A note on holographic structures

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Abstract

We show that there exists a generic structure which is holographic but not ω-categorical.

1 Holographic structures

We assume that a language L is finite and relational.

Definition 1.1 ([1]) Let M be an L-structure, and $n \in \omega$.

1. M is said to be n-oligomorphic, if the number of orbits of $\mathrm{Aut}(M)$ on M^n is finite.
2. M is said to be oligomorphic, if it is n-oligomorphic for every n.

Note 1.2 It is known that M is oligomorphic if and only if it is ω-categorical.

Definition 1.3 ([2]) An L-structure M is said to be holographic, if it is height(L)-oligomorphic, where $\mathrm{height}(L) = \max\{\mathrm{arity}(R) : R \in L\}$.

In many cases, holographic structures are ω-categorical. In [2], Kasymkanuly and Morozov construct a holographic structure which is not ω-categorical. Their example is a kind of plane structure. In this short note, we prove that there exists a generic structure which is holographic but not ω-categorical.

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2 Generic graphs

The basics of generic structures can be found in [3].

In this note, a graph means a simple graph: A structure $A = (A, R)$ with a binary relation R is a graph if and only if R is irreflexive and symmetric. Then A denotes a set of vertices in (A, R), and R^A a set of edges in (A, R).

Let $A, B, C, \cdots$ denote graphs. A predimension $\delta(A)$ of a finite graph A is defined by $\delta(A) = |A| - \alpha|R^A|$, where $0 < \alpha \leq 1$. For finite A, B, we write $\delta(A/B) = \delta(A \cup B) - \delta(A)$.

For finite A, B with $A \subset B$, A is said to be closed in B (written $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$. It is noted that if $A \leq B$ then $A \cap X \leq B \cap X$ for any $X \subset B$.

For (possibly) infinite A, B, $A \leq B$ is defined by $A \cap X \leq B \cap X$ for any finite $X \subset B$.

For $A \subset B$, it can be seen that there is the smallest C with $A \subset C \leq B$. This C is called the closure of A in B, and denoted by $\mathrm{cl}_B(A)$.

Let $\mathbf{K}$ be a class of finite graphs. Then a countable graph M is said to be a $(\mathbf{K}, \leq)$-generic, if it satisfies the following:

- if $A \subset_{\mathrm{fin}} M$ then $A \in \mathbf{K}$;
- if $A \subset M$ and $A \leq B \in \mathbf{K}$, then there is a $B' \leq M$ with $B' \cong_A B$;
- if $A \subset_{\mathrm{fin}} M$ then $\mathrm{cl}_M(A)$ is finite.

For A, B, C with $A = B \cap C$, $B \perp_A C$ is defined by $R^{B \cup C} = R^B \cup R^C$. $D = B \oplus_A C$ is a graph with $D = B \cup C$ and $R^D = R^B \cup R^C$.

Note 2.1 $B \perp_A C$ implies $\delta(B/C) = \delta(B/A)$.

Proof. Since $B \perp_A C$, we have $R^{B \cup C} - R^C = (R^B \cup R^C) - R^C = R^B - R^A$. Then $\delta(B/C) = \delta(B \cup C) - \delta(C) = |B - C| - \alpha|R^{B \cup C} - R^C| = |B - A| - \alpha|R^{B \cup C} - R^A| = \delta(B/A)$.

$(\mathbf{K}, \leq)$ is said to have the free amalgamation property (FAP), if whenever $A \leq B \in \mathbf{K}, A \leq C \in \mathbf{K}$ and $B \perp_A C$, then $B \oplus_A C \in K$.

Fact 2.2 If $(\mathbf{K}, \leq)$ has FAP, then there is a $(\mathbf{K}, \leq)$-generic graph.

Fact 2.3 Let M be a $(\mathbf{K}, \leq)$-generic graph. If $A, A' \leq_{\mathrm{fin}} M$ and $A \cong A'$ then $\mathrm{tp}(A) = \mathrm{tp}(A')$.

3 The construction

Let $\alpha = \frac{1}{2}$, i.e., $\delta(A) = |A| - \frac{1}{2}|R^A|$. Let $\mathbf{K}$ be the class of all finite structures A satisfying that

- $\emptyset \in \mathbf{K}$;
- $|A| \leq 2$, or $\delta(A') \geq 2$ for any $A' \subset A$ with $|A'| \geq 3$.

Note that $\emptyset \neq A \in \mathbf{K}$ implies $\delta(A) \geq 1$. Clearly $\mathbf{K}$ is closed under substructures.

Lemma 3.1 $(\mathbf{K}, \leq)$ has FAP.

Proof. Take A, B, C with $A \leq B \in \mathbf{K}, A \leq C \in \mathbf{K}$ and $B \perp_A C$. We want to show that $D = B \oplus_A C \in \mathbf{K}$. Take any $X \subset D$ with $|X| \geq 3$. For $Y \subset D$, let X_Y denote $X \cap Y$. Since $B \perp_A C$, we have $X_B \perp_{X_A} X_C$. We can assume that $X_B - X_A \neq \emptyset$ and $X_C - X_A \neq \emptyset$.

First, suppose that $\delta(X_B) \geq 2$. By Note 2.1, $\delta(X) = \delta(X_C/X_B) + \delta(X_B) = \delta(X_C/X_A) + \delta(X_B) \geq 0 + 2 = 2$, and hence $X \in \mathbf{K}$.

Next, suppose that $\delta(X_C) \geq 2$. Then we have $X \in \mathbf{K}$ as in the first case.

So we can suppose that $\delta(X_B) < 2$ and $\delta(X_C) < 2$. If $X_A = \emptyset$, then $\delta(X) = \delta(X_B) + \delta(X_C) \geq 1 + 1 \geq 2$, and hence $X \in \mathbf{K}$. If $X_A \neq \emptyset$, then $|X_B|, |X_C| \geq 2$, and so $\delta(X_B) = \delta(X_C) = \frac{3}{2}$. Then $X_A = \{a\}, X_B = \{b, a\}$ and $X_C = \{c, a\}$ with $D \models R(b, a)$ and $D \models R(c, a)$. So $\delta(X) = 3 - \frac{1}{2} \cdot 2 = 2$, and hence $X \in \mathbf{K}$.

By Lemma 3.1, there exists the $(\mathbf{K}, \leq)$-generic graph M.

Lemma 3.2 M is holographic.

Proof. We want to show that M is 2-oligomorphic. Take any $A, A' \subset M$ with $A \cong A'$ and $|A| = |A'| = 2$. Note that $A, A' \leq M$ since $\delta(A), \delta(A') \leq 2$. By Fact 2.3, $\mathrm{tp}(A) = \mathrm{tp}(A')$. Hence the number of orbits of $\mathrm{Aut}(M)$ on M^2 is finite.

Lemma 3.3 Let A be a graph of size 3 with no edges. Then, for each $n \in \omega$, there is a $B \in \mathbf{K}$ with $A \subset B, \mathrm{cl}_B(A) = B$ and $|B| = 3n + 4$.

Proof. Take any $n \in \omega$. Let $A = \{a_0, a_0', a_0''\}$. Take $a_1, a_1', a_1'', \cdots,$ a_n, a_n', a_n'', b satisfying

- $R(a_i, a_{i+1}), R(a_i', a_{i+1}), R(a_i', a_{i+1}'), R(a_i'', a_{i+1}'), R(a_i, a_{i+1}''), R(a_i'', a_{i+1}'')$ for each $i \leq n$;
- $R(a_n, b), R(a_n', b), R(a_n'', b)$.

Let $B = \{a_i, a_i', a_i'' : 0 \leq i \leq n\} \cup \{b\}$. Then it is easily checked that $B \in \mathbf{K}$, $\mathrm{cl}_B(A) = B$ and $|B| = 3n + 4$.

Lemma 3.4 *M is not 3-oligomorphic.*

Proof. Take any n. Let A_n be a graph of size 3 with no relations. By Lemma 3.3, we can take $B_n \in \mathbf{K}$ such that $A_n \subset B_n, \mathrm{cl}_{B_n}(A_n) = B_n$ and $|B_n| = 3n + 4$. Since $B_n \in \mathbf{K}$, we can assume that $B_n \leq M$. Then $\mathrm{cl}_M(A_n) = \mathrm{cl}_{B_n}(A_n) = B_n$. So if $n \neq m$ then $\mathrm{tp}(A_n) \neq \mathrm{tp}(A_m)$ since $|\mathrm{cl}_M(A_n)| \neq |\mathrm{cl}_M(A_m)|$. Therefore the number of orbits on M^3 is infinite. Hence M is not 3-oligomorphic.

Theorem 3.5 There is a countable graph M which is holographic but not ω-categorical.

4 Stable 1-based structures

Let T be a complete theory and $\mathcal{M}$ a big model. T is said to be 1-based, if $Cb(\bar{e}/A) \subset \mathrm{acl}(\bar{e})$ for any tuple $\bar{e} \in \mathcal{M}$ and any algebraically closed $A \subset \mathcal{M}^{eq}$. T is said to have weak elimination of imaginaries, if, for any $e \in \mathcal{M}^{eq}$ there is a tuple $\bar{c} \in M$ with $e \in \mathrm{dcl}(\bar{c})$ and $c \in \mathrm{acl}(e)$.

Theorem 4.1 Let M be a holographic structure with the following conditions:

- M is countably saturated,
- $\mathrm{Th}(M)$ is stable 1-based
- $\mathrm{Th}(M)$ has weak elimination of imaginaries.

Then M is ω-categorical.

Proof. Suppose by way of contradiction that M is not ω-categorical. Then there is $n \geq \mathrm{height}(L)$ such that M is n-oligomorphic but not $(n+1)$-oligomorphic. For simplicity, we assume that $n = 2$. Then there are elements $a, b, c_1, c_2, \ldots$ in M with $\mathrm{tp}(c_i/ab) \neq \mathrm{tp}(c_j/ab)$ and

$\mathrm{tp}(c_i) = \mathrm{tp}(c_j)$ for each i, j with $i \neq j$. For $i \in \omega$, let $E_i = Cb(c_i/ab)$. Since M is 1-based, then we have $E_i \subset \mathrm{acl}(c_i)$. On the other hand, we have $\mathrm{tp}(c_i E_i) \neq \mathrm{tp}(c_j E_j)$ since $\mathrm{tp}(c_i/ab) \neq \mathrm{tp}(c_j/ab)$. Since $\mathrm{tp}(c_1) = \mathrm{tp}(c_i)$, for each $i \in \omega$ there is an elementary map σ_i with $\sigma(c_j) = c_1$. Then we have $\mathrm{tp}(\sigma_i(E_i)/c_1) \neq \mathrm{tp}(\sigma_j(E_j)/c_1)$. Since $\mathrm{Th}(M)$ has weak elimination of imaginaries, $\mathrm{acl}(c_1)$ is infinite (in $\mathcal{M}$). Hecne M is not 2-oligomorphic. A contradiction.

References

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