

Eigenvalues of graphs and spectral Moore theorems

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Abstract

In this paper, we describe some recent spectral Moore theorems related to determining the maximum order of a connected graph of given valency and second eigenvalue. We show how these spectral Moore theorems have applications in Alon-Boppana theorems for regular graphs and in the classical degree-diameter/Moore problem.

§1 Introduction

Our graph theoretic notation is standard (see [4, 5]). Let $\Gamma = (V, E)$ be an undirected graph with vertex set V and edge set E . Given $u, v \in V$, the distance $d(u, v)$ equals the minimum length of a path between u and v if such a path exists or ∞ otherwise. If Γ is connected, then all the distances between its vertices are finite and the diameter $\text{diam}(\Gamma)$ of Γ is defined as the maximum of $d(u, v)$, where the maximum is taken over all pairs $u, v \in V$. The pairwise distance and the diameter of a connected graph can be calculated efficiently using breadth first search. The Moore or degree-diameter problem is a classical problem in combinatorics (see [31]).

Problem 1. *Given $r \geq 3$ and $D \geq 2$, what is the maximum order $n_{r,D}$ of a connected r -regular graph of diameter D ?*

There is a well known upper bound for $n_{r,D}$ known as the Moore bound $m_{r,D}$ which is obtained as follows. If Γ is a connected r -regular graph of

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diameter D , then for any given vertex x in Γ and any $1 \leq j \leq D$, the number of vertices at distance j from x is at most $r(r-1)^{j-1}$. Therefore,

$$n_{r,D} \leq 1 + r + r(r-1) + \cdots + r(r-1)^{D-1}. \quad (1)$$

We denote by $m_{r,D}$ the right hand-side of the above inequality. When $r = 2$, it is straightforward to note that $n_{2,D} = m_{2,D} = 2D + 1$ and the maximum is attained by the cycle C_{2D+1} on $2D + 1$ vertices. For $D = 1$, it is easy to see that $n_{r,1} = m_{r,1} = r + 1$ and the maximum is attained by the complete graph K_{r+1} on $r + 1$ vertices.

For $D = 2$, a classical result of Hoffman and Singleton [20] gives that $n_{r,2}$ equals the Moore bound $m_{r,2} = r^2 + 1$ only when $r = 2$ (attained by the cycle C_5), $r = 3$ (the Petersen graph), $r = 7$ (the Hoffman-Singleton graph) or possibly $r = 57$. The existence of a 57-regular graph with diameter 2 on $57^2 + 1 = 3250$ vertices is a well known open problem in this area (see [10, 24, 28]). For $D \geq 3$ and $r \geq 3$, Damerell [11] and independently, Bannai and Ito [2] proved that there are no graphs attaining the Moore bound (1).

The adjacency matrix A is the $V \times V$ matrix whose (x, y) -th entry equals the number of edges between x and y . This matrix is a real symmetric matrix and if Γ is simple (no loops nor multiple edges), then A is a $(0, 1)$ symmetric matrix. Let $r \geq 3$ be a given integer. We will use the following family of orthogonal polynomials:

$$F_0(x) = 1, F_1(x) = x, F_2(x) = x^2 - r, \quad (2)$$

$$F_j(x) = xF_{j-1}(x) - (r-1)F_{j-2}(x), \quad (3)$$

for any $j \geq 3$. Let $q = \sqrt{k-1}$. The polynomials $(F_i)_{i \geq 0}$ form a sequence of orthogonal polynomials with respect to the positive weight

$$w(x) = \frac{\sqrt{4q^2 - x^2}}{k^2 - x^2}$$

on the interval $[-2q, 2q]$ (see [23, Section 4]). The polynomials $F_i(qy)/q^i$ in y are called Geronimus polynomials [18].

For any vertices u and v of Γ and any non-negative integer ℓ , the entry (u, v) of the matrix A^ℓ equals the number of walks of length ℓ between u and v . A walk $u = u_0, u_1, \dots, u_{\ell-1}, u_\ell = v$ in G is called non-backtracking if $u_i u_{i+1} \in E$ for any $0 \leq i \leq \ell - 1$ and $u_i \neq u_{i+2}$ for any $0 \leq i \leq \ell - 2$ (when $\ell \geq 2$). The following result goes back to Singleton [38].

Proposition 2 (Singleton [38]). *Let Γ be a connected r -regular graph with adjacency matrix A . For any vertices u and v of Γ and any non-negative integer ℓ , the entry (u, v) of the matrix $F_\ell(A)$ equals the number of non-backtracking walks of length ℓ between u and v .*

The eigenvalues of A are real and we denote them by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, where $n = |V|$. Sometimes, to highlight the dependence of the eigenvalues on a particular graph, we will use $\lambda_j(\Gamma)$ for λ_j . When Γ is r -regular and connected, it is known that $\lambda_1 = r$ and that $\lambda_2 < r$. It is also known that $\lambda_j \in [-r, r]$ and that $-r$ is an eigenvalue if and only if the graph is bipartite. The smallest eigenvalue of a regular graph has been used to determine the independence number of various interesting graphs (see Godsil and Meagher [17]).

The properties of the eigenvalues of a regular graph were essential in the proofs of Hoffman and Singleton [20] as well as Damerell [11] and Bannai and Ito [2].

The spectral gap $r - \lambda_2$ is an important parameter in spectral graph theory and is closely related to the connectivity [15] and expansion properties of the graph [19]. Informally, expanders are sparse graphs with large spectral gap. More precisely, a family $(\Gamma_m)_{m \geq 1}$ of graphs is called a family of expanders if

1. there exists $r \geq 3$ such that each Γ_m is a connected r -regular graph for $m \geq 1$ and the number of vertices of Γ_m goes to infinity as m goes to infinity,
2. there is a positive constant $c_r > 0$ such that $r - \lambda_2(\Gamma_m) > c_r$ for any Γ_m .

The first condition above explains the denomination of *sparse* used at the beginning of this paragraph. This condition implies that the number of edges in Γ_m is linear in its number of vertices for every $m \geq 1$. This is best possible in order of magnitude for connected graphs.

The second condition is algebraic and is equivalent to a combinatorial condition that each Γ_m is highly connected (meaning their expansion constants are bounded away from 0) and also equivalent to the probability condition that a random walk on Γ_m converges quickly to its stationary distribution. We refer to [19] for the precise descriptions of these conditions.

A natural question arising from the previous considerations is how large can the spectral gap $r - \lambda_2(\Gamma)$ be for an r -regular connected graph Γ ? Since we are interested in situations where $r \geq 3$ is fixed, this is equivalent to asking how small can λ_2 be for a connected r -regular graph Γ . This was answered by the Alon-Boppana theorem.

Theorem 3. *Let $r \geq 3$ be a natural number.*

1. **Alon-Boppana 1986.** *If Γ is a connected r -regular graph with n vertices, then*

$$\lambda_2(\Gamma) \geq 2\sqrt{r-1} \left(1 - \frac{C}{\text{diam}^2(\Gamma)} \right) = 2\sqrt{r-1}(1 - o(1)), \quad (4)$$

where $C > 0$ is a constant and $o(1)$ is a quantity that goes to 0 as n goes to infinity.

2. **Asymptotic Alon-Boppana Theorem.** *If $(\Gamma_m)_{m \geq 1}$ is a sequence of connected r -regular graphs such that $|V(\Gamma_m)| \rightarrow \infty$ as $m \rightarrow \infty$. Then*

$$\liminf_{m \rightarrow \infty} \lambda_2(\Gamma_m) \geq 2\sqrt{r-1}. \quad (5)$$

To my knowledge, there is no paper written by Alon and Boppana which contains the theorem above. The first appearance of this result that I am aware of, is in 1986 in Alon's paper [1] where it is stated that

R. Boppana and the present author showed that for every d -regular graph G on n vertices $\lambda(G) \leq d - 2\sqrt{d-1} + O(\log_d n)^{-1}$.

Note that the λ in [1] is the smallest positive eigenvalue of the Laplacian $D - A = rI - A$ of G and it equals $r - \lambda_2(G)$. Therefore the statement above is equivalent to

$$\lambda_2(G) \geq 2\sqrt{r-1} - O(\log_r n)^{-1}. \quad (6)$$

There is a similar result to the Alon-Boppana theorem that is due to Serre [37]. The meaning of this theorem below is that large r -regular graphs tend to have a positive proportion of eigenvalues trying to be greater than $2\sqrt{r-1}$.

Theorem 4 (Serre [37]). *For any $r \geq 3, \epsilon > 0$, there exists $c = c(\epsilon, r) > 0$ such that any r -regular graph Γ on n vertices has at least $c \cdot n$ eigenvalues that are at least $2\sqrt{r-1} - \epsilon$.*

These results motivated the definition of Ramanujan graphs that was introduced by Lubotzky, Phillips and Sarnak [27]. A connected r -regular graph Γ is called Ramanujan if all its eigenvalues (with the exception of r and perhaps $-r$, if Γ is bipartite) have absolute value at most $2\sqrt{r-1}$. Lubotzky, Phillips and Sarnak [27] and independently Margulis [30] constructed infinite families of r -regular Ramanujan graphs when $r-1$ is a prime. These constructions used results from algebra and number theory closely related to a conjecture of Ramanujan regarding the number of ways of writing a natural number as a sum of four squares of a certain kind (see [27, 30] and also, [12] for a more detailed description of these results). For the longest time, it was not known whether infinite families of r -regular Ramanujan graphs exist for any $r \geq 3$. Marcus, Spielman and Srivastava [29] obtained a breakthrough result by showing that there exist infinite families of bipartite r -regular Ramanujan graphs for any $r \geq 3$. Their method of interlacing polynomials has been fundamental to this proof and has found applications in other areas of mathematics as well.

§2 Spectral Moore theorems for general graphs

Throughout the years, several proofs of the Alon-Boppana theorem have appeared (see Lubotzky, Phillips and Sarnak [27], Nilli [33], Kahale [25], Friedman [16], Feng and Li [14], Li and Solé [26], Nilli [34] and Mohar [32]). Theorem 4 was proved by Serre [37] with a non-elementary proof (see also [12]). The first elementary proofs appeared around the same time by Cioabă [6] and Nilli [34]. Richey, Stover and Shutty [36] worked to turn Serre's proof into a quantitative theorem and asked the following natural question.

Problem 5. *Given an integer $r \geq 3$ and $\theta < 2\sqrt{r-1}$, what is the maximum order $v(r, \theta)$ of a r -regular graph Γ with $\lambda_2(\Gamma) \leq \theta$?*

These authors obtained several results involving $v(r, \theta)$. In this section, we describe our recent results related to the problem above and its bipartite and hypergraph versions. See [9, 8, 7] and the references therein for more details and other related problems. The method that is fundamental to all these results is due to Nozaki [35] who proved the *linear programming bound for graphs*.

Theorem 6 (Nozaki [35]). *Let Γ be a connected r -regular graph with v vertices and distinct eigenvalues $\theta_1 = k > \theta_2 > \dots > \theta_d$. If there exists a polynomial $f(x) = \sum_{i=0}^t f_i F_i(x)$ such that $f(r) > 0$, $f(\theta_i) \leq 0$ for any $2 \leq i \leq d$, $f_0 > 0$, and $f_i \geq 0$ for any $1 \leq i \leq t$, then*

$$v \leq \frac{f(r)}{f_0}.$$

Nozaki used this result to study the following problem.

Problem 7. *Given integers $v > r \geq 3$, what is the r -regular graph Γ on v vertices that has the smallest λ_2 among all r -regular graphs on v vertices?*

While similar to it, this problem is quite different from Problem 5.

In [8], the authors used Nozaki's LP bound for graphs to obtain the following general upper bound for $v(r, \theta)$.

Theorem 8 (Cioabă, Koolen, Nozaki and Vermette [8]). *Given integers $r, t \geq 3$ and a non-negative real number c , let $T(r, t, c)$ be the $t \times t$ tridiagonal matrix:*

$$T(r, t, c) = \begin{bmatrix} 0 & r & & & & & & & & \\ 1 & 0 & r-1 & & & & & & & \\ & 1 & 0 & r-1 & & & & & & \\ & & \cdot & \cdot & \cdot & & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & & 1 & 0 & r-1 & & \\ & & & & & & c & r-c & & \end{bmatrix}$$

If θ equals the second largest eigenvalue $\lambda_2(T(r, t, c))$ of the matrix $T(r, t, c)$, then

$$v(r, \theta) \leq 1 + \sum_{i=0}^{t-3} r(r-1)^i + \frac{r(r-1)^{t-2}}{c}.$$

We sketch below the ideas of the proof of this theorem. For $j \geq 0$, denote

$$G_j = \sum_{i=0}^j F_i, \tag{7}$$

where the F_i s are the orthogonal polynomials defined in equations (2) and (3). The polynomials $(G_j)_{j \geq 0}$ also form a family of orthogonal polynomials. They satisfy the following properties:

$$G_0(x) = 1, G_1(x) = x + 1, G_2(x) = x^2 + x - (r - 1) \tag{8}$$

$$G_j(x) = xG_{j-1} - (r - 1)G_{j-2}(x), \tag{9}$$

for $j \geq 3$.

The eigenvalues of the matrix $T = T(r, t, c)$ are the roots of $(x - r)(G_{t-1} + (c - 1)G_{t-2})$ and are distinct (see [8, Theorem 2.3]). If we denote them by $r = \lambda_1 > \lambda_2 > \dots > \lambda_t$, then the polynomial $f(x) = \frac{1}{c} \cdot (x - \lambda_2) \prod_{i \geq 3} (x - \lambda_i)^2$ satisfies $f(\lambda_i) \leq 0$ for $i \geq 2$. It is a bit more involved to check the other conditions from Theorem 6 and we refer the reader to [8] for the details to see how one can apply Nozaki's LP bound to f and obtain that

$$v(r, \theta) \leq \frac{f(r)}{f_0} = \sum_{i=0}^{t-2} F_i(r) + F_{t-1}(r)/c = 1 + \sum_{i=0}^{t-3} r(r-1)^i + \frac{r(r-1)^{t-2}}{c}.$$

To make things more clear, note the following result.

Proposition 9. *Let $r \geq 3$ be an integer. For any $\theta \in [-1, 2\sqrt{r-1})$, there exists an integer t and a positive number c such that θ is the second largest eigenvalue of the matrix $M(r, t, c)$.*

Let $\lambda^{(t)}$ denote the largest root of G_t and $\mu^{(t)}$ denote the largest root of F_t . Note that $\lambda^{(1)} = -1 < 0 = \mu^{(1)}$ and

$$\lambda^{(2)} = \frac{-1 + \sqrt{4r - 3}}{2} < \sqrt{r} = \mu^{(2)}. \tag{10}$$

Bannai and Ito [3, Section III.3] showed that $\lambda^{(t)} = 2\sqrt{r-1} \cos \tau$, where $\frac{\pi}{t+1} < \tau < \frac{\pi}{t}$. Because $F_t = G_t - G_{t-1}$, one can show that $\lambda^{(t)} < \mu^{(t)}$ for any t (see [8, Prop 2.6] for other properties of these eigenvalues). From the

remarks following Theorem 8, note that the second largest eigenvalue $\lambda_2(t, c)$ of $T(r, t, c)$ equals the largest root of the polynomial $(c - 1)G_{t-1} + G_{t-2}$. Because the roots of G_{t-2} and G_{t-1} interlace, one obtains that $\lambda_2(t, c)$ is a decreasing function in c and takes values between $\lim_{c \rightarrow \infty} \lambda_2(t, c) = \lambda^{(t-2)}$ and $\lim_{c \rightarrow 0} \lambda_2(t, c) = \mu^{(t-1)}$. Taking into the account what happens when $c = 1$, namely that $\lambda_2(t, c) = \lambda^{(t-1)}$, we obtain the following result.

Proposition 10. *For $c \in [1, \infty)$, $\lambda_2(t, c)$ takes any value in the interval $[\lambda^{(t-1)}, \lambda^{(t-2)}]$.*

Putting these things together, one deduces that $\lambda_2(t, c)$ can take any possible value between $\lambda_2(2, 1) = -1$ and $\lim_{t \rightarrow \infty} \lambda_2(t, c) = 2\sqrt{r - 1}$. There are several infinite families (r, θ) for which the precise values $v(r, \theta)$ have been determined in [8], but there are several open problems for relatively small values of r and θ . For example, $v(6, 2) \geq 42$ with an example of a 6-regular graph with $\lambda_2 = 2$ on 42 vertices being the 2nd subconstituent of the Hoffman-Singleton graph. Theorem 8 can give $v(6, 2) \leq 45$ (see [7]). Also, we know that $v(3, \sqrt{2}) = 14$ (Heawood graph), but we don't know the exact value of $v(k, \sqrt{2})$ for any $k \geq 3$. Lastly, $v(k, \sqrt{k})$ has been determined for $k = 3$ (equals 18 with Pappus graph as an example attaining it) and $k = 4$ (it is 35 with the odd graph O_4 meeting it), but we don't know it for $k \geq 5$.

§3 Alon-Boppana and Serre theorems

We point out the relevance of these results in the context of Alon-Boppana and Serre theorems. A typical Alon-Boppana result is of the form: if Γ is a connected r -regular graph with diameter $D \geq 2k$, then

$$\lambda_2(\Gamma) \geq 2\sqrt{r - 1} \cos \frac{\pi}{k + 1}. \tag{11}$$

See Friedman [16, Corollary 3.6] for the inequality above or Nilli [34, Theorem 1] for a slightly weaker bound. The equivalent contrapositive formulation of inequality (11) is the following: if Γ is a connected r -regular graph of diameter D with $\lambda_2(\Gamma) < 2\sqrt{r - 1} \cos \frac{\pi}{k+1}$, then $D < 2k$. By Moore bound (1), this implies that

$$|V(\Gamma)| \leq m_{r, 2k-1} = 1 + r + r(r - 1) + \dots + r(r - 1)^{2k-2}. \tag{12}$$

Obviously, the best bound one can achieve here is obtained for that k with $2\sqrt{r - 1} \cos \frac{\pi}{k} \leq \lambda_2(\Gamma) < 2\sqrt{r - 1} \cos \frac{\pi}{k+1}$. When applying Theorem 8, let θ be a real number such that $2\sqrt{r - 1} \cos \frac{\pi}{k} \leq \theta < 2\sqrt{r - 1} \cos \frac{\pi}{k+1}$. Given the

properties of the largest roots $\lambda^{(t)}$ of the polynomials G_t (see Proposition 10 and the paragraph containing it), we must have that either $\theta \in (\lambda^{(k-1)}, \lambda^{(k)})$ or $\theta \in (\lambda^{(k)}, \lambda^{(k+1)})$. If $\theta \in (\lambda^{(k-1)}, \lambda^{(k)})$, then there exists $c_1 \geq 1$ such that θ is the largest eigenvalue of the polynomial $G_k + (c_1 - 1)G_{k-1}$. If Γ is a connected r -regular graph with $\lambda_2(\Gamma) \leq \theta$, then using Theorem 8 we get that

$$|V(\Gamma)| \leq v(r, \theta) = 1 + r + r(r - 1) + \dots + r(r - 1)^{k-2} + \frac{r(r - 1)^{k-1}}{c_1} \quad (13)$$

which is clearly better than (12). If $\theta \in (\lambda^{(k)}, \lambda^{(k+1)})$, then there exists $c_2 > 1$ such that θ is the largest eigenvalue of the polynomial $G_{k+1} + (c_2 - 1)G_k$. As above, if Γ is a connected r -regular graph with $\lambda_2(\Gamma) \leq \theta$, then Theorem 8 implies that

$$|V(\Gamma)| \leq v(r, \theta) = 1 + r + r(r - 1) + \dots + r(r - 1)^{k-1} + \frac{r(r - 1)^k}{c_2} \quad (14)$$

which is again better than (11).

§4 Spectral Moore theorems for bipartite graphs

Building on this work, the author with Koolen and Nozaki extended and refined these results to bipartite regular graphs [9]. Let $r \geq 3$ be an integer and θ be any real number between 0 and $2\sqrt{r - 1}$. Define $b(r, \theta)$ as the maximum number of vertices of a bipartite r -regular graph whose second largest eigenvalue is at most θ . Clearly, $b(r, \theta) \leq v(r, \theta)$ and a natural question is whether or not these parameters are actually the same or not. One can show that $v(3, 1) = 10$ attained by the Petersen graph and that $b(3, 1) = 8$ attained by the 3-dimensional cube. For the bipartite graphs, a linear programming bound similar to Nozaki’s Theorem 6 from [35] was obtained with the use of the following polynomials:

$$\mathcal{F}_{0,i} = F_{2i}(\sqrt{x}), \mathcal{F}_{1,i} = \frac{F_{2i+1}(\sqrt{x})}{\sqrt{x}},$$

for any $i \geq 0$, where $(F_i)_{i \geq 0}$ were defined earlier in (2) and (3). Let Γ be a bipartite connected regular graph. Its adjacency matrix A has the form $\begin{bmatrix} 0 & \mathbf{N} \\ \mathbf{N}^\top & 0 \end{bmatrix}$ and we call \mathbf{N} the biadjacency matrix of Γ . Note that $F_{2i}(A) = \begin{bmatrix} \mathcal{F}_{0,i}(\mathbf{N}\mathbf{N}^\top) & 0 \\ 0 & \mathcal{F}_{0,i}(\mathbf{N}^\top\mathbf{N}) \end{bmatrix}$ for any $i \geq 0$. The following is called the LP bound for bipartite regular graphs.

Theorem 11 (Cioabă, Koolen and Nozaki [9]). *Let Γ be a connected bipartite r -regular graph with v vertices and denote by $\{\pm\tau_0, \dots, \pm\tau_d\}$ its set of distinct eigenvalues, where $\tau_0 = r$. If there exists a polynomial $f(x) = \sum_{i=0}^t f_i \mathcal{F}_{0,i}(x)$ such that $f(r^2) > 0$, $f(\tau_i^2) \leq 0$ for each $i \in \{1, \dots, d\}$, $f_0 > 0$, and $f_j \geq 0$ for each $j \in \{1, \dots, t\}$, then*

$$v \leq \frac{2f(r^2)}{f_0}. \tag{15}$$

Equality holds if and only if for each $i \in \{1, \dots, d\}$, $f(\tau_i^2) = 0$ and for each $j \in \{1, \dots, t\}$, $\text{tr}(f_j \mathcal{F}_{0,j}(\mathbf{N}\mathbf{N}^\top)) = 0$, and $\text{tr}(f_j \mathcal{F}_{0,j}(\mathbf{N}^\top \mathbf{N})) = 0$, where \mathbf{N} is the biadjacency matrix of Γ . If equality holds and $f_j > 0$ for each $j \in \{1, \dots, t\}$, then the girth of Γ is at least $2t + 2$.

For any integers $t \geq 3, r \geq 3$ and any positive $c \leq r$, let $\mathbf{B}(r, t, c)$ be the $t \times t$ tridiagonal matrix with lower diagonal $(1, \dots, 1, c, r)$, upper diagonal $(r, r - 1, \dots, r - 1, r - c)$, and constant row sum r . Using Theorem 11, the following general upper bound for $b(r, \theta)$ was obtained in [9].

Theorem 12 (Cioabă, Koolen and Nozaki [9]). *If θ is the second largest eigenvalue of $\mathbf{B}(r, t, c)$, then*

$$b(r, \theta) \leq 2 \left(\sum_{i=0}^{t-4} (r - 1)^i + \frac{(r - 1)^{t-3}}{c} + \frac{(r - 1)^{t-2}}{c} \right) := M(r, t, c). \tag{16}$$

Equality holds if and only if there exists a bipartite distance-regular graph whose quotient matrix with respect to the distance-partition from a vertex is $\mathbf{B}(r, t, c)$ for $1 \leq c < r$ or $\mathbf{B}(r, t - 1, 1)$ for $c = r$.

Define $H_j(x) = \sum_{i=0}^{\lfloor j/2 \rfloor} F_{j-2i}(x)$ for $j \geq 0$. These are orthogonal polynomials and one can show that $H_j(x) = xH_{j-1}(x) - (r - 1)H_{j-2}(x)$ for $j \geq 2$ as well as that $H_j(x) = \frac{F_{j+2}(x) - (r-1)^2 F_i(x)}{x^2 - r^2}$. The first step in proving the above result is showing that the characteristic polynomial of $\mathbf{B}(r, t, c)$ equals $(x^2 - r^2)(H_{t-2}(x) + (c - 1)H_{t-4}(x))$. The proof proceeds in similar steps to Theorem 8, but is more technical and we refer the reader to [9] for the details. Similar to the situation for general graphs, one can show the following.

Proposition 13. *Let $r \geq 3$ be an integer. For any $\theta \in [0, 2\sqrt{r - 1})$, there exists t and c such that θ is the second largest eigenvalue of $\mathbf{B}(r, t, c)$.*

In [9], the authors also proved that for given r and θ , the upper bound obtained in Theorem 12 is better than the one in Theorem 8. Theorem 12 has applications to various areas and it improves results obtained in the context

of coding theory by Høholdt and Janwa [21] and Høholdt and Justensen [22] and design theory by Teranishi and Yasuno [39].

As in the case of Theorem 8, Theorem 12 has applications for the Alon-Boppana theorems for bipartite regular graphs. Corollary 4.11 in [9] is a consequence of Theorem 12 and states that if Γ is a bipartite r -regular graph of order greater than $M(r, t, c)$ (the right hand-side in Theorem 12), then $\lambda_2 \geq \theta$, where θ is the second largest eigenvalue of $\mathbf{B}(r, t, c)$. Li and Solé [26, Theorems 3 and 5] proved that if Γ is a bipartite r -regular graph of girth 2ℓ , then $\lambda_2(\Gamma) \geq 2\sqrt{r-1} \cos \frac{\pi}{\ell}$. This result follows from Corollary 4.11 in [9] as $2\sqrt{r-1} \cos \frac{\pi}{\ell}$ is the second largest eigenvalue of $\mathbf{B}(r, \ell + 1, 1)$ and having girth 2ℓ implies that Γ has at least $M(r, \ell + 1, 1)$ vertices.

§5 Classical Moore problem

Since the fundamental work of Singleton [38], Hoffman and Singleton [20], Bannai and Ito [2] and Damerell [11] in the 1970s, the families of orthogonal polynomials $(F_j)_{j \geq 0}$ and $(G_j)_{j \geq 0}$ have been important in the study of the Moore problem (1). It has been observed by several authors (see [13] or [31] for example) that if Γ is connected r -regular of diameter D , eigenvalues $r = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$ and $\beta = \max(|\lambda_2|, |\lambda_n|)$, then

$$|V(\Gamma)| \leq G_D(r) - G_D(\beta) = m_{r,D} - G_D(\beta), \tag{17}$$

where $m_{r,D}$ is the upper bound from the Moore bound (1). Recall that $\lambda^{(D)}$ denotes the largest root of $G_D(x)$ and satisfies

$$2\sqrt{r-1} \cos \frac{\pi}{D} < \lambda^{(D)} < 2\sqrt{r-1} \cos \frac{\pi}{D+1}. \tag{18}$$

Inequality (17) will improve the classical Moore bound (1) when $G_D(\beta) > 0$. This will happen when $\beta > \lambda^{(D)}$. When $D = 2$, from (10) we know that $\lambda^{(2)} = \frac{-1 + \sqrt{4r-3}}{2}$. Note that [13, Theorem 2] contains a typo in the numerator of the right hand-side of the inequality (the 1 in the numerator should be a -1). The informal description of the result above is that when β is large, then the order of the graph will be smaller than the Moore bound. Note however that if β is small, then $G_D(\beta)$ may be negative and inequality (17) may be worse than the classical Moore bound (1). Our results from [8] may be used to handle some cases when β is small (actually when λ_2 is small). More precisely, Theorem 8 gives an upper bound for the order of an r -regular graph with small second largest eigenvalue regardless of its diameter actually.

We explain this argument and give some numerical examples in Table 1 where we listed some values of (r, D) (these values are from [31, page 4])

(r, D)	Known	Defect	Lower	Moore	Upper
(8,2)	57	8	2.09503	2.19258	3.40512
(9,2)	74	8	2.29956	2.37228	3.53113
(10,2)	91	10	2.46923	2.54138	3.88473
(4,3)	41	12	2.11232	2.25342	2.88396
(5,3)	72	34	2.42905	2.62620	3.77862
(4,4)	98	63	2.53756	2.69963	3.44307
(5,4)	212	214	2.91829	3.12941	4.41922
(3,5)	70	24	2.32340	2.39309	2.64401
(4,5)	364	121	2.89153	2.93996	3.42069
(3,6)	132	58	2.45777	2.51283	2.75001
(4,6)	740	717	3.00233	3.08314	3.73149

Table 1: Numerical results for small (r, D)

where the maximum orders $n_{r,D}$ of r -regular graphs of diameter D are not known. For each such pair, the column labeled *Known* gives the largest known order of an r -regular graph of diameter D . The column *Defect* equals the difference between the Moore bound $m_{r,D}$ and the entry in the *Known* column. The column *Moore* contains the value of $\lambda^{(D)}$ rounded below to 5 decimal points. The column *Upper* contains the lower bound for τ that guarantees that inequality (17) will give a lower bound than the value from the *Known* column. For example, for $r = 8$ and $D = 2$, if Γ is an 8-regular graph with diameter 2 having $\tau < 3.40512$, then $|V(\Gamma)| < 57$. The column *Lower* contains an upper bound for λ_2 that guarantees that the order of such r -regular graph would be small. For example, for $r = 8$ and $D = 2$, our Theorem 8 implies that if Γ is an 8-regular graph with $\lambda_2 < 2.0953$, then $|V(\Gamma)| < 57$. Another way to interpret these results in Table 1 is that if one wants to look for a 3-regular graph of diameter 6 with more than 132 vertices, then the second largest eigenvalue of such putative graph has to be between 2.45777 and 2.75001.

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