}}\,, \quad y_3 > 1\,, \tag{12}$$

where C is a numerical constant.

(ii) *The 2×2 matrix $\overline{M} \in \mathbb{R}^{2\times 2}$ defined by $\overline{M} = (\alpha^{(j)}_i)_{1 \le i,j \le 2}$ is symmetric and positive definite.*

Some useful estimates. We state an easy lemma useful in estimating $v^{(j)}$. We omit the proof since it is just a simple computation using Proposition 2.1.

Lemma 2.3. *Fix $j \in \{1,2\}$ and let $\varepsilon \in (0,1)$ and $r \in [\varepsilon, 1]$. Then we have*

$$\int_{B^\varepsilon_{r,+}(0)} \big|(\nabla_y v^{(j)})(\frac{x}{\varepsilon})\big|^2 \,\mathrm{d}x \le C\varepsilon r^2 , \tag{13}$$

and for $m \in \{0,1,2\}$,

$$\int_{B^\varepsilon_{r,+}(0)} \big|v^{(j)}(\frac{x}{\varepsilon})\big|^{2+m} \,\mathrm{d}x \le \frac{Cr^{4-\frac{m}{2}}}{\varepsilon^{1+\frac{m}{2}}} , \tag{14}$$

where the constant C is independent of ε and r and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$.

3 Regularity for the Stokes equations

In this section we consider the Stokes equations

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1(0) \end{cases} \tag{S$^\varepsilon$}$$

in order to demonstrate how the compactness and iteration arguments work in a simpler setting. Note that a weak formulation for (S$^\varepsilon$) can be defined in a similar manner as (10) for (NS$^\varepsilon$) in the introduction. Our goal in this section is to prove the following theorem.

Theorem 3.1 (linear estimate)**.** *There exists a constant $\varepsilon^{(3)} \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ such that the following statement holds. For all $\varepsilon \in (0, \varepsilon^{(3)}]$ and $r \in [\varepsilon/\varepsilon^{(3)}, 1]$, any weak solution $u^\varepsilon = (u^\varepsilon_1(x), u^\varepsilon_2(x), u^\varepsilon_3(x))^\top \in H^1(B^\varepsilon_{1,+}(0))^3$ to (S$^\varepsilon$) satisfies*

$$\Big(\fint_{B^\varepsilon_{r,+}(0)} |u^\varepsilon|^2\Big)^{\frac{1}{2}} \le C^{(3)} r \Big(\fint_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2\Big)^{\frac{1}{2}} , \tag{15}$$

where the constant $C^{(3)}$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$.

We prove the compactness and iteration lemmas in Subsection 3.1 which are essential tools for our argument. We prove Theorem 3.1 in Subsection 3.2 using the estimates in Section 2.

3.1 Compactness and iteration lemmas

The compactness lemma is stated as follows. Let $v^{(j)} = v^{(j)}(y)$ be the weak solution to (BL$^{(j)}$) for $j \in \{1,2\}$ provided by Proposition 2.1.

Lemma 3.2. *For $\mu \in (0,1)$, there exist constants $\theta \in (0, \frac{1}{8})$ and $\varepsilon_\mu \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and μ such that the following statement holds. For $\varepsilon \in (0, \varepsilon_\mu]$, any weak solution $u^\varepsilon = (u^\varepsilon_1(x), u^\varepsilon_2(x), u^\varepsilon_3(x))^\top \in H^1(B^\varepsilon_{1,+}(0))^3$ to (S$^\varepsilon$) with*

$$\fint_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2 \le 1 \tag{16}$$

satisfies

$$\fint_{B^\varepsilon_{\theta,+}(0)} \Big|u^\varepsilon(x) - \sum_{j=1}^{2} (\overline{\partial_3 u^\varepsilon_j})_{B^\varepsilon_{\theta,+}(0)} \big(x_3 \mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2 \,\mathrm{d}x \le \theta^{2+2\mu} . \tag{17}$$

Proof. For given $\mu \in (0,1)$, we choose $\theta \in (0, \frac{1}{8})$ in the statement as follows. Let $(u^0, p^0) \in H^1(B_{\frac{1}{2},+}(0))^3 \times L^2(B_{\frac{1}{2},+}(0))$ be a weak solution to the ε-zero limit equations

$$\begin{cases} -\Delta u^0 + \nabla p^0 = 0 & \text{in } B_{\frac{1}{2},+}(0) \\ \nabla \cdot u^0 = 0 & \text{in } B_{\frac{1}{2},+}(0) \\ u^0 = 0 & \text{on } \Gamma_{\frac{1}{2}}(0) \end{cases} \tag{18}$$

with

$$\int_{B_{\frac{1}{2},+}(0)} |u^0|^2 \leq 4\,. \tag{19}$$

By the regularity theory to (18) in Appendix combined with (19), we see that $u^0 \in C^2(\overline{B_{\frac{3}{8},+}(0)})^3$. From the no-slip condition in (18), we calculate the tangential component u^0_j of u with $j \in \{1,2\}$ as

$$\begin{aligned} & u^0_j(x) - (\overline{\partial_3 u^0_j})_{B_{\theta,+}(0)}\, x_3 \\ &= \frac{x_3}{|B_{\theta,+}(0)|} \int_0^1 \int_{B_{\theta,+}(0)} \big(\partial_3 u^0_j(x', t x_3) - \partial_3 u^0_j(z)\big)\, \mathrm{d}z\, \mathrm{d}t\,, \end{aligned}$$

where $\theta \in (0, \frac{1}{4})$ is arbitrary. Thus we see that

$$\fint_{B_{\theta,+}(0)} |u^0_j(x) - (\overline{\partial_3 u^0_j})_{B_{\theta,+}(0)}\, x_3|^2\, \mathrm{d}x \leq C\theta^4 \tag{20}$$

with a constant C independent of θ. For the normal component u^0_3 of u, by the divergence-free and no-slip conditions in (18), we have

$$u^0_3(x) = -x_3 \int_0^1 \sum_{j=1}^2 \partial_j u^0_j(x', t x_3)\, \mathrm{d}t\,.$$

Since $\partial_j u^0_j = 0$ on $\Gamma_{\frac{1}{2}}(0)$ holds for $j \in \{1,2\}$, we also have

$$u^0_3(x) = -x_3^2 \int_0^1 \int_0^1 \sum_{j=1}^2 t\, \partial_3 \partial_j u^0_j(x', s t x_3)\, \mathrm{d}s\, \mathrm{d}t\,.$$

Thus there exists a constant C independent of θ such that for any $\theta \in (0, \frac{1}{4})$,

$$\fint_{B_{\theta,+}(0)} |u^0_3|^2 \leq C\theta^4\,. \tag{21}$$

Then we choose $\theta \in (0, \frac{1}{8})$ in (20) and (21) sufficiently small depending on μ so that

$$\begin{aligned} & \fint_{B_{\theta,+}(0)} \Big|u^0(x) - \sum_{j=1}^2 (\overline{\partial_3 u^0_j})_{B_{\theta,+}(0)} x_3 \mathbf{e}_j\Big|^2\, \mathrm{d}x \\ &= \fint_{B_{\theta,+}(0)} |u^0_j(x) - (\overline{\partial_3 u^0_j})_{B_{\theta,+}(0)}\, x_3|^2\, \mathrm{d}x + \fint_{B_{\theta,+}(0)} |u^0_3|^2 < \frac{\theta^{2+2\mu}}{8}\,. \end{aligned} \tag{22}$$

The rest of the proof is by contradiction. Assume that there exist sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ in $(0,1)$ with $\lim_{k\to\infty}\varepsilon_k = 0$ and $\{u^{\varepsilon_k}\}_{k=1}^{\infty}$ in $H^1(B_{1,+}^{\varepsilon_k}(0))^3$ with

$$\fint_{B_{1,+}^{\varepsilon_k}(0)} |u^{\varepsilon_k}|^2 \le 1 \tag{23}$$

satisfying both

$$\begin{cases} -\Delta u^{\varepsilon_k} + \nabla p^{\varepsilon_k} = 0 & \text{in } B_{1,+}^{\varepsilon_k}(0) \\ \nabla\cdot u^{\varepsilon_k} = 0 & \text{in } B_{1,+}^{\varepsilon_k}(0) \\ u^{\varepsilon_k} = 0 & \text{on } \Gamma_1^{\varepsilon_k}(0) \end{cases}$$

and

$$\fint_{B_{\theta,+}^{\varepsilon_k}(0)} \Big| u^{\varepsilon_k}(x) - \sum_{j=1}^{2} (\overline{\partial_3 u_j^{\varepsilon_k}})_{B_{\theta,+}^{\varepsilon_k}(0)} \big(x_3\mathbf{e}_j + \varepsilon_k v^{(j)}(\frac{x}{\varepsilon_k})\big)\Big|^2 \,\mathrm{d}x > \theta^{2+2\mu}\,. \tag{24}$$

Since $\varepsilon_k\gamma(x'/\varepsilon_k) \to 0$ uniformly in $x' \in \mathbb{R}^2$, the boundary $\Gamma_1^{\varepsilon_k}(0)$ is included in the set $(-1,1)^2\times(-\frac{1}{2},0)$ when k is sufficiently large. We extend u^{ε_k} by zero below the boundary, which is denoted again by u^{ε_k}, and we see that $u^{\varepsilon_k}\in H^1(B_1(0))^3$ for all $k\in\mathbb{N}$. Then, by the Caccioppoli inequality in Lemma A.2 with $\rho=\frac{1}{2}$ and $r=1$ in Appendix, we have from (23),

$$\fint_{B_{\frac{1}{2},+}^{\varepsilon_k}(0)} |\nabla u^{\varepsilon_k}|^2 \le C$$

with C independent of ε_k. Hence, up to a subsequence of $\{u^{\varepsilon_k}\}_{k=1}^{\infty}$, which is denoted by $\{u^{\varepsilon_k}\}_{k=1}^{\infty}$ again, there exists $u^0\in H^1(B_{\frac{1}{2}}(0))^3$ such that in the limit $k\to\infty$,

$$u^{\varepsilon_k} \to u^0 \ \text{ in } \ L^2(B_{\frac{1}{2}}(0))^3\,, \qquad \nabla u^{\varepsilon_k} \rightharpoonup \nabla u^0 \ \text{ in } \ L^2(B_{\frac{1}{2}}(0))^{3\times 3}\,,$$

and (19) holds by the assumption (23). Moreover, we have for any $\varphi\in C_0^{\infty}((-\frac{1}{2},\frac{1}{2})^2\times(-\frac{1}{2},0))^3$,

$$\int_{(-\frac{1}{2},\frac{1}{2})^2\times(-\frac{1}{2},0)} u^0\cdot\varphi = \lim_{k\to\infty}\int_{(-\frac{1}{2},\frac{1}{2})^2\times(-\frac{1}{2},0)} u^{\varepsilon_k}\cdot\varphi = 0$$

and for any $\varphi\in C_{0,\sigma}^{\infty}(B_{\frac{1}{2}}(0))^3$,

$$\int_{B_{\frac{1}{2},+}(0)} \nabla u^0\cdot\nabla\varphi = \lim_{k\to\infty}\int_{B_{1,+}^{\varepsilon_k}(0)} \nabla u^{\varepsilon_k}\cdot\nabla\varphi = 0\,.$$

We see that $u^0=0$ on $(-\frac{1}{2},\frac{1}{2})^2\times(-\frac{1}{2},0)$ and hence that $u^0=0$ on $\Gamma_{\frac{1}{2}}(0)$ from $u^0\in H^1(B_{\frac{1}{2}}(0))$. Thus u^0 is a weak solution to (18) satisfying (19). Then, from $B_{\theta,+}^{\varepsilon_k}(0) = \big(B_{\theta,+}^{\varepsilon_k}(0)\setminus B_{\theta,+}(0)\big)\cup$

$\big(B^{\varepsilon_k}_{\theta,+}(0)\cap B_{\theta,+}(0)\big)$ and $|B^{\varepsilon_k}_{\theta,+}(0)|=|B_{\theta,+}(0)|=4\theta^3$, by the triangle inequality we have

$$
\begin{aligned}
&\fint_{B^{\varepsilon_k}_{\theta,+}(0)}\big|u^{\varepsilon_k}(x)-\sum_{j=1}^{2}(\overline{\partial_3 u^{\varepsilon_k}_j})_{B^{\varepsilon_k}_{\theta,+}(0)}\big(x_3\mathbf{e}_j+\varepsilon_k v^{(j)}(\frac{x}{\varepsilon_k})\big)\big|^2\,\mathrm{d}x\\
&\le \frac{1}{4\theta^3}\int_{B^{\varepsilon_k}_{\theta,+}(0)\setminus B_{\theta,+}(0)}\big|u^{\varepsilon_k}(x)-\sum_{j=1}^{2}(\overline{\partial_3 u^{\varepsilon_k}_j})_{B^{\varepsilon_k}_{\theta,+}(0)}\big(x_3\mathbf{e}_j+\varepsilon_k v^{(j)}(\frac{x}{\varepsilon_k})\big)\big|^2\,\mathrm{d}x\\
&\quad+\frac{2}{\theta^3}\bigg(\int_{B^{\varepsilon_k}_{\theta,+}(0)\cap B_{\theta,+}(0)}\big|u^{\varepsilon_k}-u^0\big|^2+\sum_{j=1}^{2}\big|(\overline{\partial_3 u^{\varepsilon_k}_j})_{B^{\varepsilon_k}_{\theta,+}(0)}x_3\mathbf{e}_j-(\overline{\partial_3 u^{0}_j})_{B_{\theta,+}(0)}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\\
&\qquad\qquad+\varepsilon_k^2\sum_{j=1}^{2}\big|(\overline{\partial_3 u^{\varepsilon_k}_j})_{B^{\varepsilon_k}_{\theta,+}(0)}\big|^2\int_{B^{\varepsilon_k}_{\theta,+}(0)\cap B_{\theta,+}(0)}\big|v^{(j)}(\frac{x}{\varepsilon_k})\big|^2\,\mathrm{d}x\bigg)\\
&\quad+8\fint_{B_{\theta,+}(0)}\big|u^0(x)-\sum_{j=1}^{2}(\overline{\partial_3 u^{0}_j})_{B_{\theta,+}(0)}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\,.
\end{aligned}
$$

Since $u^{\varepsilon_k}\to u^0$ in $L^2(B_{\frac12}(0))^3$ and $\{\nabla u^{\varepsilon_k}\}_{k=1}^\infty$ is uniformly bounded in $L^2(B_{\frac12}(0))^{3\times3}$, from the assumption (24) we see that

$$
\begin{aligned}
\theta^{2+2\mu}&\le \varlimsup_{k\to\infty}\fint_{B^{\varepsilon_k}_{\theta,+}(0)}\big|u^{\varepsilon_k}(x)-\sum_{j=1}^{2}(\overline{\partial_3 u^{\varepsilon_k}_j})_{B^{\varepsilon_k}_{\theta,+}(0)}\big(x_3\mathbf{e}_j+\varepsilon_k v^{(j)}(\frac{x}{\varepsilon_k})\big)\big|^2\,\mathrm{d}x\\
&\le 8\fint_{B_{\theta,+}(0)}\big|u^0(x)-\sum_{j=1}^{2}(\overline{\partial_3 u^{0}_j})_{B_{\theta,+}(0)}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\,,
\end{aligned}
$$

where (14) with $m=0$ in Lemma 2.3 is applied to obtain the second line. Hence the choice of θ in (22) contradicts (24). This completes the proof of Lemma 3.2. □

The iteration lemma to (S^ε) is stated as follows. Let K_0 be the constant of the Caccioppoli inequality in Lemma A.2 in Appendix.

Lemma 3.3. *Fix $\mu\in(0,1)$ and let $\theta\in(0,\frac18)$ and $\varepsilon_\mu\in(0,1)$ be the constants in Lemma 3.2. Then for $k\in\mathbb{N}$ and $\varepsilon\in(0,\theta^{k-1}\varepsilon_\mu]$, any weak solution $u^\varepsilon=(u^\varepsilon_1(x),u^\varepsilon_2(x),u^\varepsilon_3(x))^\top\in H^1(B^\varepsilon_{1,+}(0))^3$ to (S^ε) with*

$$\fint_{B^\varepsilon_{1,+}(0)}|u^\varepsilon|^2\le 1 \tag{25}$$

satisfies

$$\fint_{B^\varepsilon_{\theta^k,+}(0)}\big|u^\varepsilon(x)-\sum_{j=1}^{2}a^\varepsilon_{k,j}\big(x_3\mathbf{e}_j+\varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x\le\theta^{(2+2\mu)k}\,. \tag{26}$$

Here the number $a^\varepsilon_{k,j}\in\mathbb{R}$, $j\in\{1,2\}$, is estimated as

$$\sum_{j=1}^{2}|a^\varepsilon_{k,j}|\le 2K_0^{\frac12}\theta^{-\frac32}(1-\theta)^{-1}\sum_{l=1}^{k}\theta^{\mu(l-1)}\,. \tag{27}$$

Proof. The proof is done by induction on $k \in \mathbb{N}$. The case $k = 1$ is valid since it is exactly (17) in Lemma 3.2 putting $a^\varepsilon_{1,j} = (\overline{\partial_3 u^\varepsilon_j})_{B^\varepsilon_{\theta,+}(0)}$, $j \in \{1, 2\}$. Indeed, by the Hölder inequality we have

$$\begin{aligned}\sum_{j=1}^{2} |a^\varepsilon_{1,j}| &\le 2|B^\varepsilon_{\theta,+}(0)|^{-\frac{1}{2}} \|\nabla u^\varepsilon\|_{L^2(B^\varepsilon_{\theta,+}(0))} \\ &\le K_0^{\frac{1}{2}} \theta^{-\frac{3}{2}} (1-\theta)^{-1} \|u^\varepsilon\|_{L^2(B^\varepsilon_{1,+}(0))}\,,\end{aligned}$$

where we have applied the Caccioppoli inequality to (S^ε) with $\rho = \theta$ and $r = 1$ in Lemma A.2 in Appendix. Thus by (25) we have (27) for $k = 1$. Next let us assume that (26) and (27) hold at rank $k \in \mathbb{N}$ and let $\varepsilon \in (0, \theta^k \varepsilon_\mu]$. Then we define new functions $U^{\varepsilon/\theta^k} = (U_1^{\varepsilon/\theta^k}(y), U_2^{\varepsilon/\theta^k}(y), U_3^{\varepsilon/\theta^k}(y))^\top$ and $P^{\varepsilon/\theta^k} = P^{\varepsilon/\theta^k}(y)$ on $B^{\varepsilon/\theta^k}_{1,+}(0)$ by

$$\begin{aligned}U^{\varepsilon/\theta^k}(y) &= \frac{1}{\theta^{(1+\mu)k}} \Big(u^\varepsilon(\theta^k y) - \sum_{j=1}^{2} \theta^k a^\varepsilon_{k,j} \big(y_3 \mathbf{e}_j + \frac{\varepsilon}{\theta^k} v^{(j)}(\frac{\theta^k y}{\varepsilon})\big)\Big)\,, \\ P^{\varepsilon/\theta^k}(y) &= \frac{1}{\theta^{\mu k}} \Big(p^\varepsilon(\theta^k y) - \sum_{j=1}^{2} a^\varepsilon_{k,j} q^{(j)}(\frac{\theta^k y}{\varepsilon})\Big)\,.\end{aligned}$$

We see that $(U^{\varepsilon/\theta^k}, P^{\varepsilon/\theta^k})$ is a weak solution to

$$\begin{cases} -\Delta_y U^{\varepsilon/\theta^k} + \nabla_y P^{\varepsilon/\theta^k} = 0 & \text{in } B^{\varepsilon/\theta^k}_{1,+}(0) \\ \nabla_y \cdot U^{\varepsilon/\theta^k} = 0 & \text{in } B^{\varepsilon/\theta^k}_{1,+}(0) \\ U^{\varepsilon/\theta^k} = 0 & \text{on } \Gamma^{\varepsilon/\theta^k}_1(0)\,. \end{cases} \tag{28}$$

From the recurrence hypothesis (26) at rank k, we have

$$⨍_{B^{\varepsilon/\theta^k}_{1,+}(0)} |U^{\varepsilon/\theta^k}|^2 \le 1 \tag{29}$$

by a change of variables. Now, since $\varepsilon/\theta^k \in (0, \varepsilon_\mu]$, we can apply Lemma 3.2 to see that

$$⨍_{B^{\varepsilon/\theta^k}_{\theta,+}(0)} \Big| U^{\varepsilon/\theta^k}(y) - \sum_{j=1}^{2} (\overline{\partial_{y_3} U_j^{\varepsilon/\theta^k}})_{B^{\varepsilon/\theta^k}_{\theta,+}(0)} \big(y_3 \mathbf{e}_j + \frac{\varepsilon}{\theta^k} v^{(j)}(\frac{\theta^k y}{\varepsilon})\big)\Big|^2 \, \mathrm{d}y \le \theta^{2+2\mu}\,.$$

A change of variables leads to

$$⨍_{B^\varepsilon_{\theta^{k+1},+}(0)} \Big| u^\varepsilon(x) - \sum_{j=1}^{2} a^\varepsilon_{k+1,j} \big(x_3 \mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2 \, \mathrm{d}x \le \theta^{(2+2\mu)(k+1)}\,, \tag{30}$$

where the number $a^\varepsilon_{k+1,j} \in \mathbb{R}$, $j \in \{1, 2\}$, is defined as

$$a^\varepsilon_{k+1,j} = a^\varepsilon_{k,j} + \theta^{\mu k} (\overline{\partial_{y_3} U_j^{\varepsilon/\theta^k}})_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\,. \tag{31}$$

The Caccioppoli inequality to (28) with $\rho = \theta$ and $r = 1$ combined with (29) leads to

$$\begin{aligned}\|\nabla_y U^{\varepsilon/\theta^k}\|_{L^2(B^{\varepsilon/\theta^k}_{\theta,+}(0))} &\le K_0^{\frac{1}{2}} (1-\theta)^{-1} \|U^{\varepsilon/\theta^k}\|_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} \\ &\le 2K_0^{\frac{1}{2}} (1-\theta)^{-1}\,.\end{aligned}$$

Therefore, from the assumption (27) for k and (31), by the Hölder inequality we obtain

$$\begin{aligned}\sum_{j=1}^{2}|a^{\varepsilon}_{k+1,j}| &\leq \sum_{j=1}^{2}|a^{\varepsilon}_{k,j}| + \theta^{\mu k}\sum_{j=1}^{2}\big|\overline{(\partial_{y_3}U_j^{\varepsilon/\theta^k})}_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\big| \\ &\leq 2K_0^{\frac12}\theta^{-\frac32}(1-\theta)^{-1}\sum_{l=1}^{k+1}\theta^{\mu(l-1)}\,,\end{aligned}$$

which with (30) proves the assertions (26) and (27) for $k+1$. This completes the proof. □

3.2 Proof of Theorem 3.1

We prove Theorem 3.1 by applying Lemma 3.3. Fix $\mu \in (0,1)$ and let $\theta \in (0,\frac18)$ and $\varepsilon_\mu \in (0,1)$ be the constants in Lemma 3.2.

Proof of Theorem 3.1: Since the equations (S^ε) are linear, it suffices to prove the estimate

$$\Big(\fint_{B^{\varepsilon}_{r,+}(0)}|u^\varepsilon|^2\Big)^{\frac12} \leq Cr\,. \tag{32}$$

Set $\varepsilon^{(3)} = \varepsilon_\mu$ and let $\varepsilon \in (0,\varepsilon^{(3)}]$. Firstly we note that if $r\in(\theta,1]$, then

$$\Big(\fint_{B^{\varepsilon}_{r,+}(0)}|u^\varepsilon|^2\Big)^{\frac12} \leq \theta^{-\frac52}r$$

holds. Thus we focus on the case $r\in[\varepsilon/\varepsilon^{(3)},\theta]$. For any given $r\in[\varepsilon/\varepsilon^{(3)},\theta]$, there exists $k\in\mathbb{N}$ with $k\geq 2$ such that $r\in(\theta^k,\theta^{k-1}]$ holds. From $\varepsilon\in(0,\theta^{k-1}\varepsilon^{(3)}]$ we apply Lemma 3.3 to see that

$$\begin{aligned}\Big(\fint_{B^{\varepsilon}_{r,+}(0)}|u^\varepsilon|^2\Big)^{\frac12} &\leq \Big(\theta^{-3}\fint_{B^{\varepsilon}_{\theta^{k-1},+}(0)}|u^\varepsilon|^2\Big)^{\frac12} \\ &\leq \theta^{-\frac32}\Big(\fint_{B^{\varepsilon}_{\theta^{k-1},+}(0)}\big|u^\varepsilon(x)-\sum_{j=1}^{2}a^{\varepsilon}_{k-1,j}\big(x_3\mathbf{e}_j+\varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x\Big)^{\frac12} \\ &\quad+\theta^{-\frac32}\Big(\sum_{j=1}^{2}|a^{\varepsilon}_{k-1,j}|\Big)\Big(\sum_{j=1}^{2}\fint_{B^{\varepsilon}_{\theta^{k-1},+}(0)}\big|x_3\mathbf{e}_j+\varepsilon v^{(j)}(\frac{x}{\varepsilon})\big|^2\,\mathrm{d}x\Big)^{\frac12} \\ &\leq \theta^{(1+\mu)(k-1)-\frac32} \\ &\quad+C\theta^{-3}(1-\theta)^{-1}(1-\theta^\mu)^{-1}\Big(\sum_{j=1}^{2}\fint_{B^{\varepsilon}_{\theta^{k-1},+}(0)}\big|x_3\mathbf{e}_j+\varepsilon v^{(j)}(\frac{x}{\varepsilon})\big|^2\,\mathrm{d}x\Big)^{\frac12}\,,\end{aligned} \tag{33}$$

where C depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$. From (14) with $m=0$ in Lemma 2.3 one has

$$\Big(\sum_{j=1}^{2}\fint_{B^{\varepsilon}_{\theta^{k-1},+}(0)}\big|x_3\mathbf{e}_j+\varepsilon v^{(j)}(\frac{x}{\varepsilon})\big|^2\,\mathrm{d}x\Big)^{\frac12} \leq C(\theta^{k-1}+\varepsilon^{\frac12}\theta^{\frac{k-1}{2}})\,.$$

Therefore, by $\theta^{k-1}\in(0,\theta^{-1}r)$ and $\varepsilon\in(0,\theta^{k-1}\varepsilon^{(3)}]$, we have from (33),

$$\begin{aligned}\Big(\fint_{B^{\varepsilon}_{r,+}(0)}|u^\varepsilon|^2\Big)^{\frac12} &\leq \theta^{-\frac52-\mu}r^{1+\mu}+C\theta^{-3}(1-\theta)^{-1}(1-\theta^\mu)^{-1}(\theta^{k-1}+\varepsilon^{\frac12}\theta^{\frac{k-1}{2}}) \\ &\leq \Big(\theta^{-\frac52-\mu}r^{\mu}+C\theta^{-4}(1-\theta)^{-1}(1-\theta^\mu)^{-1}(1+(\varepsilon^{(3)})^{\frac12})\Big)r\,.\end{aligned}$$

Hence we obtain the desired estimate (32) by letting $\mu = \frac{1}{2}$ for instance. This completes the proof of Theorem 3.1. □

4 Proof of the main results

We prove Theorem 1.1 and Theorem 1.3 in this section. As is done in Section 3, we first work out the compactness and iteration lemmas in Subsection 4.1. Contrary to the linear case, we need to carry out a careful analysis of the iteration argument due to the nonlinearity. Indeed, since we do not assume any smallness condition on solutions of (NS$^\varepsilon$), a naive iterated application of the Caccioppoli inequality leads to a blow-up of the derivative estimate in the nonlinear case. We overcome this difficulty a priori by taking the free parameter θ appearing in the compactness lemma sufficiently small depending on the bound M of the solution to (NS$^\varepsilon$). Eventually, the proof of Theorem 1.1 and Theorem 1.3 is given in Subsection 4.2.

4.1 Nonlinear compactness and iteration lemmas

We consider the modified Navier-Stokes equations:

$$\begin{cases} -\Delta U^\varepsilon + \nabla P^\varepsilon = -\nabla \cdot (U^\varepsilon \otimes b^\varepsilon + b^\varepsilon \otimes U^\varepsilon) \\ \qquad\qquad\qquad\quad - \lambda^\varepsilon U^\varepsilon \cdot \nabla U^\varepsilon + \nabla \cdot F^\varepsilon \ \text{ in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot U^\varepsilon = 0 \ \text{ in } B^\varepsilon_{1,+}(0) \\ U^\varepsilon = 0 \ \text{ on } \Gamma^\varepsilon_1(0)\,, \end{cases} \tag{MNS$^\varepsilon$}$$

where $b^\varepsilon = b^\varepsilon(x)$ is defined as

$$b^\varepsilon(x) = \sum_{j=1}^{2} C^\varepsilon_j \big(x_3 \mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\,, \quad x \in B^\varepsilon_{1,+}(0)\,. \tag{34}$$

Note that $\nabla \cdot b^\varepsilon = 0$ in $B^\varepsilon_{1,+}(0)$ and $b^\varepsilon = 0$ on $\Gamma^\varepsilon_1(0)$. The compactness lemma is stated as follows.

Lemma 4.1. *For $M \in (0,\infty)$ and $\mu \in (0,1)$, there exists a constant $\theta_0 \in (0,\frac{1}{8})$ depending on M and μ such that the following statement holds. For any $\theta \in (0,\theta_0]$, there exists $\varepsilon_\mu \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, μ, and θ such that for $\varepsilon \in (0,\varepsilon_\mu]$, $(\lambda^\varepsilon, C^\varepsilon_1, C^\varepsilon_2) \in [-1,1]^3$, and $F^\varepsilon \in L^2(B^\varepsilon_{1,+}(0))^{3\times 3}$ with*

$$\|F^\varepsilon\|_{L^2(B^\varepsilon_{1,+}(0))} \le M\varepsilon_\mu\,, \tag{35}$$

any weak solution $U^\varepsilon = (U^\varepsilon_1(x), U^\varepsilon_2(x), U^\varepsilon_3(x))^\top \in H^1(B^\varepsilon_{1,+}(0))^3$ to (MNS$^\varepsilon$) with

$$⨍_{B^\varepsilon_{1,+}(0)} |U^\varepsilon|^2 \le M^2 \tag{36}$$

satisfies

$$⨍_{B^\varepsilon_{\theta,+}(0)} \Big|U^\varepsilon(x) - \sum_{j=1}^{2} (\overline{\partial_3 U^\varepsilon_j})_{B^\varepsilon_{\theta,+}(0)} \big(x_3 \mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2 \,\mathrm{d}x \le M^2 \theta^{2+2\mu}\,. \tag{37}$$

Proof. By setting

$$V^\varepsilon = \frac{U^\varepsilon}{M}\,, \qquad Q^\varepsilon = \frac{P^\varepsilon}{M}\,, \qquad G^\varepsilon = \frac{F^\varepsilon}{M}\,,$$

we see that V^ε and G^ε satisfy

$$\fint_{B^\varepsilon_{1,+}(0)} |V^\varepsilon|^2 \le 1\,, \qquad \|G^\varepsilon\|_{L^2(B^\varepsilon_{1,+}(0))} \le \varepsilon_\mu\,,$$

and that $(V^\varepsilon, Q^\varepsilon)$ solves the equations

$$\begin{cases} -\Delta V^\varepsilon + \nabla Q^\varepsilon = -\nabla\cdot(V^\varepsilon\otimes b^\varepsilon + b^\varepsilon\otimes V^\varepsilon) \\ \qquad\qquad\qquad - M\lambda^\varepsilon V^\varepsilon\cdot\nabla V^\varepsilon + \nabla\cdot G^\varepsilon \ \text{ in } B^\varepsilon_{1,+}(0) \\ \nabla\cdot V^\varepsilon = 0 \ \text{ in } B^\varepsilon_{1,+}(0) \\ V^\varepsilon = 0 \ \text{ on } \Gamma^\varepsilon_1(0)\,. \end{cases} \tag{38}$$

In the following we consider the rescaled equations (38). Hence our goal is to obtain

$$\fint_{B^\varepsilon_{\theta,+}(0)} \Big|V^\varepsilon(x) - \sum_{j=1}^{2}(\overline{\partial_3 V^\varepsilon_j})_{B^\varepsilon_{\theta,+}(0)}\big(x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2\,\mathrm{d}x \le \theta^{2+2\mu}\,. \tag{39}$$

For given $M\in(0,\infty)$ and $\mu\in(0,1)$, we choose $\theta_0\in(0,\frac18)$ in the statement as follows. Let $(V^0,Q^0)\in H^1(B_{\frac12,+}(0))^3\times L^2(B_{\frac12,+}(0))$ be a weak solution to the ε-zero limit equations

$$\begin{cases} -\Delta V^0 + \nabla Q^0 = -\nabla\cdot\Big(V^0\otimes\big(\sum_{j=1}^{2} C^0_j x_3\mathbf{e}_j\big) + \big(\sum_{j=1}^{2} C^0_j x_3\mathbf{e}_j\big)\otimes V^0\Big) \\ \qquad\qquad\qquad - M\lambda^0 V^0\cdot\nabla V^0 \ \text{ in } B_{\frac12,+}(0) \\ \nabla\cdot V^0 = 0 \ \text{ in } B_{\frac12,+}(0) \\ V^0 = 0 \ \text{ on } \Gamma_{\frac12}(0) \end{cases} \tag{40}$$

with

$$\fint_{B_{\frac12,+}(0)} |V^0|^2 \le 4\,. \tag{41}$$

By the regularity theory to (40) in Appendix using (41), we see that $V^0\in C^2(\overline{B_{\frac38,+}(0)})^3$ and

$$\|V^0\|_{C^2(\overline{B_{\frac38,+}(0)})} \le K$$

with a constant K depending on M but independent of $(\lambda^0, C^0_1, C^0_2)\in[-1,1]^3$. Then, in the same way as in the proof of Lemma 3.2, we choose $\theta_0\in(0,\frac18)$ sufficiently small so that for any $\theta\in(0,\theta_0]$

$$\fint_{B_{\theta,+}(0)} \Big|V^0(x) - \sum_{j=1}^{2}(\overline{\partial_3 V^0_j})_{B_{\theta,+}(0)} x_3\mathbf{e}_j\Big|^2\,\mathrm{d}x < \frac{\theta^{2+2\mu}}{8} \tag{42}$$

holds. We emphasize that θ_0 depends only on M and μ. The rest of the proof is done by contradiction. Assume that there exist $\theta \in (0,\theta_0]$ and sequences $\{\varepsilon_k\}_{k=1}^\infty \subset (0,1)$ with $\lim_{k\to\infty}\varepsilon_k = 0$, $\{(\lambda^{\varepsilon_k}, C_1^{\varepsilon_k}, C_2^{\varepsilon_k})\}_{k=1}^\infty \subset [-1,1]^3$, and $\{G^{\varepsilon_k}\}_{k=1}^\infty \subset L^2(B_{1,+}^{\varepsilon_k}(0))^{3\times 3}$ with

$$\|G^{\varepsilon_k}\|_{L^2(B_{1,+}^{\varepsilon_k}(0))} \le \varepsilon_k\,.$$

Moreover, we assume that there exists $\{V^{\varepsilon_k}\}_{k=1}^\infty$ in $H^1(B_{1,+}^{\varepsilon_k}(0))^3$ with

$$⨍_{B_{1,+}^{\varepsilon_k}(0)} |V^{\varepsilon_k}|^2 \le 1 \tag{43}$$

satisfying both

$$\begin{cases} -\Delta V^{\varepsilon_k} + \nabla Q^{\varepsilon_k} = -\nabla\cdot(V^{\varepsilon_k}\otimes b^{\varepsilon_k} + b^{\varepsilon_k}\otimes V^{\varepsilon_k}) \\ \qquad\qquad\qquad - M\lambda^{\varepsilon_k} V^{\varepsilon_k}\cdot\nabla V^{\varepsilon_k} + \nabla\cdot G^{\varepsilon_k} \quad \text{in } B_{1,+}^{\varepsilon_k}(0) \\ \nabla\cdot V^{\varepsilon_k} = 0 \quad \text{in } B_{1,+}^{\varepsilon_k}(0) \\ V^{\varepsilon_k} = 0 \quad \text{on } \Gamma_1^{\varepsilon_k}(0) \end{cases}$$

and

$$⨍_{B_{\theta,+}^{\varepsilon_k}(0)} \Big|V^{\varepsilon_k}(x) - \sum_{j=1}^{2} (\overline{\partial_3 V_j^{\varepsilon_k}})_{B_{\theta,+}^{\varepsilon_k}(0)} \big(x_3\mathbf{e}_j + \varepsilon_k v^{(j)}(\frac{x}{\varepsilon_k})\big)\Big|^2 \,\mathrm{d}x > \theta^{2+2\mu}\,. \tag{44}$$

We extend V^{ε_k}, $v^{(j)}(\cdot/\varepsilon_k)$, and G^{ε_k} by zero below the boundary, which are respectively denoted by V^{ε_k}, $v^{(j)}(\cdot/\varepsilon_k)$, and G^{ε_k} again, and see that $V^{\varepsilon_k} \in H^1(B_1(0))^3$ and $G^{\varepsilon_k} \in L^2(B_1(0))^{3\times 3}$ for all $k \in \mathbb{N}$. By applying the Caccioppoli inequality in Lemma A.2 with $\rho = \frac{1}{2}$ and $r = 1$ in Appendix, we obtain

$$\|\nabla V^{\varepsilon_k}\|_{L^2(B_{\frac12,+}^{\varepsilon_k}(0))} \le C(1+M^3)$$

uniformly in k with a constant C independent of M. Here we have used (13) in Lemma 2.3 and (43). Hence, up to subsequences of $\{V^{\varepsilon_k}\}_{k=1}^\infty$, $\{(\lambda^{\varepsilon_k}, C_1^{\varepsilon_k}, C_2^{\varepsilon_k})\}_{k=1}^\infty$, and $\{G^{\varepsilon_k}\}_{k=1}^\infty$, which are respectively denoted by $\{V^{\varepsilon_k}\}_{k=1}^\infty$, $\{(\lambda^{\varepsilon_k}, C_1^{\varepsilon_k}, C_2^{\varepsilon_k})\}_{k=1}^\infty$, and $\{G^{\varepsilon_k}\}_{k=1}^\infty$ again, there exist $V^0 \in H^1(B_{\frac12}(0))^3$ and $(\lambda^0, C_1^0, C_2^0) \in [-1,1]^3$ such that in the limit $k\to\infty$,

$$\begin{aligned} &V^{\varepsilon_k} \;\to\; V^0 \;\text{ in }\; L^2(B_{\frac12}(0))^3\,, \qquad \nabla V^{\varepsilon_k} \;\rightharpoonup\; \nabla V^0 \;\text{ in }\; L^2(B_{\frac12}(0))^{3\times3}\,,\\ &(\lambda^{\varepsilon_k}, C_1^{\varepsilon_k}, C_2^{\varepsilon_k}) \to (\lambda^0, C_1^0, C_2^0) \;\text{ in }\; [-1,1]^3\,, \qquad G^{\varepsilon_k} \;\to\; 0 \;\text{ in }\; L^2(B_{\frac12}(0))^{3\times3}\,. \end{aligned}$$

On the other hand, the assumption (43) implies (41). Hence, from (14) with $m = 0$ in Lemma 2.3, by a similar reasoning as in the proof of Lemma 3.2 combined with the convergences

$$\begin{aligned} &V^{\varepsilon_k}\otimes V^{\varepsilon_k} \;\to\; V^0\otimes V^0 \;\text{ in }\; L^1(B_{\frac12}(0))^3\,,\\ &V^{\varepsilon_k}\otimes\big(\sum_{j=1}^{2} C_j^{\varepsilon_k}\varepsilon_k v^{(j)}(\frac{\cdot}{\varepsilon_k})\big) + \big(\sum_{j=1}^{2} C_j^{\varepsilon_k}\varepsilon_k v^{(j)}(\frac{\cdot}{\varepsilon_k})\big)\otimes V^{\varepsilon_k}\\ &\qquad\qquad \to\; 0 \;\text{ in }\; L^1(B_{\frac12}(0))^{3\times3}\,, \end{aligned}$$

we see that the limit V^0 gives a weak solution to (40) satisfying (41). Then, in the same way as in the proof of Lemma 3.2, we reach a contradiction to (44) from the choice of $\theta \in (0,\theta_0]$ in (42). Hence we obtain the desired estimate (39) yielding (37). This completes the proof. □

Next we prove the iteration lemma to the Navier-Stokes equations

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon & \text{in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1(0)\,. \end{cases} \tag{NS$^\varepsilon$}$$

An important step is the a priori choice of θ of Lemma 4.1 depending on the bound of the solution. Let K_0 be the constant in the Caccioppoli inequality in Appendix.

Lemma 4.2. *Fix $M \in (0,\infty)$ and $\mu \in (0,1)$, and let $\theta_0 \in (0,\frac{1}{8})$ be the constant in Lemma 4.1. Choose $\theta = \theta(M,\mu) \in (0,\theta_0]$ sufficiently small to satisfy the conditions*

$$\begin{aligned} &4K_0^{\frac12}(1-\theta^\mu)^{-1}(6+2^8M^4)^{\frac12}M\theta^{\frac12} \le 1\,, \\ &\big(C_1(1-\theta^\mu)^{-1}(6+2^8M^4)^{\frac12}M\theta^{\frac12}\big)^4 + (1-\theta)^{-\frac43}\big(C_1(1-\theta^\mu)^{-1}(6+2^8M^4)^{\frac12}M\theta^{\frac12}\big)^{\frac43} \\ &\le \frac{(1-\theta)^{-2}}{4}\,, \\ &\text{and}\quad C_2(1-\theta^\mu)^{-2}(6+2^8M^4)M\theta \le 1\,, \end{aligned} \tag{45}$$

where C_1 and C_2 are numerical constants appearing respectively in (55) *and* (56) *in the proof. Moreover, let $\varepsilon_\mu \in (0,1)$ be the corresponding constant for θ in Lemma 4.1. Then for $k \in \mathbb{N}$ and*

$$\varepsilon \in (0, \theta^{k+2(2+\mu)(1-\delta_{1k})-1}\varepsilon_\mu^{2-\delta_{1k}}]\,, \tag{46}$$

where δ_{1k} is the Kronecker delta, any weak solution $u^\varepsilon = (u_1^\varepsilon(x), u_2^\varepsilon(x), u_3^\varepsilon(x))^\top \in H^1(B^\varepsilon_{1,+}(0))^3$ to (NS$^\varepsilon$) *with*

$$⨍_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2 \le M^2 \tag{47}$$

satisfies

$$⨍_{B^\varepsilon_{\theta^k,+}(0)} \Big|u^\varepsilon(x) - \sum_{j=1}^2 a^\varepsilon_{k,j}\big(x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2 \,\mathrm{d}x \le M^2\theta^{(2+2\mu)k}\,. \tag{48}$$

Here the number $a^\varepsilon_{k,j} \in \mathbb{R}$, $j \in \{1,2\}$, is estimated as

$$\sum_{j=1}^2 |a^\varepsilon_{k,j}| \le K_0^{\frac12}\theta^{-\frac32}(1-\theta)^{-1}\big(6+2^6(1-\theta)^{-2}M^4\big)^{\frac12}M\sum_{l=1}^k \theta^{\mu(l-1)}\,. \tag{49}$$

Proof. The proof is done by induction on $k \in \mathbb{N}$. For $k=1$, from $\varepsilon \in (0,\varepsilon_\mu]$, we can apply Lemma 4.1 to (NS$^\varepsilon$) by putting in (MNS$^\varepsilon$),

$$(U^\varepsilon, P^\varepsilon) = (u^\varepsilon, p^\varepsilon)\,, \qquad \lambda^\varepsilon = 1\,, \qquad C^\varepsilon_j = 0\,, \quad j \in \{1,2\}\,, \qquad F^\varepsilon = 0\,.$$

Thus, if we set $a^\varepsilon_{1,j} = (\overline{\partial_3 u^\varepsilon_j})_{B^\varepsilon_{\theta,+}(0)}$, $j \in \{1,2\}$, the assertion (48) for the case $k=1$ follows. Moreover, from (47), the Caccioppoli inequality in Lemma A.2 with $\rho = \theta$ and $r = 1$ leads to

$$\begin{aligned} \|\nabla u^{\varepsilon_k}\|^2_{L^2(B^{\varepsilon_k}_{\theta,+}(0))} &\le K_0(1-\theta)^{-2}\big(\|u^\varepsilon\|^2_{L^2(B^\varepsilon_{1,+}(0))} + (1-\theta)^{-2}\|u^\varepsilon\|^6_{L^2(B^\varepsilon_{1,+}(0))}\big) \\ &\le K_0(1-\theta)^{-2}\big(4+2^6(1-\theta)^{-2}M^4\big)M^2\,. \end{aligned}$$

Hence we obtain (49) for $k = 1$ from

$$\begin{aligned}\sum_{j=1}^{2} |a^{\varepsilon}_{1,j}| &\le 2|B^{\varepsilon}_{\theta,+}(0)|^{-\frac{1}{2}} \|\nabla u^{\varepsilon}\|_{L^2(B^{\varepsilon}_{\theta,+}(0))} \\ &\le K_0^{\frac{1}{2}} \theta^{-\frac{3}{2}} (1-\theta)^{-1} \big(4 + 2^6 (1-\theta)^{-2} M^4\big)^{\frac{1}{2}} M \,.\end{aligned} \tag{50}$$

Next we assume that (48) and (49) hold for $k \in \mathbb{N}$ and let $\varepsilon \in (0, \theta^{k+2(2+\mu)} \varepsilon_{\mu}^2]$. We define $U^{\varepsilon/\theta^k} = U^{\varepsilon/\theta^k}(y)$ and $P^{\varepsilon/\theta^k} = P^{\varepsilon/\theta^k}(y)$ on $B^{\varepsilon/\theta^k}_{1,+}(0)$ by

$$U^{\varepsilon/\theta^k}(y) = \frac{1}{\theta^{(1+\mu)k}} \Big(u^{\varepsilon}(\theta^k y) - \sum_{j=1}^{2} \theta^k a^{\varepsilon}_{k,j} \big(y_3 \mathbf{e}_j + \frac{\varepsilon}{\theta^k} v^{(j)}(\frac{\theta^k y}{\varepsilon}) \big) \Big) \,,$$

$$P^{\varepsilon/\theta^k}(y) = \frac{1}{\theta^{\mu k}} \Big(p^{\varepsilon}(\theta^k y) - \sum_{j=1}^{2} a^{\varepsilon}_{k,j} q^{(j)}(\frac{\theta^k y}{\varepsilon}) \Big) \,.$$

After a direct computation, we see that $(U^{\varepsilon/\theta^k}, P^{\varepsilon/\theta^k})$ is a weak solution to

$$\begin{cases} -\Delta_y U^{\varepsilon/\theta^k} + \nabla_y P^{\varepsilon/\theta^k} = -\nabla_y \cdot \big(U^{\varepsilon/\theta^k} \otimes (\theta^k b^{\varepsilon/\theta^k}) + (\theta^k b^{\varepsilon/\theta^k}) \otimes U^{\varepsilon/\theta^k} \big) \\ \qquad\qquad - \theta^{(2+\mu)k} U^{\varepsilon/\theta^k} \cdot \nabla_y U^{\varepsilon/\theta^k} \\ \qquad\qquad + \nabla_y \cdot F^{\varepsilon/\theta^k} \quad \text{in } B^{\varepsilon/\theta^k}_{1,+}(0) \\ \nabla_y \cdot U^{\varepsilon/\theta^k} = 0 \quad \text{in } B^{\varepsilon/\theta^k}_{1,+}(0) \\ U^{\varepsilon/\theta^k} = 0 \quad \text{on } \Gamma^{\varepsilon/\theta^k}_1(0) \,, \end{cases} \tag{51}$$

where $b^{\varepsilon/\theta^k} = b^{\varepsilon/\theta^k}(y)$ and $F^{\varepsilon/\theta^k} = F^{\varepsilon/\theta^k}(y)$ are respectively defined by

$$b^{\varepsilon/\theta^k}(y) = \sum_{j=1}^{2} C^{\varepsilon}_{j,k} \big(y_3 \mathbf{e}_j + \frac{\varepsilon}{\theta^k} v^{(j)}(\frac{\theta^k y}{\varepsilon}) \big) \,, \qquad C^{\varepsilon}_{j,k} = \theta^k a^{\varepsilon}_{k,j} \,,$$

$$F^{\varepsilon/\theta^k}(y) = -\theta^{-\mu k} \Big(b^{\varepsilon/\theta^k}(y) \otimes b^{\varepsilon/\theta^k}(y) - \big(\sum_{j=1}^{2} C^{\varepsilon}_{j,k} y_3 \mathbf{e}_j \big) \otimes \big(\sum_{j=1}^{2} C^{\varepsilon}_{j,k} y_3 \mathbf{e}_j \big) \Big) \,.$$

Note that $\nabla_y \cdot b^{\varepsilon/\theta^k} = 0$ in $B^{\varepsilon/\theta^k}_{1,+}(0)$ and $b^{\varepsilon/\theta^k} = 0$ on $\Gamma^{\varepsilon/\theta^k}_1(0)$. Moreover, we can subtract

$$\big(\sum_{j=1}^{2} C^{\varepsilon}_{j,k} y_3 \mathbf{e}_j \big) \otimes \big(\sum_{j=1}^{2} C^{\varepsilon}_{j,k} y_3 \mathbf{e}_j \big)$$

from $b^{\varepsilon/\theta^k} \otimes b^{\varepsilon/\theta^k}$ beforehand, since it vanishes if we take its divergence. This is indeed a crucial fact in the following proof where we cancel singularities in θ^{-1} by choosing ε small with respect to θ as in (46). From the recurrence hypothesis, (48) at rank k, we also have

$$\fint_{B^{\varepsilon/\theta^k}_{1,+}(0)} |U^{\varepsilon/\theta^k}|^2 \le M^2 \tag{52}$$

by a change of variables. Let us estimate b^{ε/θ^k} and F^{ε/θ^k}. From the recurrence hypothesis

$$\sum_{j=1}^{2} |a^{\varepsilon}_{k,j}| \le 4 K_0^{\frac{1}{2}} \theta^{-\frac{3}{2}} (1 - \theta^{\mu})^{-1} (6 + 2^8 M^4)^{\frac{1}{2}} M \tag{53}$$

holds, where $(1-\theta)^{-1} \leq 2$ was used. We have uniformly in $k \in \mathbb{N}$,

$$|\theta^k C^\varepsilon_{j,k}| \leq 4K_0^{\frac{1}{2}}(1-\theta^\mu)^{-1}(6+2^8M^4)^{\frac{1}{2}}M\theta^{\frac{1}{2}} \leq 1 \tag{54}$$

by (45). Moreover, by (13) in Lemma 2.3 and $\varepsilon \in (0, \theta^{k+2(2+\mu)}\varepsilon_\mu^2]$, we see that

$$\begin{aligned}\|\nabla_y(\theta^k b^{\varepsilon/\theta^k})\|_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} &\leq C\big(\sum_{j=1}^2 |\theta^k C^\varepsilon_{j,k}|\big)(1+\varepsilon^{\frac{1}{2}}\theta^{-\frac{k}{2}}) \\ &\leq C_1(1-\theta^\mu)^{-1}(6+2^8M^4)^{\frac{1}{2}}M\theta^{\frac{1}{2}}\,,\end{aligned} \tag{55}$$

where C_1 is independent of k, M, θ, and ε, while the definition of F^{ε/θ^k} implies that for $y \in B^{\varepsilon/\theta^k}_{1,+}(0)$,

$$|F^{\varepsilon/\theta^k}(y)| \leq C\theta^{(2-\mu)k}\big(\sum_{j=1}^2|a^\varepsilon_{k,j}|\big)^2 \sum_{j=1}^2 \big((\frac{\varepsilon}{\theta^k})\big|v^{(j)}(\frac{\theta^k y}{\varepsilon})\big| + (\frac{\varepsilon}{\theta^k})^2\big|v^{(j)}(\frac{\theta^k y}{\varepsilon})\big|^2\big)\,.$$

Thus, from (14) with $m=0$ and $m=2$ in Lemma 2.3, we have again by $\varepsilon \in (0, \theta^{k+2(2+\mu)}\varepsilon_\mu^2]$,

$$\begin{aligned}\|F^{\varepsilon/\theta^k}\|_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} &\leq C\theta^{(2-\mu)k-3}(1-\theta^\mu)^{-2}(6+2^8M^4)M^2(\varepsilon^{\frac{1}{2}}\theta^{-\frac{k}{2}}+\varepsilon\theta^{-k}) \\ &\leq C\theta^{-1-\mu}(1-\theta^\mu)^{-2}(6+2^8M^4)M^2(\varepsilon_\mu\theta^{2+\mu}+\varepsilon_\mu^2\theta^{2(2+\mu)}) \\ &\leq \big(C_2(1-\theta^\mu)^{-2}(6+2^8M^4)M\theta\big)M\varepsilon_\mu\,,\end{aligned} \tag{56}$$

where C_2 is independent of k, M, θ, and ε. Then, from (52) combined with (54) and (56) under (45), since $\varepsilon/\theta^k \in (0, \varepsilon_\mu]$, we can apply Lemma 4.1 to (51) by putting

$$\begin{aligned}(U^\varepsilon, P^\varepsilon) &= (U^{\varepsilon/\theta^k}, P^{\varepsilon/\theta^k})\,, \qquad \lambda^\varepsilon = \theta^{(2+\mu)k}\,, \\ C^\varepsilon_j &= \theta^k C^\varepsilon_{j,k}\,, \quad j \in \{1,2\}\,, \qquad F^\varepsilon = F^{\varepsilon/\theta^k}\end{aligned}$$

in (MNS$^\varepsilon$) and find that

$$\begin{aligned}&\fint_{B^{\varepsilon/\theta^k}_{\theta,+}(0)} \Big|U^{\varepsilon/\theta^k}(y) - \sum_{j=1}^2 \overline{(\partial_{y_3}U_j^{\varepsilon/\theta^k})}_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\big(y_3\mathbf{e}_j + \frac{\varepsilon}{\theta^k}v^{(j)}(\frac{\theta^k y}{\varepsilon})\big)\Big|^2\,\mathrm{d}y \\ &\leq M^2\theta^{2+2\mu}\,.\end{aligned}$$

A change of variables yields that

$$\fint_{B^\varepsilon_{\theta^{k+1},+}(0)} \Big|u^\varepsilon(x) - \sum_{j=1}^2 a^\varepsilon_{k+1,j}\big(x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\Big|^2\,\mathrm{d}x \leq M^2\theta^{(2+2\mu)(k+1)}\,, \tag{57}$$

where the number $a^\varepsilon_{k+1,j}$, $j \in \{1,2\}$, is defined as

$$a^\varepsilon_{k+1,j} = a^\varepsilon_{k,j} + \theta^{\mu k}\overline{(\partial_{y_3}U_j^{\varepsilon/\theta^k})}_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\,. \tag{58}$$

Let us estimate $a^\varepsilon_{k+1,j}$. By (52) and (55) under (45) we have

$$\begin{aligned}&\Big(\|\nabla_y(\theta^k b^{\varepsilon/\theta^k})\|^4_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} + (1-\theta)^{-\frac{4}{3}}\|\nabla_y(\theta^k b^{\varepsilon/\theta^k})\|^{\frac{4}{3}}_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))}\Big)\|U^{\varepsilon/\theta^k}\|^2_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} \\ &\leq (1-\theta)^{-2}M^2\,.\end{aligned}$$

Then (56) under (45) and the Caccioppoli inequality applied to (51) with $\rho = \theta$ and $r = 1$ lead to

$$\begin{aligned}
&\|\nabla_y U^{\varepsilon/\theta^k}\|^2_{L^2(B^{\varepsilon/\theta^k}_{\theta,+}(0))} \\
&\le K_0(1-\theta)^{-2}\Big(\|U^{\varepsilon/\theta^k}\|^2_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} + (1-\theta)^{-2}\|U^{\varepsilon/\theta^k}\|^6_{L^2(B^{\varepsilon/\theta^k}_{1,+}(0))} + 2M^2\Big) \\
&\le K_0(1-\theta)^{-2}\big(6 + 2^6(1-\theta)^{-2}M^4\big)M^2\,.
\end{aligned}$$

Therefore from the Hölder inequality we obtain

$$\begin{aligned}
\sum_{j=1}^{2}\Big|\big(\overline{\partial_{y_3}U_j^{\varepsilon/\theta^k}}\big)_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\Big| &\le 2|B^{\varepsilon/\theta^k}_{\theta,+}(0)|^{-\frac12}\|\nabla_y U^{\varepsilon/\theta^k}\|_{L^2(B^{\varepsilon/\theta^k}_{\theta,+}(0))} \\
&\le K_0^{\frac12}\theta^{-\frac32}(1-\theta)^{-1}\big(6 + 2^6(1-\theta)^{-2}M^4\big)^{\frac12}M\,.
\end{aligned}$$

Thus, by the recurrence hypothesis, (49) at rank k, and (58), we have

$$\begin{aligned}
\sum_{j=1}^{2}|a^{\varepsilon}_{k+1,j}| &\le \sum_{j=1}^{2}|a^{\varepsilon}_{k,j}| + \theta^{\mu k}\sum_{j=1}^{2}\Big|\big(\overline{\partial_{y_3}U_j^{\varepsilon/\theta^k}}\big)_{B^{\varepsilon/\theta^k}_{\theta,+}(0)}\Big| \\
&\le K_0^{\frac12}\theta^{-\frac32}(1-\theta)^{-1}\big(6 + 2^6(1-\theta)^{-2}M^4\big)^{\frac12}M\sum_{l=1}^{k+1}\theta^{\mu(l-1)}\,,
\end{aligned}$$

which with (57) proves (48) and (49) at rank $k+1$. This completes the proof. □

4.2 Proof of Theorems 1.1 and 1.3

Firstly we prove Theorem 1.1 by applying Lemma 4.2. Throughout this subsection, for given $M \in (0,\infty)$ and $\mu \in (0,1)$, let $\theta \in (0,\frac18)$ and $\varepsilon_\mu \in (0,1)$ be the corresponding constants in Lemma 4.2. Note that, for any $k \in \mathbb{N}$, we have

$$(0, \theta^{k-1}(\theta^{2(2+\mu)}\varepsilon_\mu^2)] \subset (0, \theta^{k+2(2+\mu)(1-\delta_{1k})-1}\varepsilon_\mu^{2-\delta_{1k}}]\,.$$

Proof of Theorem 1.1: We fix $\mu \in (0,1)$ and set $\varepsilon^{(1)} = \theta^{2(2+\mu)}\varepsilon_\mu^2$. Let $\varepsilon \in (0,\varepsilon^{(1)}]$. As in the proof of Theorem 3.1 in Subsection 3.2, we can focus on the case $r \in [\varepsilon/\varepsilon^{(1)}, \theta]$. For any given $r \in [\varepsilon/\varepsilon^{(1)}, \theta]$, there exists $k \in \mathbb{N}$ with $k \ge 2$ such that $r \in (\theta^k, \theta^{k-1}]$. From the bound (3) and $\varepsilon \in (0, \theta^{k-1}\varepsilon^{(1)}]$, one can apply Lemma 4.2. By using an easy estimate of $a^{\varepsilon}_{k,j} \in \mathbb{R}$, $j \in \{1,2\}$:

$$\sum_{j=1}^{2}|a^{\varepsilon}_{k,j}| \le C\theta^{-\frac32}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}M \tag{59}$$

with a constant C depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, we see that

$$
\begin{aligned}
&\Big(\fint_{B^\varepsilon_{r,+}(0)} |u^\varepsilon|^2\Big)^{\frac12} \le \Big(\theta^{-3}\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} |u^\varepsilon|^2\Big)^{\frac12}\\
&\le \theta^{-\frac32}\Big(\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} \big|u^\varepsilon(x) - \sum_{j=1}^2 a^\varepsilon_{k-1,j}\big(x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x\Big)^{\frac12}\\
&\quad + \theta^{-\frac32}\Big(\sum_{j=1}^2 |a^\varepsilon_{k-1,j}|\Big)\Big(\sum_{j=1}^2\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} \big|x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x\Big)^{\frac12}\\
&\le M\theta^{(1+\mu)(k-1)-\frac32}\\
&\quad + C\theta^{-3}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}M\Big(\sum_{j=1}^2\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} \big|x_3\mathbf{e}_j + \varepsilon v^{(j)}(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x\Big)^{\frac12}.
\end{aligned}
$$

Then, in the same way as in the proof of Theorem 3.1, we have

$$
\Big(\fint_{B^\varepsilon_{r,+}(0)} |u^\varepsilon|^2\Big)^{\frac12} \le \Big(\theta^{-\frac52-\mu}r^\mu + C\theta^{-4}(1-\theta^\mu)^{-1}(1+(\varepsilon^{(1)})^{\frac12})(1+M^4)^{\frac12}\Big)Mr\,.
$$

Hence we obtain the assertion (4) by letting $\mu = \frac12$ for instance and by defining $C^{(1)}_M$ by

$$
C^{(1)}_M = \Big(\theta^{-3} + C\theta^{-4}(1-\theta^{\frac12})^{-1}(1+(\varepsilon^{(1)})^{\frac12})(1+M^4)^{\frac12}\Big)M\,.
$$

Indeed, it is easy to see that $C^{(1)}_M$ increases in M if one chooses θ to be the supremum of the numbers θ satisfying (45) with $\mu = \frac12$. Moreover $C^{(1)}_M$ converges to zero when $M \to 0$ from this choice of θ. The proof is complete if we combine the trivial estimate for $r \in (\theta, 1]$. □

Next we prove Theorem 1.3. Let $\alpha^{(j)} \in \mathbb{R}^3$, $j \in \{1,2\}$, be the constant vector in Proposition 2.2.

Proof of Theorem 1.3: As in the proof of Theorem 1.1, we set $\varepsilon^{(2)} = \theta^{2(2+\mu)}\varepsilon_\mu^2$ and take $\varepsilon \in (0, \varepsilon^{(2)}]$.

(i) We focus on the case $r \in [\varepsilon/\varepsilon^{(1)}, \theta]$ again as in the proof of Theorem 1.1. Since every $r \in [\varepsilon/\varepsilon^{(2)}, \theta]$ satisfies $r \in (\theta^k, \theta^{k-1}]$ with some $k \in \mathbb{N}$ satisfying $k \ge 2$ we have

$$
\begin{aligned}
&\Big(\fint_{B^\varepsilon_{r,+}(0)} \big|u^\varepsilon(x) - \sum_{j=1}^2 a^\varepsilon_{k-1,j}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\Big)^{\frac12}\\
&\le \Big(\theta^{-3}\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} \big|u^\varepsilon(x) - \sum_{j=1}^2 a^\varepsilon_{k-1,j}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\Big)^{\frac12}\\
&\le M\theta^{(1+\mu)(k-1)-\frac32}\\
&\quad + C\theta^{-3}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}M\varepsilon\Big(\sum_{j=1}^2\fint_{B^\varepsilon_{\theta^{k-1},+}(0)} \big|v^{(j)}(\frac{x}{\varepsilon})\big|^2\,\mathrm{d}x\Big)^{\frac12},
\end{aligned}
$$

where Lemma 4.2 has been applied in the third line. The estimate (59) for $a^\varepsilon_{k-1,j} \in \mathbb{R}$, $j \in \{1,2\}$, is also used in the same line. Then (14) with $m=0$ in Lemma 2.3 and $\theta^{k-1}\in(0,\theta^{-1}r)$ lead to

$$\begin{aligned}&\Big(\fint_{B^\varepsilon_{r,+}(0)}\big|u^\varepsilon(x)-\sum_{j=1}^{2}a^\varepsilon_{k-1,j}x_3\mathbf{e}_j\big|^2\,\mathrm{d}x\Big)^{\frac12}\\&\le\Big(\theta^{-\frac52-\mu}r^{1+\mu}+C\theta^{-\frac72}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}\varepsilon^{\frac12}r^{\frac12}\Big)M\,.\end{aligned}$$

Hence we obtain the assertion (5) by defining $c^\varepsilon_{r,j}$ and $C^{(2)}_M$ by

$$c^\varepsilon_{r,j}=a^\varepsilon_{k-1,j}\,,\qquad C^{(2)}_M=\Big(\theta^{-\frac52-\mu}+C\theta^{-\frac72}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}\Big)M\,,\tag{60}$$

and by combining the trivial estimate for $r\in(\theta,1]$.

(ii) In a similar way as in (i), for $r\in[\varepsilon/\varepsilon^{(2)},\theta]$ with $r\in(\theta^k,\theta^{k-1}]$, we have

$$\begin{aligned}&\Big(\fint_{B^\varepsilon_{r,+}(0)}\big|u^\varepsilon(x)-\sum_{j=1}^{2}c^\varepsilon_{r,j}(x_3\mathbf{e}_j+\varepsilon\alpha^{(j)})\big|^2\,\mathrm{d}x\Big)^{\frac12}\\&\le M\theta^{(1+\mu)(k-1)-\frac32}\\&\quad+C\theta^{-3}(1-\theta^\mu)^{-1}(1+M^4)^{\frac12}M\varepsilon\Big(\sum_{j=1}^{2}\fint_{B^\varepsilon_{\theta^{k-1},+}(0)}\big|v^{(j)}(\tfrac{x}{\varepsilon})-\alpha^{(j)}\big|^2\,\mathrm{d}x\Big)^{\frac12},\end{aligned}\tag{61}$$

where Lemma 4.2 and the estimate (59) are applied again. Moreover, the notation $c^\varepsilon_{r,j}=a^\varepsilon_{k-1,j}$ in (60) is used. Then (14) with $m=0$ in Lemma 2.3 and (12) in Proposition 2.2 lead to

$$\begin{aligned}&\Big(\sum_{j=1}^{2}\fint_{B^\varepsilon_{\theta^{k-1},+}(0)}\big|v^{(j)}(\tfrac{x}{\varepsilon})-\alpha^{(j)}\big|^2\,\mathrm{d}x\Big)^{\frac12}\\&\le\theta^{-\frac32(k-1)}\Big(\sum_{j=1}^{2}\int_{B^\varepsilon_{2\varepsilon,+}(0)}\big|v^{(j)}(\tfrac{x}{\varepsilon})-\alpha^{(j)}\big|^2\,\mathrm{d}x\Big)^{\frac12}\\&\quad+\theta^{-\frac32(k-1)}\Big(\sum_{j=1}^{2}\int_{(-\theta^{k-1},\theta^{k-1})^2}\int_{\varepsilon}^{\theta^{k-1}}\big|v^{(j)}(\tfrac{x}{\varepsilon})-\alpha^{(j)}\big|^2\,\mathrm{d}x_3\,\mathrm{d}x'\Big)^{\frac12}\\&\le C(\varepsilon^{\frac32}\theta^{-\frac32(k-1)}+\varepsilon^{\frac12}\theta^{-\frac12(k-1)})\,.\end{aligned}$$

Hence, by $\theta^{k-1}\in(0,\theta^{-1}r)$, $r^{-1}\in[\theta^{-(k-1)},\theta^{-k})$, and $\varepsilon\in(0,\theta^{k-1}\varepsilon^{(2)}]$, from (61) we find

$$\begin{aligned}&\Big(\fint_{B^\varepsilon_{r,+}(0)}\big|u^\varepsilon(x)-\sum_{j=1}^{2}c^\varepsilon_{r,j}(x_3\mathbf{e}_j+\varepsilon\alpha^{(j)})\big|^2\,\mathrm{d}x\Big)^{\frac12}\\&\le\Big(\theta^{-\frac52-\mu}r^{1+\mu}+C\theta^{-3}(1-\theta^\mu)^{-1}(1+\varepsilon^{(2)})(1+M^4)^{\frac12}\varepsilon^{\frac32}r^{-\frac12}\Big)M\,.\end{aligned}$$

The assertion (6) follows by setting

$$\widetilde{C^{(2)}_M}=\Big(\theta^{-\frac52-\mu}+C\theta^{-3}(1-\theta^\mu)^{-1}(1+\varepsilon^{(2)})(1+M^4)^{\frac12}\Big)M\,.$$

This completes the proof by using the trivial estimate for $r\in(\theta,1]$. □

A Appendix

In this appendix we state a few technical lemmas without proof. The proofs can be found in [19, Appendices A and B]. The first one is about the regularity results for

$$\begin{cases} -\Delta u + \nabla p = -\nabla \cdot (u \otimes b + b \otimes u) - \lambda u \cdot \nabla u & \text{in } B_{\frac{1}{2},+}(0) \\ \nabla \cdot u = 0 & \text{in } B_{\frac{1}{2},+}(0) \\ u = 0 & \text{on } \Gamma_{\frac{1}{2}}(0)\,, \end{cases} \tag{62}$$

where $b = b(x)$ is defined as $b(x) = \sum_{j=1}^{2} C_j x_3 \mathbf{e}_j$.

Lemma A.1. *Let $(\lambda, C_1, C_2) \in \mathbb{R}^3$ and let $(u, p) \in H^1(B_{\frac{1}{2},+}(0))^3 \times L^2(B_{\frac{1}{2},+}(0))$ be a weak solution to (62). Then for all $r \in (0, \frac{7}{16})$, we have*

$$u \in C^\infty(\overline{B_{r,+}(0)})^3\,, \qquad p \in C^\infty(\overline{B_{r,+}(0)})\,, \tag{63}$$

and for all $k \in \mathbb{N} \cup \{0\}$, we have

$$\|u\|_{C^k(\overline{B_{r,+}(0)})} \leq K_1\,, \tag{64}$$

where the constant K_1 depends nonlinearly on (λ, C_1, C_2), $\|u\|_{L^2(B_{\frac{1}{2},+}(0))}$, and k.

The second one is about the Caccioppoli inequality for

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -\nabla \cdot (b^\varepsilon \otimes u^\varepsilon + u^\varepsilon \otimes b^\varepsilon) - \lambda^\varepsilon (u^\varepsilon \cdot \nabla u^\varepsilon) + \nabla \cdot F^\varepsilon & \text{in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1(0)\,. \end{cases} \tag{65}$$

Lemma A.2. *Let $\varepsilon \in [0, 1)$, $b^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ with $b^\varepsilon = 0$ on $\Gamma^\varepsilon_1(0)$, $\lambda^\varepsilon \in [0, 1]$, and $F^\varepsilon \in L^2(B^\varepsilon_{1,+}(0))^{3\times3}$, and let $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ be a weak solution to (65). Then we have for all $0 < \rho < r \leq 1$,*

$$\begin{aligned} \|\nabla u^\varepsilon\|^2_{L^2(B^\varepsilon_{\rho,+}(0))} \leq K_0 \Bigg(& \frac{1}{(r-\rho)^2} \|u^\varepsilon\|^2_{L^2(B^\varepsilon_{r,+}(0))} \\ & + \left(\|\nabla b^\varepsilon\|^4_{L^2(B^\varepsilon_{r,+}(0))} + \frac{\|\nabla b^\varepsilon\|^{\frac{4}{3}}_{L^2(B^\varepsilon_{r,+}(0))}}{(r-\rho)^{\frac{4}{3}}} \right) \|u^\varepsilon\|^2_{L^2(B^\varepsilon_{r,+}(0))} \\ & + \frac{(\lambda^\varepsilon)^4}{(r-\rho)^4} \|u^\varepsilon\|^6_{L^2(B^\varepsilon_{r,+}(0))} + \|F^\varepsilon\|^2_{L^2(B^\varepsilon_{r,+}(0))} \Bigg)\,, \end{aligned} \tag{66}$$

where the constant K_0 depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$. In particular it is independent of ε, b^ε, λ^ε, ρ, and r.

References

[1] Armstrong, S., Kuusi, T., Mourrat, J.-C.: Quantitative stochastic homogenization and large-scale regularity. Springer, Cham (2019)

[2] Armstrong, S., Kuusi, T., Mourrat, J.-C., Prange, C.: Quantitative analysis of boundary layers in periodic homogenization. Arch. Ration. Mech. Anal. **226**(2), 695–741 (2017)

[3] Armstrong, S. N., Shen, Z.: Lipschitz estimates in almost-periodic homogenization. Comm. Pure Appl. Math. **69**(10), 1882–1923 (2016)

[4] Avellaneda, M., Lin, F.-H.: Compactness methods in the theory of homogenization. Comm. Pure Appl. Math. **40**(6), 803–847 (1987)

[5] Avellaneda, M., Lin, F.-H.: Compactness methods in the theory of homogenization. II. Equations in nondivergence form. Comm. Pure Appl. Math. **42**(2), 139–172 (1989)

[6] Beirão da Veiga, H., Berselli, L. C.: Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary. J. Differential Equations **246**(2), 597–628 (2009)

[7] Constantin, P., Fefferman, C.: Direction of vorticity and the problem of global regularity for the Navier-Stokes equations. Indiana Univ. Math. J. **42**(3), 775–789 (1993)

[8] Fransson, J. H. M., Talamelli, A., Brandt, L., Cossu, C.: Delaying transition to turbulence by a passive mechanism. Phys. Rev. Lett. **96**(064501), 1–4 (2006)

[9] Geng, J., Shen, Z.: Homogenization of parabolic equations with non-self-similar scales. Arch. Ration. Mech. Anal. **236**, 145188 (2020).

[10] Gérard-Varet, D.: The Navier wall law at a boundary with random roughness. Comm. Math. Phys. **286**(1), 81–110 (2009)

[11] Gérard-Varet, D., Lacave C., Nguyen T. T., Rousset, F.: The vanishing viscosity limit for 2D Navier-Stokes in a rough domain. J. Math. Pures Appl. (9) **119**, 45–84 (2018)

[12] Gérard-Varet, D., Masmoudi, N.: Relevance of the slip condition for fluid flows near an irregular boundary. Comm. Math. Phys. **295**(1), 99–137 (2010)

[13] Gérard-Varet, D., Masmoudi, N.: Homogenization and boundary layers. Acta Math. **209**(1), 133–178 (2012)

[14] Giaquinta, M., Modica, G.: Nonlinear systems of the type of the stationary Navier-Stokes system. J. Reine Angew. Math. **330**, 173–214 (1982)

[15] Gloria, A., Neukamm S., Otto F.: A regularity theory for random elliptic operators. Milan J. Math. (2020)

[16] Gu, S., Shen Z.: Homogenization of Stokes systems and uniform regularity estimates. SIAM J. Math. Anal. **47**(5), 4025–4057 (2015)

[17] Gu, S., Zhuge J.: Periodic homogenization of Green's functions for Stokes systems. Calc. Var. Partial Differential Equations **58**(3), Art. 114, 46 (2019)

[18] Higaki, M.: Navier wall law for nonstationary viscous incompressible flows. J. Differential Equations **260**(10), 7358–7396 (2016)

[19] Higaki, M., Prange, C.: Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law. arXiv e-prints, page arXiv:1911.12609 (Nov 2019)

[20] Iftimie, D., Sueur, F.: Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. Arch. Ration. Mech. Anal. **199**(1), 145–175 (2011)

[21] Jäger, W., Mikelić, A.: On the roughness-induced effective boundary conditions for an incompressible viscous flow. J. Differential Equations **170**(1), 96–122 (2001)

[22] Jiménez, J.: Turbulent flows over rough walls. Annu. Rev. Fluid Mech. **36**(1), 173–196 (2004)

[23] Kenig, C., Prange, C.: Uniform Lipschitz estimates in bumpy half-spaces. Arch. Ration. Mech. Anal. **216**(3), 703–765 (2015)

[24] Kenig, C., Prange, C.: Improved regularity in bumpy Lipschitz domains. J. Math. Pures Appl. (9) **113**, 1–36 (2018)

[25] Kenig, C. E., Shen, Z.: Homogenization of elliptic boundary value problems in Lipschitz domains. Math. Ann. **350**(4), 867–917 (2011)

[26] Kenig, C. E., Shen, Z.: Layer potential methods for elliptic homogenization problems. Comm. Pure Appl. Math. **64**(1), 1–44 (2011)

[27] Ladyženskaja, O. A.: The mathematical theory of viscous incompressible flow. 2nd edition. Science Publishers, New York-London-Paris (1969)

[28] Lin, F.: A new proof of the Caffarelli-Kohn-Nirenberg theorem. Comm. Pure Appl. Math. **51**(3), 241–257 (1998)

[29] Mikelić, A.: Rough boundaries and wall laws. In: Qualitative properties of solutions to partial differential equations. Volume 5 of Jindřich Nečas Cent. Math. Model. Lect. Notes, pp. 103–134. Matfyzpress, Prague (2009)

[30] Mikelić, A., Nečasová, Š., Neuss-Radu, M.: Effective slip law for general viscous flows over an oscillating surface. Math. Methods Appl. Sci. **36**(15), 2086–2100 (2013)

[31] Nguyen, T. T.: The inviscid limit of Navier-Stokes with critical Navier-slip boundary conditions for analytic data. arXiv e-prints, page arXiv:1904.12943 (Apr 2019)

[32] Niu, W., Shen Z., Xu Y.: Quantitative Estimates in Reiterated Homogenization. arXiv e-prints, page arXiv:1909.09513 (Sep 2019)

[33] Schlichting, H., Gersten, K.: Boundary-layer theory. Springer, Berlin-Germany (2016)

[34] Shen, Z.: Boundary estimates in elliptic homogenization. Anal. PDE **10**(3), 653–694 (2017)

[35] Shen, Z., Zhuge J.: Boundary layers in periodic homogenization of Neumann problems. Comm. Pure Appl. Math. **71**(11), 2163–2219 (2018)

Department of Mathematics
Kobe University
1-1 Rokkodai, Nada-ku, Kobe 657-8501
JAPAN
E-mail address: higaki@math.kobe-u.ac.jp