The Weisfeiler–Leman stability: the case of trees

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$M_X(\mathbb{C})$: the full matrix algebra $/\mathbb{C}$
rows + columns indexed by X
w.r.t. the ordinary matrix product

$$(AB)_{xy} = \sum_{z \in X} A_{xz} \cdot B_{zy}$$

$M_X(\mathbb{C})^\circ$: the commutative alg. $/\mathbb{C}$
w.r.t. the Hadamard product $\circ$

$M_X(\mathbb{C})^\circ = M_X(\mathbb{C})$ as vector spaces

$$(A \circ B)_{xy} = A_{xy} \, B_{xy}$$

$M_X(\mathbb{C}) \supseteq \mathcal{M}$ <u>coherent algebra</u> 3

$\overset{\text{def}}{\Leftrightarrow}$ (0) $\mathcal{M}$: a subspace as a vector space

(i) closed under transpose – conjugate

$$A \in \mathcal{M} \implies {}^t\overline{A} \in \mathcal{M}$$

(ii) closed under $\cdot$ & $\circ$

$$A, B \in \mathcal{M} \implies AB \in \mathcal{M}$$
$$A \circ B \in \mathcal{M}$$

i.e. $\mathcal{M} \leq M_X(\mathbb{C})$ subalg.

and $\mathcal{M} \leq M_X(\mathbb{C})^{\circ}$ subalg.

B4

$$\mathrm{Span}\{I, J\} \subseteq \mathcal{M} \subseteq M_X(\mathbb{C})$$

smallest coherent alg. — coherent alg. — largest coherent alg.

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : \text{identity of } M_X(\mathbb{C})$$

$$J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{pmatrix} : \text{identity of } M_X(\mathbb{C})^{\circ}$$

<u>coherent algebra</u>

$$\mathcal{M} = \mathrm{Span}\{A_\alpha \mid \alpha \in \Lambda\}$$

A_α : $(0,1)$-matrix

$$I = \sum_{\alpha \in \Lambda_0} A_\alpha \quad (\text{some } \Lambda_0 \subseteq \Lambda)$$

$$J = \sum_{\alpha \in \Lambda} A_\alpha$$

$${}^{t}A_\alpha = A_{\alpha'} \quad \text{for some } \alpha' \in \Lambda$$

$$A_\alpha A_\beta = \sum_{\gamma \in \Lambda} p_{\alpha\beta}^{\gamma} A_\gamma$$

<u>coherent configuration</u> (D. G. Higman) 1974 5

$$\mathcal{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda})$$

If $M = \mathrm{Span}\{A_\alpha \mid \alpha \in \Lambda\}$

is a coherent configuration $\subseteq M_X(\mathbb{C})$

$$\begin{array}{ccc} X \times X & & M_X(\mathbb{C}) \\ \cup & & \in \\ R_\alpha & \longleftrightarrow & A_\alpha \end{array}$$ adjacency matrix

$$(A_\alpha)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_\alpha \\ 0 & \text{otherwise} \end{cases}$$

coherent configuration $\longleftrightarrow$ coherent algebra

Example

$G \subseteq \mathrm{Sym}(X)$

$G \longrightarrow X \times X \qquad a\cdot(x,y) = (ax, ay)$

$\{R_\alpha\}_{\alpha\in\Lambda}$: the G-orbits on $X \times X$

$\mathcal{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ coherent conf.

Schurian

$\mathrm{Aut}\,\mathcal{X}$ is transitive on each R_α $(\alpha\in\Lambda)$.

$$\mathrm{Aut}\,\mathcal{X} = \left\{ a \in \mathrm{Sym}(X) \;\middle|\; \begin{array}{c} (x,y) \in R_\alpha \\ \Updownarrow \\ (ax, ay) \in R_\alpha \\ \text{for all } \alpha\in\Lambda \end{array} \right\}$$

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$\Gamma = (\underset{\text{vertex set}}{X}, \underset{\text{edge set}}{R})$ graph

$$R \subseteq X \times X - \underset{\text{diagonal}}{\Delta}$$

$M_X(\mathbb{C}) \ni A$
adjacency matrix of R

$M_X(\mathbb{C}) \geq \mathcal{A}_0 = \langle A, {}^tA \rangle \ni I$
subalg. generated by A, ${}^tA$

$M_X(\mathbb{C})^\circ \geq \mathcal{A}_1 = \langle \mathcal{A}_0 \rangle^\circ \ni J$
subalg. generated by $\mathcal{A}_0$

$M_X(\mathbb{C}) \geq \mathcal{A}_2 = \langle \mathcal{A}_1 \rangle$
subalg. generated by $\mathcal{A}_1$

$M_X(\mathbb{C})^\circ \geq \mathcal{A}_3 = \langle \mathcal{A}_2 \rangle^\circ$
subalg. generated by $\mathcal{A}_2$

$\vdots$

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$$\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots \subseteq M_X(\mathbb{C})$$

sequence of subalgebras

$$\mathcal{A}_1 \subseteq \mathcal{A}_3 \subseteq \mathcal{A}_5 \subseteq \cdots \subseteq M_X(\mathbb{C})^\circ$$

sequence of subalgebras.

<u>Weisfeiler-Lehman stabilization</u>

late 60's

$\exists\, r$ s.t.

$$\mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_{r-1} \subsetneq \mathcal{A}_r = \mathcal{A}_{r+1}$$

$$\mathcal{A} = \bigcup_{i=0}^{\infty} \mathcal{A}_i = \mathcal{A}_r \subseteq M_X(\mathbb{C})$$

the smallest coherent algebra $\supseteq \mathcal{A}_0$

<u>the coherent closure</u>

$$\mathcal{A} = \bigcap_{\substack{\mathcal{A}_0 \subseteq \mathcal{M} \subseteq M_X(\mathbb{C}) \\ \text{coherent alg.}}} \mathcal{M}$$

$r = r(\Gamma)$: <u>coherent length</u>

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$$\mathcal{A} \longleftrightarrow \mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$$

coherent closure of $\mathcal{A}_0 = \langle A, {}^tA \rangle$

coherent conf.

coherent closure of Γ

Fact $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathfrak{X})$

Remark $\mathfrak{X}$ is NOT Schurian in general, i.e.,

$\mathrm{Aut}(\mathfrak{X})$ is Not transitive on each R_α in general

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Theorem (joint with 徐静, 李双东)

Γ : tree

Then $r \leq 7$.

$$A_0 \subseteq A_2 \subseteq A_4 \subseteq \cdots \subseteq M_X(\mathbb{C})$$

sequence of subalgebras

semi-simple alg. (representations)

$$A_1 \subseteq A_3 \subseteq A_5 \subseteq \cdots \subseteq M_X(\mathbb{C})^{\circ}$$

sequence of subalgebras

semi-simple alg. (combinatorics)

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$X \ni \exists! x_0$ the centre of Γ

$$D(x_0) < D(x) = \max\{\partial(x,y) \mid y \in X\}$$

all $x \in X$, $x \neq x_0$

or $\exists! x_0, x_1$ the adjacent centre of Γ

$$D(x_0) = D(x_1) < D(x).$$

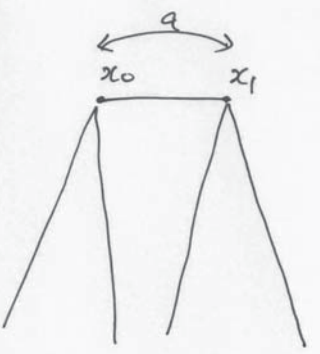
all $x \in X$, $x \neq x_0, x_1$

$G = \mathrm{Aut}(\Gamma)$

Set $X_0 = \{x_0\}$ if $ax_0 = x_0$ all $a \in G$

$X_0 = \{x_0, x_1\}$ if $ax_0 = x_1$ (some $a \in G$)

$ax_1 = x_0$



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<u>Terwilliger algebra</u> $T = T(x_0)$

$$X_i = \{ x \in X \mid \partial(x_0, x) = i \}, \quad 0 \le i \le D$$

$V = \mathbb{C}X \quad (X : \text{orthonormal basis})$

standard module

$$= \bigoplus_{i=0}^{D} V_i^*, \qquad V_i^* = \mathbb{C}X_i$$

$E_i^* : V \longrightarrow V_i^*$ orthogonal projection

$$T = \langle A, E_i^* \mid 0 \le i \le D \rangle \subseteq M_X(\mathbb{C})$$

A: adjacency alg. of Γ

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$$S = \mathrm{Hom}_G(V, V)$$

$$= \{ f : V \to V \text{ linear mapping} \mid f(av) = af(v), \text{ all } a \in G, \text{ all } v \in V \}$$

the centralizer algebra of G

$$\mathcal{A}_0 = \langle A \rangle \subseteq T \subseteq S \subseteq M_X(\mathbb{C})$$

Want to show

$$T \subseteq \mathcal{A}_6$$

$$T \circ T = S = \mathcal{A}_7$$

particularly, the coherent (configuration) closure $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda})$ is Schurian.

Isomorphism classes of Irreducible T-modules

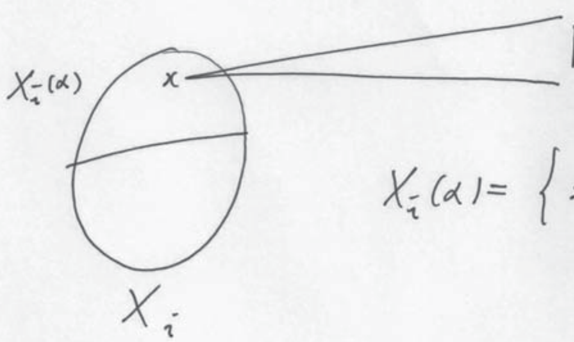
$X_i \ni x, y$

$x \sim y \iff \Gamma^{(x)} \simeq \Gamma^{(y)}$

as rooted trees

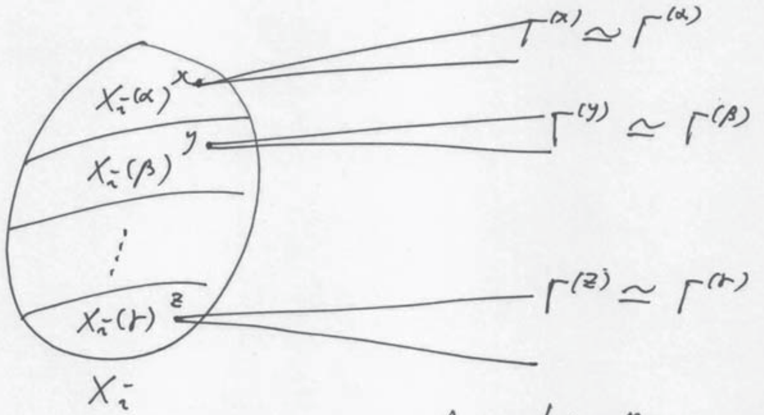
$\{\Gamma^{(\alpha)} \mid \alpha \in \Lambda_i\}$ representatives of the equivalence classes $\Gamma^{(x)}$, $x \in X_i$

$= \{\Gamma^{(x)} \mid x \in X_i\} / \sim$



$\Gamma^{(x)} \simeq \Gamma^{(\alpha)}$

$X_i(\alpha) = \{x \in X_i \mid \Gamma^{(x)} \simeq \Gamma^{(\alpha)}\}$



$$\Lambda_i^- = \{\alpha, \beta, \cdots, \gamma\}$$

$$X_i^- = X_i^-(\alpha) \cup X_i^-(\beta) \cup \cdots \cup X_i^-(\gamma)$$

$$V_i^{*} = \mathbb{C} X_i^- = V_i^{*}(\alpha) \oplus V_i^{*}(\beta) \oplus \cdots \oplus V_i^{*}(\gamma)$$

Def

$$V_i^{*}(\alpha) = V_i^{*(0)}(\alpha) \oplus V_i^{*(1)}(\alpha) \qquad \text{orthogonal sum}$$

$$V_i^{*(1)}(\alpha) = \ker \left. E_{i-1}^{*} A E_i^{*} \right|_{V_i^{*}(\alpha)}$$

Remark $V_i^{*(1)}(\alpha) = 0$ may happen!

In this case, $V_i^{*(0)}(\alpha) = V_i^{*}(\alpha)$.

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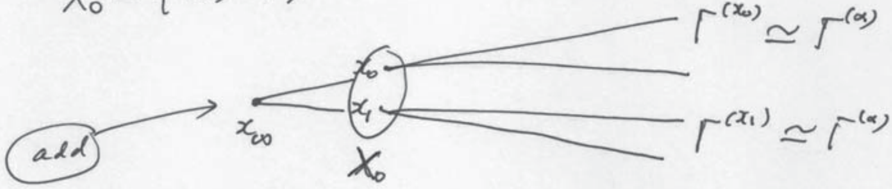
$\underline{i=0}$ $\quad |\Lambda_0| = 1$, $\quad \Lambda_0 = \{\alpha\}$

$X_0 = \{x_0\}$. $\quad V_0^{*(0)}(\alpha) = 0$

$V_0^{*(1)}(\alpha) = V_0^{*}$

$X_0 = \{x_0, x_1\}$



$X_{-1} = \{x_\infty\}$

$$V_0^{*(1)}(\alpha) \underset{\text{def}}{=} \mathbb{C}(x_0 - x_1)$$

Set $V_{-1}^{*(1)}(\alpha) = V_0^{*(0)}(\alpha) \underset{\text{def}}{=} \mathbb{C}(x_0 + x_1)$

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Classification of Irreducible T-modules

Theorem

(i) $V_i^{*(1)}(\alpha) \ni w \neq 0 \implies W = Tw$ irred. T-module

(ii) $V_i^{*(1)}(\alpha) \ni w \neq 0, \quad W = Tw$

$V_j^{*(1)}(\beta) \ni w' \neq 0, \quad W' = Tw'$

Then

$W \simeq W'$ as T-modules

$\iff \quad i = j, \quad \alpha = \beta$

(iii) $V \supseteq {}^{\forall} W$ irred. T-module

$\exists\, 0 \neq w \in V_i^{*(1)}(\alpha)$ s.t. $W = Tw$.

Cor

$$T = \bigoplus_{\substack{(i,\alpha) \\ V_i^{*(\iota)}(\alpha) \neq 0}} E_i^{*(\iota)}(\alpha) \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

semi simple

direct sum of simple algebras

$$\subseteq M_X(\mathbb{C})$$

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<u>Notation</u>

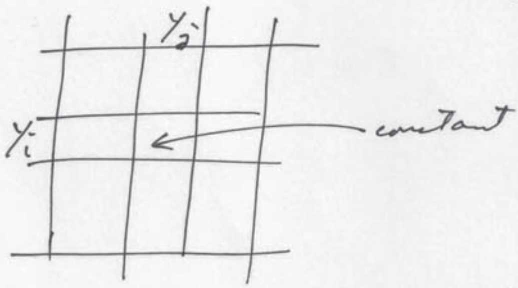
(i) $X \supseteq Y$

$$M_X(\mathbb{C}) \supseteq M_Y(\mathbb{C}) = \begin{array}{c|c|c} & Y & \\ \hline Y & /// & 0 \\ \hline & 0 & 0 \end{array}$$

(ii) $Y/\!\sim \; = Y_1 \cup Y_2 \cup \cdots \cup Y_r$

partition

$$M_{Y/\sim}(\mathbb{C}) = \{ a \in M_Y(\mathbb{C}) \mid a \text{ is constant on } Y_i \times Y_j, \text{ all } i,j \}$$



(iii) $\Gamma^{(\alpha)}/\!\sim$: orbits of $\mathrm{Aut}\,\Gamma^{(\alpha)}$ on $\Gamma^{(\alpha)}$

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(iv) $$\bigcup_{x \in X_i(\alpha)} \Gamma^{(x)} = X_i(\alpha) \times \Gamma^{(\alpha)}$$

identified

(v)
$$V_i^*(\alpha) = \mathbb{C} X_i(\alpha) = V_i^{*(0)}(\alpha) \oplus V_i^{*(1)}(\alpha)$$

$$E_i^{*(1)}(\alpha) : V_i^*(\alpha) \longrightarrow V_i^{*(1)}(\alpha)$$

orthogonal projection

$$\cap$$

$$M_{X_i(\alpha)}(\mathbb{C})$$

So $$E_i^{*(1)}(\alpha) \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

$$\subseteq M_{X_i(\alpha) \times \Gamma^{(\alpha)}}(\mathbb{C}) \subseteq M_X(\mathbb{C})$$

$S = \mathrm{Hom}_G(V, V)$ centralizer algebra

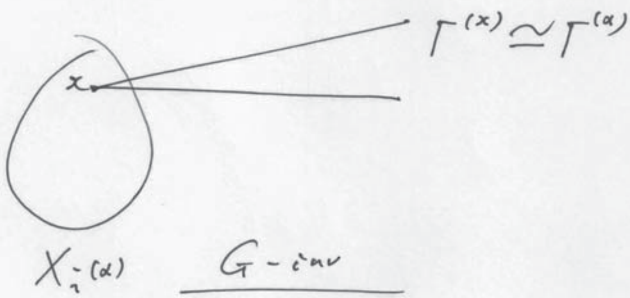
$G = \mathrm{Aut}\,\Gamma$

$V = \mathbb{C}X$

$T \subseteq S$

$X/\sim$: the G-orbits

$= \bigcup_{i,\alpha} X_i(\alpha)/\sim$



<u>G-inv</u>

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$$X_i(\alpha)/\sim \ \ni y \qquad G\text{-orbit}$$

$$V_y = \mathbb{C}y$$
$$= V_y^{(0)} \oplus V_y^{(1)} \quad \text{orthogonal sum}$$

$$V_y^{(1)} = \ker E_{i-1}^* A E_i^* \big|_{V_y}$$

Then

$$V_i^*(\alpha) = \mathbb{C}X_i(\alpha)$$
$$= \bigoplus_{y \in X_i(\alpha)/\sim} V_y$$

$$\begin{cases} V_i^{*(0)}(\alpha) = \bigoplus_{y \in X_i(\alpha)/\sim} V_y^{(0)} \\ V_i^{*(1)}(\alpha) = \bigoplus_{y \in X_i(\alpha)/\sim} V_y^{(1)} \end{cases}$$

Classification of Irreducible S-modules

Theorem

(i) $V_y^{(1)} \ni w \neq 0 \Rightarrow W = Sw$
$(y \in X/\sim)$ irred. S-module

(ii) $V_y^{(1)} \ni w \neq 0$, $W = Sw$ $(y \in X/\sim)$

$V_z^{(1)} \ni w' \neq 0$, $W' = Sw'$ $(z \in X/\sim)$

Then $W \simeq W'$ as S-modules

$\Longleftrightarrow y = z$

(iii) $V \supseteq {}^{\forall}W$ irred. S-module

$\exists_{0 \neq} w \in V_y^{(1)}$ s.t. $W = Sw$

Cor

$$S = \bigoplus_{\substack{Y \in X_{\frac{?}{2}}(\alpha)/\sim \\ V_Y^{(1)} \neq 0}} E_Y^{(1)} \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

direct sum of simple algebras

where $E_Y^{(1)}$: $V_Y = \mathbb{C}Y \longrightarrow V_Y^{(1)}$ orthogonal proj.

$\cap$

$M_Y(\mathbb{C})$

$$\underline{X_0} = \begin{cases} x_0 & \text{if } X_0 = \{x_0\} \\ x_0 + x_1 & \text{if } X_0 = \{x_0, x_1\} \end{cases}$$

$$V_0 = T\,\underline{X_0} = S\,X_0 = \mathrm{Span}\left\{ \underline{Y} \mid Y \in X/\sim \right\}$$

principal module

irreducible T-module

unique

Then

$$T \supseteq T|_{V_0} = S|_{V_0} = M_{X/\sim}$$

Observe

$$Y \times \Gamma^{(\alpha)}/\sim \;\subseteq\; X/\sim$$

$Y \times \Gamma^{(\alpha)}$: union of G-orbits

$$J_{Y\times\Gamma^{(\alpha)}} = \begin{array}{c|c} & Y\times\Gamma^{(\alpha)} \\ \hline Y\times\Gamma^{(\alpha)} & \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & J & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \end{array} \in T_0 = T|_{V_0}$$

$$J_{Y\times\Gamma^{(\alpha)}} \circ T = E_Y^{(1)} \otimes M_{\Gamma^{(\alpha)}/\sim}(\mathbb{C})$$

So

$$(T_0) \circ T = S$$