Relative t-designs in Johnson association schemes for P-polynomial structure

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1 Introduction

Relative t-designs were defined in Q-polynomial and P-polynomial association schemes respectively by Delsarte [6] in 1977 and Bannai–Bannai–Suda–Tanaka [1] in 2015. Actually, relative t-designs in P-polynomial association schemes were first introduced by Delsarte-Seidel [7] in 1998 for binary Hamming association scheme $H(n,2)$. They called such designs regular t-wise balanced designs which are equivalent to relative t-designs in $H(n,2)$ for P-polynomial structure with respect to the fixed point $(0,0,\ldots,0)$. In 2015, Bannai-Bannai-Suda-Tanaka [1] proposed the definition for general P-polynomial association schemes and proved that the concepts of relative t-designs in Hamming association schemes $H(n,q)$ $(q \geq 2)$ for P-polynomial structure and Q-polynomial structure are equivalent. Using this good property, Bannai–Bannai–Zhu [2] proved a necessary and sufficient condition for relative t-designs on two shells of $H(n,2)$. In addition, they proved that if (Y,w) is a relative t-design in $H(n,2)$ on p shells then the subset of (Y,w) on each shell must be a usual (weighted) combinatorial $(t+1-p)$-design. It is an interesting question to ask how the situation is in the case of Johnson association scheme $J(v,k)$. Each nontrivial shell X_r of $J(v,k)$ is known to be a commutative association scheme which is the product of two smaller Johnson association schemes. Bannai–Zhu [4] studied relative t-designs on one shell X_r in $J(v,k)$ for Q-polynomial structure and proved that they are $\mathcal{T}$-designs in X_r for $\mathcal{T} = \{(t_1,t_2) \mid 0 \leq t_1 + t_2 \leq t\}$. In particular, if the weight is constant, then relative t-designs on one shell in $J(v,k)$ are mixed t-designs in $J(k,k_1) \otimes J(v-k,k_2)$ which were introduced and studied by Martin [8]. It is well known that Johnson association schemes are both P- and Q-polynomial association schemes. Therefore it would be interesting to ask the similar question for P-polynomial structure whether we can expect to regard relative t-designs on one shell of $J(v,k)$ as weighted $\mathcal{T}$-designs in X_r for some set $\mathcal{T}$.

In this report, we investigate relative t-designs in $J(v,k)$ for P-polynomial structure and give the answer to this question. The main result is that if (Y,w) is a relative t-design supported by one shell X_r in $J(v,k)$ for P-polynomial structure, then (Y,w) is a weighted $\mathcal{T}$-design in X_r with $\mathcal{T} = \{(t_1,t_2) \mid 0 \leq t_1, t_2 \leq t\}$. We also discuss the existence problem of tight relative t-designs. We make an algorithm to construct tight relative 2-designs in one shell and obtain many examples. In addition, we can construct some tight relative 3-designs on one shell X_{2u} in $J(8u,4u)$ for integer $u \geq 1$.

2 Preliminaries

In this section, we will give the definition of t-designs in Q-polynomial association schemes introduced by Delsarte [5], designs in product of Q-polynomial association schemes by Martin [9] and relative t-designs in P-polynomial association schemes by Bannai–Bannai–Suda–Tanaka [1]. (Please refer to [3] for more information on P-polynomial or Q-polynomial association schemes.)

Throughout this report (Y, w) is assumed to be a weighted subset of X, namely, Y is a non-empty finite subset of X and $w : Y \longrightarrow \mathbb{R}_{>0}$.

2.1 Definition of t-designs

Definition 2.1 ([5, Theorem 3.10])**.** Let $\mathfrak{X} = (X, \{R_r\}_{r=0}^{k})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_k$. A weighted subset (Y, w) of X is called a weighted t-design in $\mathfrak{X}$ if $E_j\chi_{(Y,w)} = 0$ for all $1 \leq j \leq t$, where $\chi_{(Y,w)}$ is the weighted characteristic vector of (Y, w) defined by

$$\chi_{(Y,w)} = \begin{cases} w(y), & \text{if } y \in Y, \\ 0, & \text{if } y \notin Y. \end{cases}$$

Definition 2.2. Let $X = \binom{V}{k}$ be the set of all k-subsets of V with $|V| = v$. A weighted subset (Y, w) of X is called a weighted t-(v, k, λ_t) design if for any $z \in \binom{V}{t}$ the following value

$$\sum_{y \in Y, z \subset y} w(y) = \lambda_t$$

is a constant depending only on t but not on the choice of z.

Remark 1. Delsarte [5, Theorem 4.7] proved that a t-design in Johnson association scheme $J(v, k)$ for Q-polynomial structure, which is a weighted t-design with constant weight $w = 1$, is equivalent to a combinatorial t-(v, k, λ_t) design. We should remark that this result is also true for (non-constant) weighted t-designs in $J(v, k)$ and weighted t-(v, k, λ_t) designs.

2.2 Designs in product of Q-polynomial association schemes

In this sunsection, we recall the concept of designs in product of Q-polynomial association schemes introduced by Martin [9]. For any positive integer k_i, let $\mathcal{C}_i$ be the totally ordered chain on $\{0, 1, \ldots, k\}$ and set $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$. Consider the poset $(\mathcal{C}, \trianglelefteq)$ defined by

$$\mathcal{C} = \{\underline{\ell} = (\ell_1, \ell_2) \mid 0 \leq \ell_i \leq k_i, i = 1, 2\}$$

with partial order $\underline{\ell}' \trianglelefteq \underline{\ell}$ if $\ell_1' \leq \ell_1$ and $\ell_2' \leq \ell_2$. A subset $\mathcal{T}$ of $\mathcal{C}$ is called a *downset* in $(\mathcal{C}, \trianglelefteq)$ if $\underline{\ell} \in \mathcal{T}$ and $\underline{\ell}' \trianglelefteq \underline{\ell}$ imply $\underline{\ell}' \in \mathcal{T}$. For any set $\mathcal{E} \subset \mathcal{C}$, denote

$$\mathcal{E} + \mathcal{E} := \{(\ell_1 + \ell_1', \ell_2 + \ell_2') \mid (\ell_1, \ell_2), (\ell_1', \ell_2') \in \mathcal{E}\}.$$

Let $\mathfrak{X}^{(i)} = (X^{(i)}, \{R_r^{(i)}\}_{r=0}^{k_i})$ be a Q-polynomial association scheme and $E_0^{(i)}, E_1^{(i)}, \ldots, E_{k_i}^{(i)}$ its Q-polynomial ordering of primitive idempotents for $i = 1, 2$. For two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from $X^{(1)} \times X^{(2)}$, define a relation $R_{(h_1,h_2)}$ on $X^{(1)} \times X^{(2)}$ by

$$(x, y) \in R_{(h_1,h_2)} \quad \text{if} \quad (x_1, y_1) \in R_{h_1}^{(1)}, (x_2, y_2) \in R_{h_2}^{(2)}.$$

Define $\mathfrak{X} = \mathfrak{X}^{(1)} \otimes \mathfrak{X}^{(2)} = (X^{(1)} \times X^{(2)}, \mathcal{R})$ as the product of these two association schemes, where the relations set $\mathcal{R}$ is

$$\mathcal{R} = \{R_{(h_1,h_2)} \mid 0 \le h_1 \le k_1, 0 \le h_2 \le k_2\}.$$

In particular, if $\mathfrak{X} = J(v_1, k_1) \otimes J(v_2, k_2)$, then, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, $(x, y) \in R_{(h_1,h_2)}$ means $|x_1 \cap y_1| = k_1 - h_1$ and $|x_2 \cap y_2| = k_2 - h_2$. Moreover, for a set $Y \subset \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, we define the *distance set* of Y by

$$A(Y) = \left\{(h_1, h_2) \mid (x, y) \in R_{(h_1,h_2)}, x, y \in Y, x \neq y\right\}.$$

Note that $(0, 0) \notin A(Y)$.

Definition 2.3 ([9, Theorem 2.3])**.** Let $\mathcal{T}$ be a downset of $\mathcal{C}$. The weighted subset (Y, w) of $X = X^{(1)} \times X^{(2)}$ is called a weighted $\mathcal{T}$-design in $\mathfrak{X} = \mathfrak{X}^{(1)} \otimes \mathfrak{X}^{(2)}$ if

$$(E_{t_1}^{(1)} \otimes E_{t_2}^{(2)})\chi_{(Y,w)} = 0, \quad \text{for all } (t_1, t_2) \in \mathcal{T} \backslash \{(0, 0)\}.$$

In particular, if $\mathfrak{X} = J(v_1, k_1) \otimes J(v_2, k_2)$ with points set $X = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, then we have another equivalent definition of $\mathcal{T}$-designs given by Martin [9, Lemma 2.2]. Define a partial order $\preceq$ on the set $\tilde{X} = \binom{V_1}{\le k_1} \times \binom{V_2}{\le k_2}$ by $z \preceq y$ if $z_1 \subseteq y_1$ and $z_2 \subseteq y_2$ for $z = (z_1, z_2)$ and $y = (y_1, y_2)$ in $\tilde{X}$.

Definition 2.4 ([8])**.** Let (Y, w) be a weighted subset of $\binom{V_1}{k_1} \times \binom{V_2}{k_2}$. The pair (Y, w) is called a weighted (t_1, t_2)-$(v_1, k_1, v_2, k_2, \lambda_{(t_1,t_2)})$ design if for any $(z_1, z_2) \in \binom{V_1}{t_1} \times \binom{V_2}{t_2}$, the following value

$$\sum_{\substack{(y_1,y_2) \in Y \\ (z_1,z_2) \preceq (y_1,y_2)}} w(y_1, y_2) = \lambda_{(t_1,t_2)}$$

is a constant depending only on the pair (t_1, t_2) but not on the choice of (z_1, z_2).

By Definition 2.4, we obtain the following lemma (see also [8]).

Lemma 2.5. *Let (Y, w) be a weighted (t_1, t_2)-$(v_1, k_1, v_2, k_2, \lambda_{(t_1,t_2)})$ design. Then it is also a weighted (s_1, s_2)-$(v_1, k_1, v_2, k_2, \lambda_{(s_1,s_2)})$ design satisfying*

$$\binom{v_1 - s_1}{t_1 - s_1}\binom{v_2 - s_2}{t_2 - s_2}\lambda_{(t_1,t_2)} = \binom{k_1 - s_1}{t_1 - s_1}\binom{k_2 - s_2}{t_2 - s_2}\lambda_{(s_1,s_2)}, \tag{2.1}$$

whenever $0 \le s_1 \le t_1$ and $0 \le s_2 \le t_2$. In particular, $\lambda_{(0,0)} = |Y|$.

Lemma 2.5 implies that a weighted (t, t)-$(v_1, k_1, v_2, k_2, \lambda_{(t,t)})$ design is exactly a weighted $\mathcal{T}$-design in $J(v_1, k_1) \otimes J(v_2, k_2)$ with $\mathcal{T} = \{(t_1, t_2) \mid 0 \le t_1, t_2 \le t\}$.

2.3 Relative t-designs in P-polynomial association schemes

Now let us give the definition of relative t-designs in P-polynomial association schemes introduced by Bannai–Bannai–Suda–Tanaka [1]. Let $\mathfrak{X} = \left(X, \{R_r\}_{r=0}^k\right)$ be a P-polynomial association scheme and u_0 a fixed point in X. For $0 \le r \le k$, we call the subset $X_r := \{x \in X \mid (x, u_0) \in R_r\}$ the r-th shell of $\mathfrak{X}$. Let $\mathbb{R}^{|X|}$ be the space consisting of column vectors indexed by the points in X. Given a point $z \in X_j$, define a vector $f_z \in \mathbb{R}^{|X|}$ whose x-th entry is

$$f_z(x) = \begin{cases} 1 & \text{if } x \in X_i, (x,z) \in R_{i-j}, i \ge j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{Hom}_j(X) := \mathrm{span}\{f_z \mid z \in X_j\}$ for $0 \le j \le k$, then

$$\mathbb{R}^{|X|} = \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_k(X).$$

We say Y is a subset of X on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ if $\{r \mid Y \cap X_r \ne \emptyset\} = \{r_1, r_2, \ldots, r_p\}$. Denote $Y_{r_\nu} = Y \cap X_{r_\nu}$ for $1 \le \nu \le p$.

Definition 2.6 ([1, Definition 1.1])**.** Let $\mathfrak{X} = \left(X, \{R_r\}_{r=0}^k\right)$ be a P-polynomial association scheme and u_0 a fixed point in X. A weighted subset (Y, w) is called a relative t-design on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ in $\mathfrak{X}$ with respect to u_0 if the following

$$\sum_{\nu=1}^{p} \frac{W(Y_{r_\nu})}{|X_{r_\nu}|} \sum_{x \in X_{r_\nu}} f(x) = \sum_{y \in Y} w(y) f(y) \tag{2.2}$$

holds for any $f \in \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_t(X)$, where $W(Y_{r_\nu}) = \sum_{y \in Y_{r_\nu}} w(y)$.

3 Relative t-designs in one shell of $J(v,k)$

In this section, we discuss relative t-designs on one shell of Johnson association scheme $J(v,k)$ for P-polynomial structure. We first describe the structure of each nontrivial shell of $J(v,k)$. Let $X = \binom{V}{k}$ be the set of all k-subsets of $V = \{1, \ldots, v\}$ with $v \ge 2k$ and $J(v,k) = (X, \{R_r\}_{r=0}^k)$ Johnson association scheme. For any fixed point $u_0 \in X$, denote $X_r := \{x \in X \mid |x \cap u_0| = k - r\}$. Without loss of generality, we may assume $u_0 = \{1, 2, \ldots, k\}$. Let $V_1 = u_0$, $V_2 = V \backslash u_0$ and denote $v_i = |V_i|$ for $i = 1, 2$.

Proposition 3.1. *Each nontrivial shell X_r of $J(v,k)$ is identified with the product of two smaller Johnson association schemes $J(k,k_1) \otimes J(v-k,k_2)$. More explicitly,*

(1) $(X_r, \{R_h\}_{h=0}^k) \cong J(k,r) \otimes J(v-k,r)$ *if* $1 \le r \le \frac{k}{2}$.

(2) $(X_r, \{R_h\}_{h=0}^k) \cong J(k,k-r) \otimes J(v-k,r)$ *if* $\frac{k}{2} < r \le \frac{v-k}{2}$.

(3) $(X_r, \{R_h\}_{h=0}^k) \cong J(k,k-r) \otimes J(v-k,v-k-r)$ *if* $\frac{v-k}{2} < r \le k$.

Proof. For cases (1), (2) and (3), define the bijection $\phi : X_r \longrightarrow \binom{V_1}{k_1} \times \binom{V_2}{k_2}$ respectively by

$$\phi(x) = (V_1 \backslash x, V_2 \cap x), \quad (V_1 \cap x, V_2 \cap x), \quad (V_1 \backslash x, V_2 \backslash x).$$

We can check that, for $x, y \in X_r$, $(x,y) \in R_{h_1+h_2}$ if and only if $(\phi(x), \phi(y)) \in R_{(h_1,h_2)}$. Therefore $(X_r, \{R_h\}_{h=0}^k) \cong J(k,k_1) \otimes J(v-k,k_2)$. $\square$

Remark 2. Without confusion, we identify X_r with the product association scheme $J(k,k_1)\otimes J(v-k,k_2)$ whenever we mention designs in X_r. In addition, if (Y,w) is a weighted subset of one shell X_r of $J(v,k)$, then the above map ϕ preserves the weight as well, namely, $w(y)=w(\phi(y))$ for any $y\in Y$.

Now we are ready to give our main theorem. By definition 2.6, if $p=1$, i.e., $Y\subset X_r$, then we have the specific condition for relative t-designs supported by one shell X_r as follows.

$$\frac{W(Y_r)}{|X_r|}\sum_{x\in X_r} f_z(x)=\sum_{y\in Y} w(y)f_z(y),\quad \forall z\in X_0\cup X_1\cup\cdots\cup X_t. \tag{3.1}$$

Theorem 3.2. *If (Y,w) is a relative t-design in $J(v,k)$ supported by one shell X_r, then (Y,w) is a weighted $\mathcal{T}$-design in X_r with $\mathcal{T}=\{(t_1,t_2)\mid 0\le t_1,t_2\le t\}$.*

Using Lemma 2.5, it is enough to prove that (Y,w) is a (t,t)-$(v_1,k_1,v_2,k_2,\lambda_{(t,t)})$ design for some constant $\lambda_{(t,t)}$ if $(X_r,\{R_h\}_{h=0}^k)\cong J(v_1,k_1)\otimes J(v_2,k_2)$. More precisely, we need to prove that $\sum_{\substack{y\in\phi(Y)\\(z_1,z_2)\preceq y}} w(y)$ is constant for any $(z_1,z_2)\in\binom{V_1}{t}\times\binom{V_2}{t}$.

Remark 3. In Theorem 3.2, (Y,w) is a weighted $\mathcal{T}$-design in X_r means $(\phi(Y),w)$ is a weighted $\mathcal{T}$-design in $J(k,k_1)\otimes J(v-k,k_2)$, where ϕ is the bijection defined in the proof of Proposition 3.1.

4 Lower bound for relative t-designs on one shell

In this section, we give the lower bound for relative t-designs on one shell X_r of $J(v,k)$. By Theorem 3.2, it is equivalent to obtain the lower bound for weighted $\mathcal{T}$-designs in product association scheme $J(k,k_1)\otimes J(v-k,k_2)$ with $\mathcal{T}=\{(t_1,t_2)\mid 0\le t_1,t_2\le t\}$. The following lower bound of designs in product association schemes was proved by Martin [8].

Lemma 4.1 ([8, Theorem 3.2])**.** *Let $\mathcal{T}$ be a downset in $(\mathcal{C},\trianglelefteq)$ and $\mathcal{E}$ a set satisfying $(\mathcal{E}+\mathcal{E})\cap\mathcal{C}\subseteq\mathcal{T}$. If Y is a $\mathcal{T}$-design in $J(v_1,k_1)\otimes J(v_2,k_2)$, then*

$$|Y|\ge\sum_{(j_1,j_2)\in\mathcal{E}}\left[\binom{v_1}{j_1}-\binom{v_1}{j_1-1}\right]\cdot\left[\binom{v_2}{j_2}-\binom{v_2}{j_2-1}\right]. \tag{4.1}$$

Moreover, if equality holds, then for any $(h_1,h_2)\in A(Y)$ we have

$$\sum_{(j_1,j_2)\in\mathcal{E}} Q_{j_1}^{(1)}(h_1)Q_{j_2}^{(2)}(h_2)=0,$$

where $Q_{j_i}^{(i)}(x)$ is the following Hahn polynomial corresponding to $J(v_i,k_i)$

$$Q_j^{(i)}(x)=\left(\binom{v_i}{j}-\binom{v_i}{j-1}\right)\ {}_3F_2\Big(\begin{matrix}-x,-j,-v_i-1+j\\ -k_i,-v_i+k_i\end{matrix};1\Big).$$

From the proof of Theorem 3.2 in [8], one can check that inequality (4.1) also holds for weighted $\mathcal{T}$-designs in $J(v_1, k_1) \otimes J(v_2, k_2)$.

Corollary 4.2. *If (Y, w) is a relative $2e$-design in $J(v, k)$ on one shell X_r with P-polynomial structure, then*

$$|Y| \geq \binom{k}{e}\binom{v-k}{e}. \tag{4.2}$$

It follows from Theorem 3.2 that (Y, w) is a weighted $\mathcal{T}$-design for $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq 2e\}$. Take $\mathcal{E} = \{(j_1, j_2) \mid 0 \leq j_1, j_2 \leq e\}$. Using Lemma 4.1, we obtain

$$|Y| \geq \sum_{j_1=0}^{e}\sum_{j_2=0}^{e}\left[\binom{k}{j_1} - \binom{k}{j_1-1}\right]\cdot\left[\binom{v-k}{j_2} - \binom{v-k}{j_2-1}\right] = \binom{k}{e}\binom{v-k}{e}.$$

Proposition 4.3. *If (Y, w) is a relative $(2e+1)$-design in $J(v, k)$ on one shell X_r with P-polynomial structure, then*

$$|Y| \geq 4\binom{k-1}{e}\binom{v-k-1}{e}. \tag{4.3}$$

A relative t-design on one shell of $J(v, k)$ is called *tight* if equality holds in (4.2) or (4.3).

At the end of this subsection, we introduce the concept of projections for designs which will be used later. Given a (t_1, t_2)-$(v_1, k_1, v_2, k_2, \lambda_{(t_1,t_2)})$ design Y, define the left and right projection of Y as follows.

$$Y^{(L)} = \{y^{(L)} \mid (y^{(L)}, y^{(R)}) \in Y\}.$$
$$Y^{(R)} = \{y^{(R)} \mid (y^{(L)}, y^{(R)}) \in Y\}.$$

Then $Y^{(L)}$ are t_1-$(v_1, k_1, \lambda_{(t_1,0)})$ designs and $Y^{(R)}$ are t_2-$(v_2, k_2, \lambda_{(0,t_2)})$ designs.

5 Tight relative 2-designs

In this section, we give an algorithm to construct tight relative 2-designs (with constant weight) on one shell of $J(v, k)$. We also provide two explicit examples.

Assume the weight function is constant, i.e., $w = 1$. Let $(Y, 1)$ be a tight relative 2-design on one shell X_r of $J(v, k)$. Using Lemma 4.1, we obtain the distance set $A(Y)$ from the zeros of following equation.

$$F(h_1, h_2) = \sum_{\underline{j} \in \mathcal{E}} Q_{\underline{j}}(\underline{h}) = 1 + Q_1^{(1)}(h_1) + Q_1^{(2)}(h_2) + Q_1^{(1)}(h_1)Q_1^{(2)}(h_2),$$

where $\mathcal{E} = \{(0,0), (0,1), (1,0), (1,1)\}$. More explicitly, we have

$$F(h_1, h_2) = \frac{v_1 v_2\left(v_1 h_1 - h_1 - v_1 k_1 + k_1^2)\right)\left(h_2 v_2 - h_2 - k_2 v_2 + k_2^2\right)}{k_1 k_2\left(v_1 - k_1\right)\left(v_2 - k_2\right)}. \tag{5.1}$$

Then we know that

$$A(Y) \subseteq \left\{ \left(\frac{k_1(v_1-k_1)}{v_1-1}, h_2 \right), \left(h_1, \frac{k_2(v_2-k_2)}{v_2-1} \right) \mid 0 \leq h_1 \leq k_1, 0 \leq h_2 \leq k_2, (h_1,h_2) \neq (0,0) \right\}$$

The construction of explicit tight relative 2-designs on one shell X_r in $J(v,k)$ (i.e. tight $(2,2)$-$(v_1,k_1,v_2,k_2,\lambda_{(2,2)})$ designs) Y is equivalent to the following problem.

Problem. For a point set $X = X^{(L)} \times X^{(R)} = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, find a pair of 2-$(v_1,k_1,\lambda_{(2,0)})$ design $Y^{(L)} = \{y_1^{(L)}, \ldots, y_{\lambda_{(0,0)}}^{(L)}\} \subset X^{(L)}$ and 2-$(v_2,k_2,\lambda_{(0,2)})$ design $Y^{(R)} = \{y_1^{(R)}, \ldots, y_{\lambda_{(0,0)}}^{(R)}\} \subset X^{(R)}$ with the same cardinality, so that $Y = \{(y_i^{(L)}, y_i^{(R)}) \mid 1 \leq i \leq \lambda_{(0,0)}\} \subset Y^{(L)} \times Y^{(R)}$ satisfies the condition that, for each $\{p_1,q_1\} \subset V_1$ and $\{p_2,q_2\} \subset V_2$, there exists exactly $\lambda_{(2,2)}$ element(s) $y \in Y$ such that $(\{p_1,q_1\},\{p_2,q_2\}) \preceq y$. The partial order $\preceq$ means $\{p_1,q_1\} \subset y^{(L)}$ and $\{p_2,q_2\} \subset y^{(R)}$. Moreover, the parameters $\lambda_{(0,0)}$, $\lambda_{(2,0)}$, $\lambda_{(0,2)}$ and $\lambda_{(2,2)}$ satisfy the relation given by Eq. (2.1).

5.1 Basic idea for the construction

Now we explain the basic idea of an algorithm to construct $(2,2)$-$(v_1,k_1,v_2,k_2,\lambda_{(2,2)})$ designs for a given 2-$(v_1,k_1,\lambda_{(2,0)})$ design as the left projection $Y^{(L)}$. The algorithm is applicable for cases when the size of 2-$(v_2,k_2,\lambda_{(2,2)})$ design is equal to $\lambda_{(2,0)}$. Define $\tilde{Y}^{(L)} = (y_1^{(L)}, y_2^{(L)}, \ldots, y_{\lambda_{(0,0)}}^{(L)})$ and $\tilde{Y}^{(R)} = (y_1^{(R)}, y_2^{(R)}, \ldots, y_{\lambda_{(0,0)}}^{(R)})$. We introduce the notation of $\rho(p,q;z)$ for a given $z = (z_1, z_2, \ldots, z_{\lambda_{(0,0)}})$ and $p, q \in V_1$ $(p \neq q)$, defined as

$$\rho(p,q;z) = (z_{i_1}, z_{i_2}, \ldots, z_{i_{\lambda_{(2,0)}}}) \text{ for } \{i_1, i_2, \ldots, i_{\lambda_{(2,0)}}\} = \{i \mid \{p,q\} \subset y_i^{(L)} \in \tilde{Y}^{(L)}\}.$$

We observe that $\rho\left(p,q;\tilde{Y}^{(R)}\right)$ is a 2-$(v_2,k_2,\lambda_{(2,2)})$ design. This will be the key fact in the algorithm. We use this fact as a condition to fix the right (ordered) projection $\tilde{Y}^{(R)}$.

The essential idea of the algorithm is the following. We choose one 2-$(v_1,k_1,\lambda_{(2,0)})$ design as the left (ordered) projection $Y^{(L)}$. The right (ordered) projection $\tilde{Y}^{(R)}$ is unknown. Denote $c = (\emptyset, \ldots, \emptyset)$ of length $\lambda_{(0,0)}$. Since $\rho\left(p,q;\tilde{Y}^{(R)}\right)$ for $\{p,q\} \subset V_1$ is a 2-$(v_2,k_2,\lambda_{(2,2)})$ design, we assign one 2-$(v_2,k_2,\lambda_{(2,2)})$ design to $\rho(p,q;c)$. We do this for all choices of $\{p,q\} \subset V_1$. If we find a c that is consistent with all conditions from $\rho(p,q;c)$, then we take it as $\tilde{Y}^{(R)}$.

5.2 Examples for tight relative 2-designs

Example 1. One trivial example for tight relative 2-design (i.e., (2,2)-$(v_1,k_1,v_2,k_2,\lambda_{(2,2)})$ design) Y is the product of a symmetric 2-(v_1,k_1,λ_1) design $Y^{(L)}$ and a symmetric 2-(v_2,k_2,λ_2) design $Y^{(R)}$ with the same cardinality and $\lambda_1 \cdot \lambda_2 = \lambda_{(2,2)}$. Namely,

$$Y = \{(y_i, y_j) \mid y_i \in Y^{(L)}, y_j \in Y^{(R)}, 1 \leq i \leq v_1, 1 \leq j \leq v_2\}.$$

Example 2. Tight relative 2-designs on X_3 in $J(14,7)$, i.e., tight $(2,2)$-$(7,3,7,3,1)$ designs.

Using the algorithm, we obtain many tight $(2,2)$-$(7,3,7,3,1)$ designs. Since both the left and right projections of a $(2,2)$-$(7,3,7,3,1)$ design are 2-$(7,3,7)$ designs, take the second 2-$(7,3,7)$ design on Spence's homepage [10] as the left projection. We give two examples which is constructed from that 2-$(7,3,7)$ design. Let

$$Y_1 = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{10}, y_{j_1}) \mid i_\ell \in I_\ell, j_\ell \in J_\ell, \ell = 1,2\},$$
$$Y_2 = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{10}, y_{j_3})) \mid i_\ell \in I_\ell, j_m \in J_m, \ell = 1,2, m = 1,1,3.\}$$

where $I_1 = \{1,2,3,4,7,9\}$, $I_2 = \{5,6,8,11\}$, $J_1 = \{1,2,3,4,7,9\}$, $J_2 = \{10\}$, $J_3 = \{1,12,13,14,15,9\}$. Here y_i denotes the i-th block of the list given below.

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,4,7\},\{2,5,6\},\{2,5,7\},\{3,4,6\},\{3,4,7\},\{3,5,6\},$$
$$\{3,5,7\},\{1,4,6\},\{1,5,7\},\{2,4,5\},\{2,6,7\}.$$

Then both Y_1 and Y_2 are tight $(2,2)$-$(7,3,7,3,1)$ designs with the following distance sets.

$$A(Y_1) = \{(0,2),(2,0),(2,2),(1,2),(3,2)\},$$
$$A(Y_2) = \{(0,2),(2,0),(2,2),(2,1),(2,3),(1,2),(3,2)\}.$$

Example 3. A tight relative 2-design on X_4 in $J(26,13)$, i.e., tight $(2,2)$-$(13,4,13,4,1)$ design.

Both the left and right projections of a $(2,2)$-$(13,4,13,4,1)$ design are 2-$(13,4,13)$ designs. It is known that there is a unique symmetric 2-$(13,4,1)$-design $\mathcal{D}_{13}$ up to automorphism. Take twelve copies of $\mathcal{D}_{13}$ and one more design by permuting the points of $\mathcal{D}_{13}$ labelled by 12 and 13, then we have a 2-$(13,4,13)$ design as the left projection $Y^{(L)}$. Using the algorithm, we construct a tight $(2,2)$-$(13,4,13,4,1)$ design Y in the following.

$$Y = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{i_3}, y_{18}) \mid i_\ell \in I_\ell, j_m \in J_m, \ell = 1,2,3, m = 1,2\},$$

where $I_1 = \{1,4,7,8,10,22,30\}$, $I_2 = \{13,16,20,23,26,29\}$, $I_3 = \{14,15,19,24,27,28\}$, $J_1 = \{1,4,7,8,10,13,16,20,22,23,26,29,30\}$, $J_2 = \{2,3,5,6,9,11,12,17,21,25,31,32\}$. Here y_i denotes the i-th block of the list given below

$$\{1,2,3,4\},\{1,2,3,13\},\{1,4,7,12\},\{1,5,6,7\},\{1,5,9,11\},\{1,6,8,10\},\{1,8,9,10\},$$
$$\{1,11,12,13\},\{2,4,9,10\},\{2,5,8,11\},\{2,5,8,12\},\{2,6,7,11\},\{2,6,9,12\},\{2,6,9,13\},$$
$$\{2,7,10,12\},\{2,7,10,13\},\{3,4,8,11\},\{3,5,7,10\},\{3,5,9,12\},\{3,5,9,13\},\{3,6,9,12\},$$
$$\{3,6,10,11\},\{3,7,8,12\},\{3,7,8,13\},\{4,5,6,13\},\{4,5,10,12\},\{4,5,10,13\},\{4,6,8,12\},$$
$$\{4,6,8,13\},\{4,7,9,11\},\{7,8,9,13\},\{10,11,12,13\}.$$

The distance set of Y equals

$$A(Y) = \{(0,3),(1,3),(2,3),(3,0),(3,1),(3,2),(3,3),(3,4),(4,3)\}.$$

5.3 Tight relative 3-designs on one shell of $J(v,k)$

In this subsection, we will discuss the existence problem of tight relative 3-designs (Y,w) in $J(v,k)$ on one shell with constant weight, i.e., $w=1$.

Let $V_1=u_0$, $V_2=V\backslash u_0$. We introduce the notation

$$Y_{(W_1,W_2)}^{(U_1,U_2)}=\{(y_1-W_1,y_2-W_2)\mid (y_1,y_2)\in\phi(Y),W_i\subset y_i\subset V_i-U_i\},$$

where W_i and U_i are subsets of V_i such that $W_i\cap U_i=\emptyset$ for $i=1,2$. Using the proof of Proposition 4.3, if there exists a tight relative 3-design Y on one shell X_r of $J(v,k)$, then we have four tight $\{(2,2)\}$-designs $Y_{(\alpha,\beta)}^{(\emptyset,\emptyset)}$, $Y_{(\alpha,\emptyset)}^{(\emptyset,\beta)}$, $Y_{(\emptyset,\beta)}^{(\alpha,\emptyset)}$ and $Y_{(\emptyset,\emptyset)}^{(\alpha,\beta)}$ respectively in product association schemes $J(v_1',k_1-1)\otimes J(v_2',k_2-1)$, $J(v_1',k_1-1)\otimes J(v_2',k_2)$, $J(v_1',k_1)\otimes J(v_2',k_2-1)$ and $J(v_1',k_1)\otimes J(v_2',k_2)$, where $v_1'=k-1$, $v_2'=v-k-1$ and

$$(k_1,k_2)=\begin{cases}(r,r), & \text{if } 1\le r\le \frac{k}{2},\\ (k-r,r), & \text{if } \frac{k}{2}<r\le\frac{v-k}{2},\\ (k-r,v-k-r), & \text{if } \frac{v-k}{2}<r\le k-1.\end{cases}$$

According to the computer search, there is only one family of possible parameters, i.e., $(v,k,r)=(8u,4u,2u)$ when $v\le 3000$.

We can construct a tight relative 3-design on one shell X_{2u} in $J(8u,4u)$. Choose any tight relative 2-design on one shell X_{2u-1} in $J(8u-2,4u-1)$ as $Y_{(\alpha,\beta)}^{(\emptyset,\emptyset)}$. Replacing the left (resp. right) projection of $Y_{(\alpha,\beta)}^{(\emptyset,\emptyset)}$ by its complementary design, then this new design is $Y_{(\alpha,\emptyset)}^{(\emptyset,\beta)}$ (resp. $Y_{(\emptyset,\beta)}^{(\alpha,\emptyset)}$). Replace both the left and right projection of $Y_{(\alpha,\beta)}^{(\emptyset,\emptyset)}$ by their complementary designs and denote this new design as $Y_{(\emptyset,\emptyset)}^{(\alpha,\beta)}$. Define the sets Y_1,Y_2,Y_3,Y_4 as follows.

$$\begin{aligned}
Y_1&=\left\{(Y'\cup\{\alpha\},Y''\cup\{\beta\})\mid (Y',Y'')\in Y_{(\alpha,\beta)}^{(\emptyset,\emptyset)}\right\},\\
Y_2&=\left\{(Y'\cup\{\alpha\},Y'')\mid (Y',Y'')\in Y_{(\alpha,\emptyset)}^{(\emptyset,\beta)}\right\},\\
Y_3&=\left\{(Y',Y''\cup\{\beta\})\mid (Y',Y'')\in Y_{(\emptyset,\beta)}^{(\alpha,\emptyset)}\right\},\\
Y_4&=Y_{(\emptyset,\emptyset)}^{(\alpha,\beta)}.
\end{aligned}$$

Then $Y=Y_1\cup Y_2\cup Y_3\cup Y_4$ is a tight relative 3-design on the shell X_{2u} in $J(8u,4u)$.

5.4 Further problem

According to the computer search, if $r\le\frac{k}{2}$, then we get the following table for the possible parameters of tight relative 2-designs on one shell X_r in $J(v,k)$, (i.e., tight (2,2)-$(k,r,v-k,r,\lambda_{(2,2)})$ designs).

v	14	22	26	29	30	32	32	37	37	38	$\cdots$
k	7	11	13	13	15	11	16	15	16	19	$\cdots$
r	3	5	4	6	7	5	6	7	6	9	$\cdots$
$\lambda_{(2,2)}$	1	4	1	5	9	2	4	6	3	16	$\cdots$

There are two interesting two families in the table above, namely, $(k, r, \lambda_{(2,2)}) = (u^2+u+1, u+1, 1)$ and $(k, r, \lambda_{(2,2)}) = (4u-1, 2u-1, (u-1)^2)$ for integer $u \geq 2$. We obtained some non-trivial examples for the first family when $u = 2, 3, 4, 5$. **Problem one** is the general construction for tight relative 2-designs with the possible parameters from the above two families.

According to the computer search for $v \leq 3000$, there is only one family of parameters $(v, k, r) = (8u, 4u, 2u)$ for tight relative 3-designs and we can construct such designs if the tight relative 2-designs involved exist. **Problem two** is whether all the tight relative 3-designs on one shell of $J(v, k)$ have the parameter $(v, k, r) = (8u, 4u, 2u)$.

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