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CONTINUOUS DESCENT METHODS FOR NONSMOOTH MINIMIZATION

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ABSTRACT. We consider continuous descent methods for the minimization of convex functions and Lipschitz functions defined on a general Banach space. We present several generic and nongeneric convergence theorems. Nongeneric convergence theorems are obtained for those methods which are generated by either regular or super-regular vector fields.

1. INTRODUCTION

The study of discrete and continuous descent methods is an important topic in optimization theory and in dynamical systems. See, for example, [7, 12, 14, 15, 16]. Given a continuous convex function f on a Banach space X, we associate with f a complete metric space of vector fields $V : X \to X$ such that $f^0(x, Vx) \leq 0$ for all $x \in X$. Here $f^0(x, h)$ is the right-hand derivative of f at x in the direction $h \in X$. To each such vector field there correspond two gradient-like iterative processes. In two recent papers [15, 16] it is shown that for most of the vector fields in this space, both iterative processes generate sequences $\{x_n\}_{n=1}^{\infty}$ such that the sequences $\{f(x_n)\}_{n=1}^{\infty}$ tend to $\inf(f)$ as $n \to \infty$. Analogous results for Lipschitz functions which are not necessarily convex are obtained in [17]. In this paper we discuss continuous descent methods for convex functions as well as for Lipschitz functions which are not necessarily convex.

When we say that most of the elements of a complete metric space Y enjoy a certain property, we mean that the set of points which have this property contains a G_δ everywhere dense subset of Y. In other words, this property holds generically. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y, has already been successfully applied in many areas of Analysis. See, for example, [8-10, 13, 21] and the references therein.

We now recall the concept of porosity [5, 9, 10, 16, 17, 19, 21] which enables us to obtain even more refined results.

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. We say that a subset $E \subset Y$ is porous in (Y, d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

Other notions of porosity have been used in the literature [5, 19]. We use the rather strong notion which appears in [9, 10, 16, 17].

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space R^n, then σ-porous sets are of Lebesgue measure 0. The existence of a non-σ-porous set $P \subset R^n$, which is of the first Baire category and of Lebesgue measure 0, was established in [19]. It is easy to see that for any σ-porous set $A \subset R^n$, the set $A \cup P \subset R^n$ also belongs to the family $\mathcal{E}$ consisting of all the non-σ-porous subsets of R^n which are of the first Baire category and have Lebesgue measure 0. Moreover, if $Q \in \mathcal{E}$ is a countable union of sets $Q_i \subset R^n$, $i = 1, 2, \ldots$, then there is a natural number j for which the set Q_j is non-σ-porous. Evidently, this set Q_j also belongs to $\mathcal{E}$. Thus one sees that the family $\mathcal{E}$ is quite large. Also, every complete metric space without isolated points contains a closed nowhere dense set which is not σ-porous [20].

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To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

Our paper is organized as follows. In the next section we apply continuous descent methods to the minimization of convex functions. Section 3 is devoted to Lipschitz functions. In Section 4 we study the behavior of approximate solutions to evolution equations governed by regular vector fields. Finally, in the last section we examine continuous descent methods which are generated by super-regular vector fields.

2. Convex functions

Let $(X^*, ||\cdot||_*)$ be the dual space of the Banach space $(X, ||\cdot||)$, and let $f : X \to R^1$ be a convex continuous function which is bounded from below. Recall that for each pair of sets $A, B \subset X^*$,

$$H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} ||x - y||_*, \sup_{y\in B} \inf_{x\in A} ||x - y||_*\}$$

is the Hausdorff distance between A and B.

For each $x \in X$, let

$$f^0(x, u) = \lim_{t\to 0^+} [f(x + tu) - f(x)]/t, \; u \in X,$$

and

$$\partial f(x) = \{l \in X^* : f(y) - f(x) \geq l(y - x) \text{ for all } y \in X\}$$

be the directional derivative of f at x in the direction u and the subdifferential of f at x, respectively. It is well known that the set $\partial f(x)$ is nonempty and norm-bounded. Set

$$\inf(f) = \inf\{f(x) : x \in X\}.$$

Denote by $\mathcal{A}$ the set of all mappings $V : X \to X$ such that V is bounded on every bounded subset of X (i.e., for each $K_0 > 0$ there is $K_1 > 0$ such that $||Vx|| \leq K_1$ if $||x|| \leq K_0$), and for each $x \in X$ and each $l \in \partial f(x)$, $l(Vx) \leq 0$. We denote by $\mathcal{A}_c$ the set of all continuous $V \in \mathcal{A}$, by $\mathcal{A}_u$ the set of all $V \in \mathcal{A}$ which are uniformly continuous on each bounded subset of X, and by $\mathcal{A}_{au}$ the set of all $V \in \mathcal{A}$ which are uniformly continuous on the subsets

$$\{x \in X : ||x|| \leq n \text{ and } f(x) \geq \inf(f) + 1/n\}$$

for each integer $n \geq 1$. Finally, let $\mathcal{A}_{auc} = \mathcal{A}_{au} \cap \mathcal{A}_c$.

Next we endow the set $\mathcal{A}$ with a metric ρ: for each $V_1, V_2 \in \mathcal{A}$ and each integer $i \geq 1$, we first set

$$\rho_i(V_1, V_2) = \sup\{||V_1x - V_2x|| : x \in X \text{ and } ||x|| \leq i\}$$

and then define

$$\rho(V_1, V_2) = \sum_{i=1}^{\infty} 2^{-i}[\rho_i(V_1, V_2)(1 + \rho_i(V_1, V_2))^{-1}].$$

Clearly, $(\mathcal{A}, \rho)$ is a complete metric space. It is also not difficult to see that the collection of the sets

$$E(N, \epsilon) = \{(V_1, V_2) \in \mathcal{A} \times \mathcal{A} : ||V_1x - V_2x|| \leq \epsilon, \; x \in X, \; ||x|| \leq N\},$$

where $N, \epsilon > 0$, is a base for the uniformity generated by the metric ρ. Evidently, $\mathcal{A}_c$, $\mathcal{A}_u$, $\mathcal{A}_{au}$ and $\mathcal{A}_{auc}$ are all closed subsets of the metric space $(\mathcal{A}, \rho)$. In the sequel we assign to all these spaces the same metric ρ.

To compute $\inf(f)$, we associate in [15, 16] with each vector field $W \in \mathcal{A}$ two gradient-like iterative processes. Note that the counterexample studied in Section 2.2 of Chapter VIII of [12] shows that, even for two-dimensional problems, the simplest choice for a descent direction, namely the normalized steepest descent direction,

$$V(x) = \text{argmin}\{\max_{l\in\partial f(x)} < l, d >: ||d|| = 1\},$$

may produce sequences the functional values of which fail to converge to the infimum of f. This vector field V belongs to $\mathcal{A}$ and the Lipschitz function f attains its infimum. The steepest descent scheme (Algorithm 1.1.7) presented in Section 1.1 of Chapter VIII of [12] corresponds to any of the two iterative processes considered in [15, 16].

In infinite dimensions the minimization problem is even more difficult and less understood. Moreover, positive results usually require special assumptions on the space and the functions. However, as shown in [15] (under certain assumptions on the function f), for an arbitrary Banach space X and a generic vector field $V \in \mathcal{A}$, the values of f tend to its infimum for both processes.

In [16] we introduced the class of regular vector fields $V \in \mathcal{A}$ which will be described below and established (under the two mild assumptions A(i) and A(ii) on f stated below) that the complement of the set of regular vector fields is not only of the first category, but also σ-porous in each of the spaces $\mathcal{A}$, $\mathcal{A}_c$, $\mathcal{A}_u$, $\mathcal{A}_{au}$ and $\mathcal{A}_{auc}$. We then showed in [16] that for any regular vector field $V \in \mathcal{A}_{au}$, the values of the function f tend to its infimum for both processes if f also satisfies an additional assumption. The last result in [16] is a stability theorem for regular vector fields.

The results of [16] are valid in any Banach space and for those convex functions which satisfy the following two assumptions.

A(i) There exists a bounded set $X_0 \subset X$ such that

$$\inf(f) = \inf\{f(x) : \ x \in X\} = \inf\{f(x) : \ x \in X_0\};$$

A(ii) for each $r > 0$, the function f is Lipschitz on the ball $\{x \in X : \ ||x|| \leq r\}$.

Note that assumption A(i) clearly holds if $\lim_{||x|| \to \infty} f(x) = \infty$.

We say that a mapping $V \in \mathcal{A}$ is regular if for any natural number n there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying

$$||x|| \leq n \text{ and } f(x) \geq \inf(f) + 1/n,$$

and each $l \in \partial f(x)$, we have

$$l(Vx) \leq -\delta(n).$$

Denote by $\mathcal{F}$ the set of all regular vector fields $V \in \mathcal{A}$.

It is not difficult to verify the following property of regular vector fields. It means, in particular, that $\mathcal{A} \setminus \mathcal{F}$ is a face of the convex cone $\mathcal{A}$.

Proposition 2.1. *Assume that $V_1, V_2 \in \mathcal{A}$, V_1 is regular, $\phi : X \to [0,1]$, and that for each integer $n \geq 1$,*

$$\inf\{\phi(x) : \ x \in X \text{ and } ||x|| \leq n\} > 0.$$

Then the mapping $x \to \phi(x)V_1x + (1 - \phi(x))V_2x$, $x \in X$, also belongs to $\mathcal{F}$.

The following result obtained in [16] shows that in a very strong sense most of the vector fields in $\mathcal{A}$ are regular.

Theorem 2.1. *Assume that both A(i) and A(ii) hold. Then $\mathcal{A} \setminus \mathcal{F}$ (respectively, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_{au} \setminus \mathcal{F}$ and $\mathcal{A}_{auc} \setminus \mathcal{F}$) is a σ-porous subset of the space $\mathcal{A}$ (respectively, $\mathcal{A}_c$, $\mathcal{A}_{au}$ and $\mathcal{A}_{auc}$). Moreover, if f attains its infimum, then the set $\mathcal{A}_u \setminus \mathcal{F}$ is also a σ-porous subset of the space $\mathcal{A}_u$.*

We let $x \in W^{1,1}(0,T;X)$, i.e. (see, e.g., [6]),

$$x(t) = x_0 + \int_0^t u(s)ds, \ t \in [0,T],$$

where $T > 0$, $x_0 \in X$ and $u \in L^1(0,T;X)$. Then $x : [0,T] \to X$ is absolutely continuous and $x'(t) = u(t)$ for a.e. $t \in [0,T]$.

In the sequel we denote by $\mu(E)$ the Lebesgue measure of a Lebesgue measurable $E \subset R^1$. The following two results were established in [18].

Theorem 2.2. *Let $V \in \mathcal{A}$ be regular, let $x \in W^{1,1}_{loc}([0,\infty); X)$ and suppose that*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Assume that there exists a positive number r such that

$$\mu(\{t \in [0,T] : ||x(t)|| \le r\}) \to \infty \text{ as } T \to \infty.$$

Then $\lim_{t\to\infty} f(x(t)) = \inf(f)$.

Theorem 2.3. *Let $V \in \mathcal{A}$ be regular, let f be Lipschitz on bounded subsets of X, and assume that $\lim_{||x||\to\infty} f(x) = \infty$. Let K_0 and ϵ be positive. Then there exist $N_0 > 0$ and $\delta > 0$ such that for each $T \ge N_0$ and each mapping $x \in W^{1,1}(0,T;X)$ satisfying*

$$||x(0|| \le K_0 \text{ and } ||x'(t) - V(x(t))|| \le \delta \text{ for a.e. } t \in [0,T],$$

the following inequality holds for all $t \in [N_0, T]$:

$$f(x(t)) \le \inf(f) + \epsilon.$$

Theorems 2.1-2.3 show that most of continuous descent methods for the minimization of convex functions converge. However, in these results it is assumed that the convex function f is Lipschitz on all bounded subsets of X. No such assumption is needed in [3], the main result of which we will now present.

More precisely, let $(X, ||\cdot||)$ be a Banach space and let $f : X \to R^1$ be a convex continuous function which satisfies the following conditions:

C(i) $\lim_{||x||\to\infty} f(x) = \infty$;
C(ii) there is $\bar{x} \in X$ such that $f(\bar{x}) \le f(x)$ for all $x \in X$;
C(iii) if $\{x_n\}_{n=1}^{\infty} \subset X$ and $\lim_{n\to\infty} f(x_n) = f(\bar{x})$, then

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = 0.$$

By C(iii), the point $\bar{x}$, where the minimum of f is attained, is unique.

For each $x \in X$ and $r > 0$, set

$$B(x,r) = \{z \in X : ||z - x|| \le r\} \text{ and } B(r) = B(0,r).$$

For each mapping $A : X \to X$ and each $r > 0$, put

$$\mathrm{Lip}(A,r) = \sup\{||Ax - Ay||/||x - y|| : x, y \in B(r) \text{ and } x \ne y\}.$$

Denote by $\mathcal{A}_l$ the set of all mappings $V : X \to X$ such that $\mathrm{Lip}(V,r) < \infty$ for each positive r (this means that the restriction of V to any bounded subset of X is Lipschitz) and $f^0(x, Vx) \le 0$ for all $x \in X$.

For the set $\mathcal{A}_l$ we consider the uniformity determined by the base

$$E_s(n,\epsilon) = \{(V_1,V_2) \in \mathcal{A}_l \times \mathcal{A}_l : \mathrm{Lip}(V_1 - V_2, n) \le \epsilon$$

$$\text{and } ||V_1x - V_2x|| \le \epsilon \text{ for all } x \in B(n)\}.$$

Clearly, this uniform space $\mathcal{A}_l$ is metrizable and complete. The topology induced by this uniformity in $\mathcal{A}_l$ will be called the strong topology.

We will also equip the space $\mathcal{A}_l$ with the uniformity determined by the base

$$E_w(n,\epsilon) = \{(V_1,V_2) \in \mathcal{A}_l \times \mathcal{A}_l : ||V_1x - V_2x|| \le \epsilon$$

$$\text{for all } x \in B(n)\}$$

where $n, \epsilon > 0$. The topology induced by this uniformity will be called the weak topology.

Before stating the main theorem of [3], we present the following existence result which is also proved in [3].

Proposition 2.2. *Let $x_0 \in X$ and $V \in \mathcal{A}_l$. Then there exists a continuously differentiable mapping $x : [0, \infty) \to X$ such that*

$$x'(t) = V(x(t)),\ t \in [0, \infty),$$
$$x(0) = x_0.$$

The following theorem is the main result of [3].

Theorem 2.4. *There exists a set $\mathcal{F} \subset \mathcal{A}_l$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}_l$ such that for each $V \in \mathcal{F}$ the following property holds:*

For each $\epsilon > 0$ and each $n > 0$, there exist $T_{\epsilon n} > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}_l$ with the weak topology such that for each $W \in \mathcal{U}$ and each differentiable mapping $y : [0, \infty) \to X$ satisfying

$$|f(y(0))| \le n \text{ and } y'(t) = W(y(t)) \text{ for all } t \ge 0,$$

the inequality $||y(t) - \bar{x}|| \le \epsilon$ holds for all $t \ge T_{\epsilon n}$.

3. Lipschitz functions

Let $(X, ||\cdot||)$ be a Banach space, $(X^*, ||\cdot||_*)$ its dual space, and let $f : X \to R^1$ be a function which is bounded from below and Lipschitz on bounded subsets of X. Recall that for each pair of sets $A, B \subset X^*$,

$$H(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} ||x - y||_*,\ \sup_{y \in B} \inf_{x \in A} ||x - y||_*\}$$

is the Hausdorff distance between A and B. We denote by $\mathrm{cl}(E)$ the closure of a set $E \subset X$ in the norm topology.

For each $x \in X$, let

$$f^0(x,h) = \limsup_{t \to 0^+,\ y \to x} [f(y + th) - f(y)]/t,\ h \in X,$$

be Clarke's generalized directional derivative of f at the point x and let

$$\partial f(x) = \{l \in X^* :\ f^0(x,h) \ge l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x. We also define

$$\Xi(x) = \inf\{f^0(x,h) :\ h \in X \text{ and } ||h|| = 1\}.$$

It is well known that the set $\partial f(x)$ is nonempty and bounded. Set

$$\inf(f) = \inf\{f(x) :\ x \in X\}.$$

From now on, we denote by $\mathcal{A}$ the set of all mappings $V : X \to X$ such that V is bounded on every bounded subset of X, and for each $x \in X$, $f^0(x, Vx) \le 0$. We denote by $\mathcal{A}_c$ the set of all continuous $V \in \mathcal{A}$ and by $\mathcal{A}_b$ the set of all $V \in \mathcal{A}$ which are bounded on X. Finally, let $\mathcal{A}_{bc} = \mathcal{A}_b \cap \mathcal{A}_c$. Next, we endow the set $\mathcal{A}$ with two metrics, ρ_s and ρ_w. To define ρ_s, we set, for each $V_1, V_2 \in \mathcal{A}$,

$$\tilde{\rho}_s(V_1, V_2) = \sup\{||V_1 x - V_2 x|| :\ x \in X\}$$

and

$$\rho_s(V_1, V_2) = \tilde{\rho}_s(V_1, V_2)(1 + \tilde{\rho}_s(V_1, V_2))^{-1}.$$

(Here we use the convention that $\infty/\infty = 1$.) Clearly, $(\mathcal{A}, \rho_s)$ is also a complete metric space. To define ρ_w, we set, for each $V_1, V_2 \in \mathcal{A}$ and each integer $i \ge 1$,

$$\rho_i(V_1, V_2) = \sup\{||V_1 x - V_2 x|| :\ x \in X \text{ and } ||x|| \le i\}$$

and

$$\rho_w(V_1, V_2) = \sum_{i=1}^{\infty} 2^{-i}[\rho_i(V_1, V_2)(1 + \rho_i(V_1, V_2))^{-1}].$$

Clearly, $(\mathcal{A}, \rho_w)$ is a complete metric space. It is also not difficult to see that the collection of the sets

$$E(N,\epsilon) = \{(V_1,V_2) \in \mathcal{A} \times \mathcal{A} : \ ||V_1x - V_2x|| \leq \epsilon, \ x \in X, \ ||x|| \leq N\},$$

where $N, \epsilon > 0$, is a base for the uniformity generated by the metric ρ_w. It is easy to see that

$$\rho_w(V_1,V_2) \leq \rho_s(V_1,V_2) \text{ for all } V_1,V_2 \in \mathcal{A}.$$

The metric ρ_w induces on $\mathcal{A}$ a topology which is called the weak topology and ρ_s induces a topology which is called the strong topology. Clearly, $\mathcal{A}_c$ is a closed subset of $\mathcal{A}$ with the weak topology while $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ are closed subsets of $\mathcal{A}$ with the strong topology. We consider the subspaces $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ with the metrics ρ_s and ρ_w which induce the strong and the weak topologies, respectively.

To minimize a convex function f, one usually looks for a sequence $\{x_i\}_{i=1}^{\infty}$ which tends to a minimum point of f (if such a point exists) or at least such that $\lim_{i\to\infty} f(x_i) = \inf(f)$. If f is not necessarily convex, but X is finite-dimensional, then we expect to construct a sequence which tends to a critical point z of f, namely, a point z for which $0 \in \partial f(z)$. If f is not necessarily convex and X is infinite-dimensional, then the problem is more difficult and less understood because we cannot guarantee, in general, the existence of a critical point and a convergent subsequence. To partially overcome this difficulty, we have introduced the function $\Xi : X \to R^1$. Evidently, a point z is a critical point of f if and only if $\Xi(z) \geq 0$. Therefore we say that z is ϵ-critical for a given $\epsilon > 0$ if $\Xi(z) \geq -\epsilon$. In [17] we looked for sequences $\{x_i\}_{i=1}^{\infty}$ such that either $\liminf_{i\to\infty} \Xi(x_i) \geq 0$ or at least $\limsup_{i\to\infty} \Xi(x_i) \geq 0$. In the first case, given $\epsilon > 0$, all the points x_i, except possibly a finite number of them, are ϵ-critical, while in the second case this holds for a subsequence of $\{x_i\}_{i=1}^{\infty}$.

In [17] it was shown, under certain assumptions on f, that for most (in the sense of Baire's categories) vector fields $W \in \mathcal{A}$, certain discrete iterative processes yield sequences with the desirable properties. Moreover, it was shown there that the complement of the set of "good" vector fields is not only of the first category, but also σ-porous. As a matter of fact, we used there the concept of porosity with respect to a pair of metrics, which was introduced in [21].

Recall that when (Y, d) is a metric space we denote by $B_d(y, r)$ the closed ball of center $y \in Y$, and radius $r > 0$. Assume that Y is a nonempty set and $d_1, d_2 : Y \times Y \to [0, \infty)$ are two metrics which satisfy $d_1(x, y) \leq d_2(x, y)$ for all $x, y \in Y$.

A subset $E \subset Y$ is called porous with respect to the pair (d_1, d_2) (or just porous if the pair of metrics is fixed) if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there is $z \in Y$ for which $d_2(z, y) \leq r$ and

$$B_{d_1}(z, \alpha r) \cap E = \emptyset.$$

A subset of the space Y is called σ-porous with respect to (d_1, d_2) (or just σ-porous if the pair of metrics is understood) if it is a countable union of porous (with respect to (d_1, d_2)) subsets of Y.

Note that if $d_1 = d_2$, then by Proposition 1.1 of [21] our definitions reduce to those in [9, 10, 16]. We use porosity with respect to a pair of metrics because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, it is shown in [21, Proposition 1.2] that if a subset $E \subset Y$ is porous with respect to (d_1, d_2), then E is porous with respect to any metric which is weaker than d_2 and stronger than d_1. For each subset $E \subset X$, we denote by $cl(E)$ the closure of E in the norm topology. The results of [17] were established in any Banach space and for those functions which satisfy the following two assumptions.

B(i) For each $\epsilon > 0$, there exists $\delta \in (0, \epsilon)$ such that

$$cl(\{x \in X : \ \Xi(x) < -\epsilon\}) \subset \{x \in X : \ \Xi(x) < -\delta\};$$

B(ii) For each $r > 0$, the function f is Lipschitz on the ball $\{x \in X : \ ||x|| \leq r\}$.

We say that a mapping $V \in \mathcal{A}$ is regular if for any natural number n there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying $||x|| \leq n$ and $\Xi(x) < -1/n$, we have $f^0(x, Vx) \leq -\delta(n)$.

We denote by $\mathcal{F}$ the set of all regular vector fields $V \in \mathcal{A}$.

The following result was established in [17].

Theorem 3.1. *Assume that both B(i) and B(ii) hold. Then $\mathcal{A} \setminus \mathcal{F}$ (respectively, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_b \setminus \mathcal{F}$ and $\mathcal{A}_{bc} \setminus \mathcal{F}$) is a σ-porous subset of the space $\mathcal{A}$ (respectively, $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$) with respect to the pair (ρ_w, ρ_s).*

In the sequel we will also make use of the following assumption:

B(iii) For each integer $n \geq 1$, there exists $\delta > 0$ such that for each $x_1, x_2 \in X$ satisfying $||x_1||, ||x_2|| \leq n$, $\min\{\Xi(x_i) : i = 1, 2\} \leq -1/n$, and $||x_1 - x_2|| \leq \delta$, the following inequality holds: $H(\partial f(x_1), \partial f(x_2)) \leq 1/n$.

Throughout this section, as we did in Section 2, we let $x \in W^{1,1}(0, T; X)$, i.e. (see, e.g., [6]),

$$x(t) = x_0 + \int_0^t u(s)ds, \ t \in [0, T],$$

where $T > 0$, $x_0 \in X$ and $u \in L^1(0, T; X)$. Then $x : [0, T] \to X$ is absolutely continuous and $x'(t) = u(t)$ for a.e. $t \in [0, T]$.

Now we are ready to state three convergence theorems which are established in [1].

Theorem 3.2. *Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular and let*

$$x \in W^{1,1}_{loc}([0, \infty); X).$$

Assume that

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0, \infty)$$

and that the function $x(t)$, $t \in [0, \infty)$, is bounded. Then for each $\epsilon > 0$,

$$\lim_{T \to \infty} \mu(\{t \in [T, \infty) : \Xi(x(t)) < -\epsilon\}) = 0.$$

Theorem 3.3. *Let $V \in \mathcal{A}$ be regular, let B(i), B(ii) and B(iii) hold, and let $x \in W^{1,1}_{loc}([0, \infty); X)$ be a bounded function which satisfies*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0, \infty).$$

Then

$$\liminf_{t \to \infty} \Xi(x(t)) \geq 0.$$

Theorem 3.4. *Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular, and suppose that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let K_0 and ϵ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $T \geq N_0$, each $W \in \mathcal{U}$, and each mapping $x \in W^{1,1}(0, T; X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0, T],$$

the following inequality holds:

$$\mu\{t \in [0, T] : \Xi(x(t)) < -\epsilon\} \leq N_0.$$

Corollary 3.1. *Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular, and suppose that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let K_0, ϵ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $W \in \mathcal{U}$ and each mapping $x \in W^{1,1}_{loc}([0,\infty); X)$ satisfying

$$||x(0)|| \le K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,\infty),$$

the following inequality holds:

$$\mu\{t \in [0,\infty) : \Xi(x(t)) < -\epsilon\} \le N_0.$$

This corollary, which is an extension of Theorem 3.2, follows immediately from Theorem 3.4.

It was shown in [1] that if f satisfies a Palais-Smale type condition, then we can obtain extensions of Theorems 3.2-3.4. In these extensions, instead of studying the asymptotic behavior of $\Xi(x(t))$ as $t \to \infty$, we study the asymptotic behavior of $x(t)$ itself.

Let $f : X \to R^1$ be a locally Lipschitz function which is bounded from below.

In our setting we say that the function f satisfies the Palais-Smale (P-S) condition if each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\sup\{|f(x_n)| : n = 1,2,\dots\} < \infty$$

and $\limsup_{n\to\infty} \Xi(x_n) \ge 0$ has a norm convergent subsequence.

Note that this is a generalization of the classical Palais-Smale condition to locally Lipschitz functions.

Define

$$\mathrm{Cr}(f) = \{x \in X : \Xi(x) \ge 0\}.$$

For each $x \in X$ and $A \subset X$, set

$$d(x, A) = \inf\{||x - y|| : y \in A\}.$$

The next three theorems are also established in [1].

Theorem 3.5. *Let f satisfy B(i), B(ii) and the (P-S) condition, let $V \in \mathcal{A}$ be regular, and let $x \in W^{1,1}_{loc}([0,\infty); X)$ be a bounded mapping which satisfies*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then for each $\epsilon > 0$,

$$\lim_{T\to\infty} \mu(\{t \in [0,\infty) : d(x(t), Cr(f)) > \epsilon\}) = 0.$$

Theorem 3.6. *Let f satisfy the (P-S) condition, let $V \in \mathcal{A}$ be regular, let B(i), B(ii) and B(iii) hold, and let $x \in W^{1,1}_{loc}([0,\infty); X)$ be a bounded mapping which satisfies*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then

$$\limsup_{t\to\infty} d(x(t), Cr(f)) = 0.$$

Theorem 3.7. *Let f satisfy the (P-S) condition, B(i) and B(ii), and suppose that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let $V \in \mathcal{A}$ be regular, and let K_0 and γ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $T \ge N_0$, each $W \in \mathcal{U}$, and each mapping $x \in W^{1,1}(0,T; X)$ satisfying

$$||x(0)|| \le K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,T],$$

the following inequality holds:

$$\mu(\{t \in [0,T] : d(x(t), Cr(f)) > \gamma\} \le N_0.$$

Corollary 3.2. *Let f satisfy the (P-S) condition, B(i) and B(ii), and suppose that*

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let $V \in \mathcal{A}$ be regular, and let K_0 and γ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $W \in \mathcal{U}$ and each mapping $x \in W^{1,1}_{loc}([0,\infty);X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,\infty),$$

the following inequality holds:

$$\mu\{t \in [0,\infty) : d(x(t), Cr(f)) > \gamma\} \leq N_0.$$

This corollary, which is an extension of Theorem 3.5, is a consequence of Theorem 3.7.

4. Approximate solutions to evolution equations governed by regular vector fields

Let $(X, ||\cdot||)$ be a Banach space, $(X^*, ||\cdot||_*)$ its dual space, and let $f : X \to R^1$ be a function which is bounded from below and Lipschitz on bounded subsets of X. In this section we use the notation and definitions introduced in Section 3.

Once again, let $x \in W^{1,1}(0,T;X)$, i.e.,

$$x(t) = x_0 + \int_0^t u(s)ds, \; t \in [0,T],$$

where $T > 0$, $x_0 \in X$ and $u \in L^1(0,T;X)$.

The following results were established in [2].

Theorem 4.1. *Let B(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let K_0 and ϵ be positive numbers. Then there exist $N_0 > 0$ and $\tilde{K} > 0$ such that the following property holds:

For each $T \geq N_0$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0,T;X)$ satisfies

$$||x(0)| \leq K_0$$

and

$$||x'(t) - V(x(t))|| \leq \gamma \text{ for a.e. } t \in [0,T],$$

then

$$||x(t)|| \leq \tilde{K}, \; t \in [0,T],$$

and

$$\mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\} \leq N_0.$$

Theorem 4.2. *Let B(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let $\gamma : [0,\infty) \to [0,1]$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If

$$x \in W^{1,1}_{loc}([0,\infty);X)$$

is bounded and satisfies

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ a.e. } t \in [0,\infty), \tag{4.1}$$

then for each $\epsilon > 0$, there exists $N_\epsilon > 0$ such that the following property holds:

for each $\Delta \geq N_\epsilon$, there is $t_\Delta > 0$ such that if $s \geq t_\Delta$, then

$$\mu\{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\} \leq N_\epsilon.$$

Theorem 4.3. *Let B(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let a function $\gamma : [0, \infty) \to [0, 1]$ satisfy $\lim_{t\to\infty} \gamma(t) = 0$. If $x \in W^{1,1}_{loc}([0, \infty); X)$ is bounded and satisfies (4.1), then for each $\epsilon > 0$,

$$\lim_{T \to \infty} \mu\{t \in [0, T] : \Xi(x(t)) < -\epsilon\}/T = 0.$$

Recall that

$$\mathrm{Cr}(f) = \{x \in X : \Xi(x) \geq 0\},$$

and for each $x \in X$ and $A \subset X$, set

$$d(x, A) = \inf\{||x - y|| : y \in A\}.$$

We are now ready to present the three convergence results obtained in [2] regarding functions satisfying the Palais-Smale condition.

Theorem 4.4. *Let B(ii) hold and let $V \in \mathcal{A}$ be regular. Assume that*

$$\lim_{||x|| \to \infty} f(x) = \infty,$$

and that f satisfies the (P-S) condition. Let K_0 and ϵ be positive numbers. Then there exist $N_*, \tilde{K} > 0$ such that the following property holds:

for each $T \geq N_*$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0, T; X)$ satisfies

$$||x(0)|| \leq K_0$$

and

$$||x'(t) - V(x(t))|| \leq \gamma \text{ for a.e. } t \in [0, T],$$

then

$$||x(t)|| \leq \tilde{K}, \; t \in [0, T],$$

and

$$\mu\{t \in [0, T] : d(x(t), Cr(f)) > \epsilon\} \leq N_*.$$

Theorem 4.5. *Let B(ii) hold and let $V \in \mathcal{A}$ be regular. Assume that f satisfies the (P-S) condition and that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let $\gamma : [0, \infty) \to [0, \infty)$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If

$$x \in W^{1,1}_{loc}([0, \infty); X)$$

is bounded and satisfies

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ for a.e. } t \in [0, \infty),$$

then for each $\delta > 0$, there exists $N_0 > 0$ such that the following property holds:

for each $\Delta \geq N_0$, there is $t_\Delta > 0$ such that if $s \geq t_\Delta$, then

$$\mu\{t \in [s. s + \Delta] : d(x(t), Cr(f)) > \delta\} \leq N_0.$$

Theorem 4.6. *Let B(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Let $\gamma : [0, \infty) \to [0, 1]$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If

$$x \in W^{1,1}_{loc}([0, \infty); X)$$

is bounded and satisfies

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ for a.e. } t \in [0, \infty),$$

then for each $\delta > 0$,

$$\lim_{T \to \infty} \mu\{t \in [0, T] : d(x(t), Cr(f)) > \delta\}/T = 0.$$

5. Super-regularity and evolution equations governed by super-regular vector fields

In this section we continue to examine continuous descent methods for the minimization of Lipschitz functions defined on a general Banach space. We present several convergence theorems for those methods which are generated by super-regular vector fields, a notion which is introduced in [4]. We use the notation and the definitions from Section 3.

Let $(X, ||\cdot||)$ be a Banach space, $(X^*, ||\cdot||_*)$ its dual space, and let $f : X \to R^1$ be a function which is bounded from below and Lipschitz on bounded subsets of X.

A mapping $V \in \mathcal{A}$ is called super-regular if for any natural number n, there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying $\Xi(x) < -1/n$, we have $f^0(x, Vx) \leq -\delta(n)$.

Denote by $\mathcal{G}$ the set of all super-regular vector fields $V \in \mathcal{A}$.

The following results have been established in [4].

Theorem 5.1. *Assume that B(i) holds and f is Lipschitz on X. Then the set $\mathcal{A}\setminus\mathcal{G}$ (respectively, $\mathcal{A}_c\setminus\mathcal{G}$. $\mathcal{A}_b\setminus\mathcal{G}$, $\mathcal{A}_{bc}\setminus\mathcal{G}$) is a σ-porous set of the space $\mathcal{A}$ (respectively, $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$) with respect to the pair (ρ_s, ρ_s).*

Theorem 5.2. *Assume that f is Lipschitz on X and let $V \in \mathcal{A}$ be super-regular.*

Let $K_0, \epsilon > 0$. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $(\mathcal{A}, \rho_s)$ such that for each $T \geq N_0$, each $W \in \mathcal{U}$, and each $x \in W^{1,1}(0,T;X)$ which satisfies

$$||x(0)|| \leq K_0 \tag{5.1}$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,T],$$

the following inequality holds:

$$\mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\} \leq N_0.$$

Corollary 5.1. *Assume that f is Lipschitz on X and let $V \in \mathcal{A}$ be super-regular. Let $K_0, \epsilon > 0$. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $(\mathcal{A}, \rho_s)$ such that for each $W \in \mathcal{U}$ and each $x \in W^{1,1}_{loc}([0,\infty);X)$ which satisfies (5.1) and*

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,\infty),$$

the following inequality holds

$$\mu\{t \in [0,\infty) : \Xi(x(t)) < -\epsilon\} \leq N_0.$$

Corollary 5.2. *Assume that f is Lipschitz on X, $V \in \mathcal{A}$ is super-regular and that $x \in W^{1,1}_{loc}([0,\infty);X)$ satisfies*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then for each $\epsilon > 0$,

$$\mu\{t \in [0,\infty) : \Xi(x(t)) < -\epsilon\}$$

is finite.

Theorem 5.3. *Assume that f is Lipschitz on X, $V \in \mathcal{A}$ is super-regular and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of V in $(\mathcal{A}, \rho_s)$ such that for each $W \in \mathcal{U}$ and each $x \in W^{1,1}_{loc}([0,\infty);X)$ satisfying*

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,\infty),$$

the following inequality holds:

$$\mu\{t \in [0,\infty) : \Xi(x(t)) < -\epsilon\} < \infty.$$

Theorem 5.4. *Assume that f is Lipschitz on X and that $V \in \mathcal{A}$ is super-regular. Let $K_0, \epsilon > 0$. Then there exists $N_\epsilon > 0$ such that the following property holds:*

For each $T \geq N_\epsilon$, there is $\delta > 0$ such that if $x \in W^{1,1}(0,T;X)$ satisfies

$$||x(0)|| \leq K_0$$

and

$$||x'(t) - V(x(t))|| \leq \delta \text{ for a.e. } t \in [0,T],$$

then

$$\mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\} \leq N_\epsilon.$$

Theorem 5.5. *Assume that f is Lipschitz on X, $V \in \mathcal{A}$ is super-regular, and $\gamma : [0,\infty) \to [0,1]$ satisfies $\lim_{t\to\infty} \gamma(t) = 0$. Assume also that*

$$x \in W^{1,1}_{loc}([0,\infty);X)$$

satisfies

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ a.e. } t \in [0,\infty)$$

and that x is bounded.

Then for each $\epsilon > 0$, there exists $N_\epsilon > 0$ such that the following property holds:

For each $\Delta \geq N_\epsilon$, there is $t_\Delta > 0$ such that if $s \geq t_\Delta$, then

$$\mu\{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\} \leq N_\epsilon.$$

Theorem 5.6. *Assume that f is Lipschitz on X and $V \in \mathcal{A}$ is super-regular, and let a function $\gamma : [0,\infty) \to [0,1]$ satisfy $\lim_{t\to\infty} \gamma(t) = 0$. If*

$$x \in W^{1,1}_{loc}([0,\infty);X)$$

is bounded and satisfies

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ for a.e. } t \in [0,\infty),$$

then for each $\epsilon > 0$,

$$\lim_{T\to\infty} \mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\}/T = 0.$$

Theorem 5.7. *Let f be Lipschitz on X and satisfy the (P-S) condition, and let $V \in \mathcal{A}$ be super-regular.*

Let $K_0, \delta > 0$. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $(\mathcal{A}, \rho_s)$ such that for each $T \geq N_0$, $W \in \mathcal{U}$, and each $x \in W^{1,1}(0,T;X)$ which satisfies

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,T],$$

the following inequality holds:

$$\mu\{t \in [0,T] : d(x(t), Cr(f)) > \delta\} \leq N_0.$$

Corollary 5.3. *Let f be Lipschitz on X and satisfy the (P-S) condition, and let $V \in \mathcal{A}$ be super-regular. Let $K_0, \delta > 0$. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $(\mathcal{A}, \rho_s)$ such that for each $W \in \mathcal{U}$ and each $x \in W^{1,1}_{loc}([0,\infty);X)$ which satisfies*

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)),\ t \in [0,\infty),$$

we have

$$\mu\{t \in [0,\infty) : d(x(t), Cr(f)) > \delta\} \leq N_0.$$

Corollary 5.4. *Assume that f is Lipschitz on X and satisfies the (P-S) condition. Let $V \in \mathcal{A}$ be super-regular and let $x \in W^{1,1}_{loc}([0,\infty); X)$ satisfy*

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then for each $\delta > 0$,

$$\mu\{t \in [0,\infty) : d(x(t), Cr(f)) > \delta\}$$

is finite.

Theorem 5.8. *Assume that f is Lipschitz on X and satisfies the (P-S) condition, and that $V \in \mathcal{A}$ is super-regular. Let $K_0, \delta > 0$. Then there exists $N_\delta > 0$ such that the following property holds:*

For each $T \geq N_\delta$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0,T;X)$ satisfies

$$||x(0)|| \leq K_0$$

and

$$||x'(t) - V(x(t))|| \leq \gamma \text{ for a.e. } t \in [0,T],$$

then

$$\mu\{t \in [0,T] : d(x(t), Cr(f)) > \delta\} \leq N_\delta.$$

Theorem 5.9. *Assume that f is Lipschitz on X and satisfies the (P-S) condition. Let $V \in \mathcal{A}$ be super-regular and let $\gamma : [0,\infty) \to [0,1]$ satisfy $\lim_{\to\infty} \gamma(t) = 0$. Assume that $x \in W^{1,1}_{loc}([0,\infty); X)$ is bounded and satisfies*

$$||x'(t) - V(x(t))|| \leq \gamma(t) \text{ for a.e. } t \in [0,\infty). \tag{5.2}$$

Then for each $\delta > 0$, there exists $N_\delta > 0$ such that the following property holds:

For each $\Delta \geq N_\delta$, there is $t_\Delta > 0$ such that if $s \geq t_\Delta$, then

$$\mu\{t \in [s, s+\Delta] : d(x(t), Cr(f)) > \delta\} \leq N_\delta.$$

Theorem 5.10. *Assume that f is Lipschitz on X and satisfies the (P-S) condition. Let $V \in \mathcal{A}$ be super-regular and let $\gamma : [0,\infty) \to [0,1]$ satisfy $\lim_{t\to\infty} \gamma(t) = 0$. Assume that $x \in W^{1,1}_{loc}([0,\infty); X)$ is bounded and satisfies (5.2). Then for each $\delta > 0$,*

$$\lim_{T\to\infty} \mu\{t \in [0,T] : d(x(t), Cr(f)) > \delta\}/T = 0.$$

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