Proof of the Mordell-Lang Conjecture for function fields: 20 years later

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Abstract

Hrushovski's celebrated Theorem 1.1 in [H96] proved the function field Mordell-Lang conjecture. In his proof highly sophisticated model theoretic arguments of Zariski geometry played crucial roles. As a continuous effort of removing Zariski geometry arguments, the authors of [BBPa], [BBPb], [BBPc] succeeded recently to prove Hrushovski's theorem by reducing to the function field Manin-Mumford conjecture without the dichotomy theorem of Zariski geometry. I report here outline of their arguments.

1 Hrushovski's Theorem

We start with a few definitions.

Definition 1. *(1) A group Γ is p'-finitely generated*

- *if* $\mathrm{ch} = 0$ *and* $\mathbb{Q} \otimes \Gamma$ *is finitely generated as a* $\mathbb{Q}$*-module,*
- *if* $\mathrm{ch} > 0$ *and* $Q_p \otimes \Gamma$ *is finitely generated as a* Q_p*-module where* $Q_p = \{m/n \in \mathbb{Q} : n \text{ prime to } p\}$. *(This condition is valid for finitely generated Abelian groups, and for prime-to-p-torsion groups.)*

(2) A semi-Abelian variety is an extension of an Abelian variety by an algebraic torus.

Hrushovski's Theorem 1.1 in [H96] is the following statement. Suppose K/k is a field extension and k is an algebraically closed field. Assume:

- S is a semi-Abelian variety defined over K,
- X is a subvariety of S.
- Γ is a p'-finitely generated subgroup of S.

Assume further $\boxed{X \cap \Gamma \text{ is Zariski dense in } X.}$

Then there exists

- S_0 a semi-Abelian variety defined over k,
- X_0 a subvariety of S_0 defined over k, and
- a rational homomorphism h from a group subvariety of S into S_0, such that $\boxed{X \text{ is a translate of } h^{-1}(X_0).}$

2 Proof without Zariski geometries

In the introduction of [BBPb], the authors explain the motivation of proving Model-Lang conjecture for function fields without appealing Zariski geometries;

> $\cdots$. But in the positive characteristic case, *type-definable* Zariski geometries are the relevant objects. The needed dichotomy in this case is a combination of a complicated axiomatic account of a field construction in [17] [1] , together with arguments in [15] [2] showing that the axioms are satisfied for the particular minimal types in separably closed fields that we are interested in.
>
> $\cdots$
>
> In both cases, the dichotomy theorem is difficult and its proof impenetrable for non model theory experts (as well as for many model-theorists). $\cdots$

2.1 fML and fMM

Here we state four conjectures; function field Mordel-Lang conjectures in characteristic zero and positive characteristic, and similarly function field Manin-Mamford conjecture.

2.1.1 Function field Mordell-Lang in $\mathrm{ch} = 0$

Let $K = \mathbb{C}(t)^{\mathrm{alg}}$. Let A be an abelian variety over K with $\mathbb{C}$-trace 0, i.e, with no nonzero homomorphic images defined over $\mathbb{C}$.
$\boxed{\text{Assume:}}$

- X : an irreducible subvariety of A (defined over K),
- Γ : a "finite-rank" subgroup of $A(K)$, i.e., Γ is contained in the division points of a finitely generated subgroup of $A(K)$.
- Suppose $X \cap \Gamma$ is Zariski-dense in X.

[1] E. Hrushovski, B. Zilber, Zariski Geometries, J. AMS, 1996
[2] [Hr96]

Then X is a translate of an abelian subvariety of A.

2.1.2 Function field Manin-Mumford in $\mathrm{ch} = 0$

Let $K = \mathbb{C}(t)^{\mathrm{alg}}$. Let A be an abelian variety over K with $\mathbb{C}$-trace 0, i.e, with no nonzero homomorphic images defined over $\mathbb{C}$.

Assume:

- X : an irreducible subvariety of A (defined over K),
- $\Gamma \subseteq \mathrm{Tor}(A)$
- Suppose $X \cap \Gamma$ is Zariski-dense in X.

Then X is a translate of an abelian subvariety of A.

2.1.3 Function field Mordell-Lang in $\mathrm{ch}\, p > 0$.

Let $K = (\mathbb{F}_p^{\mathrm{alg}}(t))^{\mathrm{sep}}$, the separable closure of $\mathbb{F}_p^{\mathrm{alg}}(t)$. Let A be an abelian variety over K with $\mathbb{F}_p^{\mathrm{alg}}$-trace 0.

Assume:

- X : an irreducible subvariety of A, defined over K,
- Γ : a subgroup of $A(K)$ contained in the prime-to-p-division points of a finitely generated subgroup.
- Suppose that $X \cap \Gamma$ is Zariski-dense in X.

Then X is a translate of an abelian subvariety of A.

2.1.4 Function field Manin-Mumford in $\mathrm{ch}\, p > 0$.

Let $K = (\mathbb{F}_p^{\mathrm{alg}}(t))^{\mathrm{sep}}$, the separable closure of $\mathbb{F}_p^{\mathrm{alg}}(t)$. Let A be an abelian variety over K with $\mathbb{F}_p^{\mathrm{alg}}$-trace 0.

Assume:

- X : an irreducible subvariety of A, defined over K,
- $\Gamma \subseteq \mathrm{Tor}(A)$
- Suppose that $X \cap \Gamma$ is Zariski-dense in X.

Then X is a translate of an abelian subvariety of A.

2.2 Strategy

In [BBPa], [BBPb], the authors reduce first the proof of Theorem 1.1 [H96] to fML then secondly the proof of fML to the proof of fMM by model theoretic aruguments without the dichotomy theorem of Zariski geometry.

Hrushovski's Theorem 1.1 in [H96]

$\Uparrow_2$

function field Mordell-Lang conjecture (fML)

$\Uparrow_1$

function field Manin-Mumford conjecture (fMM)

Finaly note that fMM has been proved without model theory, by Pink, Corpit, Rössler [PR], [R1, R2].

2.3 Embed the algebraic-geometric set-up in a differential algebraic one.

The function field Mordell-Lang conjecture is an algebraic-geometric statement. Inspired by Buium's approach [Bu] to the conjecture in charachteristic zero, Hrushovski used model theory of differentially closed fields and separably closed fields together with the dichotomy theorem of Zariski geometry to prove function field Mordell-Lang conjecture.

In [BBPb] to prove function field Mordell-Lang conjecture from function field Manin-Mumford conjecture, the authors use in addition to the basic properties of model theory of differentially closed fields and separably closed fields, and both in $\mathrm{ch} = 0$ and $\mathrm{ch} > 0$, they need

- the Theorem of the Kernel
- g-minimal groups of finite Morley rank.

Moreover, in $\mathrm{ch} = 0$,

- Weak socle theorem.

In $\mathrm{ch} > 0$,

- Quantifier elimination for $A^\sharp = p^\infty(A(\mathcal{U}))$.

2.4 Definitions

Let G be a commutative connected group of finite Morley rank, definable in some ambient stable structure M.

We say that G is generated (abstractly) by some sets $X_1, \cdots, X_n$ if $G \subset \mathrm{acl}(F \cup X_1 \cup \cdots \cup X_n)$, where F is a finite set.

G is said to be almost strongly minimal, if $G \subset \mathrm{acl}(F \cup X)$, where X is a strongly minimal set and F is a finite set.

Definition 2 (Socle$S(G)$)**.** *The socle $S(G)$ of G, is the greatest connected definable subgroup of G which is generated by strongly minimal definable subsets of G, i.e, contained in* $\mathrm{acl}(F \cup X_1, \cup \cdots \cup X_n)$ *where each X_i is a strongly minimal definable subset and F is a finite set.*

Definition 3 (Stabilizer)**.** *For X a definable subset of G with Morley degree* 1*, define the model theoretic stabilizer of X in G,*

$$Stab_G(X) := \{g \in G : MR(X \cap (X+g)) = MR(X)\}$$

(so the stabilizer of the generic type of X over M).

Definition 4 (Rigid)**.** *H is a definable subgroup of G, defined over some B. H is rigid if, passing to a saturated model, all connected definable subgroups of H are defined over* $\mathrm{acl}(B)$*.*

2.5 The group $A^\sharp$, Def 2.1 in [BBPb]

Let $\mathcal{U}$ be a monster model.
(i) In $\mathrm{ch} = 0$,
$A^\sharp$ denotes the "Kolchin closure of the torsion", i.e., the smallest definable (in the sense of differentially closed fields) subgroup of $A(\mathcal{U})$ which contains the torsion subgroup ($A^\sharp$ is definable over K.)

$$\boxed{\mathrm{Tor}(A) \subseteq A^\sharp \subseteq A(\mathcal{U})}$$

(ii) In $\mathrm{ch} > 0$,
$A^\sharp$ denotes $p^\infty(A(\mathcal{U})) = \bigcap_n p^n(A(\mathcal{U}))$
(an infinitely definable subgroup over K).

2.6 Properties of $A^\sharp$, [H96]

Here is a list of properties of $A^\sharp$, [H96].

(i) $A^\sharp$ is the smallest Zariski dense (infinitely) definable subgroup of $A(\mathcal{U})$.

(ii) $A^\sharp$ is connected (no relatively definable subgroup of finite index), and of finite U-rank in $\mathrm{ch} > p$, and finite Morley rank in $\mathrm{ch} = 0$.

(iii) If A is a simple abelian variety, $A^\sharp$ has no proper infinite type-definable subgroup.

(iv) If A is the sum of simple A_i then $A^\sharp$ is the sum of the $A_i^\sharp$.

2.7 Theorem of the Kernel

In all characteristics: K/k : both algebraically closed fields.

Theorem 5. *Let A be an abelian variety over K with k-trace 0, then*

$$A^\sharp(K) \subseteq \mathrm{Tor}(A)$$

where $\mathrm{Tor}(A)$ *is the group of torsion points of A.*

2.8 g-minimal groups

Let G be a group (with additional structure). Assume $\mathrm{RM}(G) < \infty$. commutative and connected

Definition 6. *G is g-minimal if it has no proper nontrivial connected definable subgroup (equivalently, no proper infinite definable subgroup).*

Theorem 7 (Wagner)**.** *Suppose that G is g-minimal. Then any infinite algebraically closed subset of G is an elementary substructure of G.*

3 From fMM to fML in $ch = 0$

Here I explain how to prove fML from fMM in case of ch = 0.

Lemma 8 (Lemma 4.6 in [H96])**.** *The socle $S(G)$ is an almost direct sum of pairwise orthogonal definable groups G_i, where each G_i is almost strongly minimal.*

Theorem 9 (Prop. 4.3 in Hr96, Weak socle theorem)**.** *Let G be a commutative connected group of finite Morley rank. Suppose that the socle $S(G)$ of G is rigid. Suppose further that X is a definable subset of G of Morley degree 1 with finite $Stab_G(X)$.*

Then, some translate of X is contained in $S(G)$. [3]

Fact 10. *If H is a connected finite Morley rank definable subgroup of $A = A(\mathcal{U})$ containing $A^\sharp$, then the socle $S(H) = A^\sharp$.*

We now prove fML from fMM. Assume:

- $K = \mathbb{C}(t)^{\mathrm{alg}}$, and A is an abelian variety over K with $\mathbb{C}$-trace 0.
- X : an irreducible subvariety of A (defined over K),
- Γ : a "finite-rank" subgroup of $A(K)$.
- Suppose $X \cap \Gamma$ is Zariski-dense in X.

We want to show that X is a translate of an abelian subvariety of A. Recall properties of $A^\sharp$:

- $A^\sharp := A^\sharp(\mathcal{U})$, the smallest definable subgroup of $A(\mathcal{U}) \supseteq \mathrm{Tor}(A)$.
- $A^\sharp$ is definable over K, connected with finite Morley rank.
- if A is simple, $A^\sharp$ is a g-minimal group.
- $A/A^\sharp$ definably embeds via some μ in a vector group.
- $\mu(\Gamma)$ is contained in a finite-dimensional vector space over $\mathcal{C}$, the preimage of which we call H.

[3] up to subtracting a definable subset X' of X with $\mathrm{RM}(X') < \mathrm{RM}(X)$.

- H is a definable subgroup of A, connected, $\mathrm{RM}(H) < \infty$, containing both $A^\sharp$ and Γ, and defined over K.

$$\begin{array}{ccccc} A^\sharp & & & & \\ & \subsetneq & & & \\ & & H & \subseteq & A \\ & \subsetneq & & & \\ \Gamma & & & & \end{array}$$

Claim : $A^\sharp(K) = A^\sharp(K^{\mathrm{diff}})$
Proof: $A^\sharp(K) \subseteq A^\sharp(K^{\mathrm{diff}})$ is clear. If $b \in A^\sharp(K^{\mathrm{diff}})$ then $tp(b/K)$ is an isolated type with respect to DCF_0. But then $tp(b/\mathcal{A}(K))$ is isolated in the structure $\mathcal{A}$. As

$$\mathcal{A}(K) \preceq \mathcal{A},$$

we have $b \in A^\sharp(K)$. Hence $A^\sharp(K^{\mathrm{diff}}) \subseteq A^\sharp(K)$. ■

Note that X is now defined over K^{diff}, and $X \cap A^\sharp(K^{\mathrm{diff}})$ is Zariski-dense in X. By the Claim, we have

$$A^\sharp(K^{\mathrm{diff}}) = A^\sharp(K)$$

(so in fact X is defined over K) so, by the Theorem of the Kernel, we have

$$A^\sharp(K^{\mathrm{diff}}) = \mathrm{Tor}(A),$$

hence by function field Manin-Mumford, X is a translate of an abelian subvariety of A.

4 From fMM to fML in $ch > 0$

In this section we prove fML from fMM in case of ch > 0.

4.1 Quntifier elimination

Key theorem is the following:

Theorem 11 (Thm 3.1 [BBPb])**.** Let K be a separably closed field of imperfection 1. and $\mathrm{ch}(K) > 0$**.** Assume A is an abelin variety over K, and $\mathcal{U}$ is a saturated elementary extension of K.
Let $A^\sharp = p^\infty A(\mathcal{U})$ and set

$$\mathcal{A}^\sharp := (A^\sharp, \text{with all relatively definable sets (parameters from } K))$$

Then $\mathrm{Th}(\mathcal{A}^\sharp)$ eliminates quantifiers as $L_{\lambda,t}$-theory.

The proof is very long and technical. As a corollary we have;

Theorem 12 (Lemma 4.6 [BBPb])**.** $\mathcal{A}^\sharp$ and $\mathcal{A}_i^\sharp$ are both connected and of finite Morley rank. Each $\mathcal{A}_i^\sharp$ is g-minimal. Moreover we have $\mathcal{A}(K) \preceq \mathcal{A}$.

4.2 From fMM to fML

Now we show that fML holds from fMM.

Let $K = (\mathbb{F}_p^{\mathrm{alg}}(t))^{\mathrm{sep}}$ be the separable closure of $\mathbb{F}_p^{\mathrm{alg}}(t)$. Let A be a $\mathbb{F}_p^{\mathrm{alg}}$-trace zero abelian variety over K. Assume;

- X is an irreducible subvariety of A and defined over K,
- There exists a finitely generated subgroup H of $A(K)$ such that $\Gamma \subseteq \mathrm{Tor}_{p'}(H) \subseteq A(K)$.
- $X \cap \Gamma$ is Zariski dense in X.

We want to prove that X is a translate of a commutative subvariety of A.

Key point:

Construct an appropriate $\mathcal{E}_0$. Then replace $X \cap \Gamma$ with $X \cap \mathcal{E}_0$ which is Zariski dense in X. Then apply fMM.

Proof : Note that for each $n > 0$ we have $[p^n\Gamma : p^{n+1}\Gamma] < \infty$. Since $X \cap \Gamma$ is Zariski dense in X, there exists a coset D_i of $p^i\Gamma$ in Γ such that

- $i < j \Longrightarrow D_i \supseteq D_j$,
- each $X \cap D_i$ is Zarsiki dense in X.

Hence there is a descending sequence of cosets E_i of $p^iA(K)$ in $A(K)$ with each $X \cap E_i$ is Zariski dense in X. We now work over $\mathcal{U}$, and set

$$E := \bigcap_i E_i.$$

then E is a translate of $A^\sharp$, and $X \cap E$ is Zariski dense in X.

In order to apply fMM and the Theorem of the Kernel (Theorem 5), we need that $X \cap E(K)$ is Zariski dense in X inside the field K. Since the field K is not saturated we do not know whether $E(K) \neq \emptyset$.

Thus we need to consider a translate of $A^\sharp$ rather than a translate of $E(K)$. Put

$$M := (\mathcal{A}, \mathcal{E}) = \begin{array}{l}(A^\sharp(\mathcal{U}), E(\mathcal{U})) \text{ the structure} \\ \text{adding relations and functions definable over } K\end{array}$$

Then by Theorem 12 which is a corollary to the quantifier elimination of $\mathrm{Th}(\mathcal{A}^\sharp)$ we have that Morley rank of $\mathrm{Th}(M)$ is finite.

Let $M_0 \prec M$ be a prime model (atomic model) over $\mathcal{A}(K)$. Then there exists $\mathcal{E}_0$ such that $\mathcal{E}_0 \preceq \mathcal{E}$ and $M_0 = (\mathcal{A}(K), \mathcal{E}_0)$.

Lemma 13. *$X \cap \mathcal{E}_0$ is Zariski dense in X.*

Proof: By contradiction. So suppose $X \cap \mathcal{E}_0$ is not Zariski dense in X. There is a proper subvariety Z of X such that $X \cap \mathcal{E}_0 \subseteq Z \subsetneq X$. By replacing Z with the Zariski closure of $X \cap \mathcal{E}_0$, we view Z as defined over $\mathcal{E}_0$. Note we have $X \cap \mathcal{E}_0 = Z \cap \mathcal{E}_0$.

Consider $Z \cap \mathcal{E}$ as a definable subset in $\mathcal{E}$. Then $Z \cap \mathcal{E}$ is definable over $\mathcal{E}_0$.

Since $\mathcal{E}_0 \prec \mathcal{E}$, we have $X \cap \mathcal{E} = Z \cap \mathcal{E}$. This contradicts with the fact that $X \cap \mathcal{E}$ is Zariski dense in X. Therefore $X \cap \mathcal{E}_0$ must be Zariski dense in X. ■

Now we complete the proof as follows, Let $a \in X \cap \mathcal{E}_0$ and set $X_1 = X - \{a\}$. Then $X_1 \cap A^\sharp(K)$ is Zariski dense in X_1, and X_1 is definable over K. From Theorem of the Kernel we have $A^\sharp(K) = \mathrm{Tor}(A)$, and we also have that $X_1 \cap \mathrm{Tor}(A)$ is Zariski dense in X_1. Hence by applying fMM to $A^\sharp(K)$, we see that X_1 is a translate of commutative subvariety of A. It follows that X is a translate of commutative subvariety of A as well. Now we are done.

5 Final remarks

In this note I only explain briefly how to prove fML from fMM, i.e., $\Uparrow_1$ [BBPb] in the following diagram:

$$\begin{array}{c}\text{Hrushovski's Theorem 1.1 in [H96]}\\ \Uparrow_2 \\ \text{function field Mordell-Lang conjecture (fML)}\\ \Uparrow_1 \\ \text{function field Manin-Mumford conjecture (fMM)}\end{array}$$

Compare the assumptions on Γ in fML and fMM:

(1) in fML (i) in ch $= 0$. $\Gamma \subseteq \mathrm{Tor}(G)$ where G is a finitely generated subgroup of $A(K)$, (ii) in ch > 0, $\Gamma \subseteq \mathrm{Tor}_{p'}(G)$ where G is a finitely generated suggroup of $A(K)$.

(2) in fMM, $\Gamma \subseteq \mathrm{Tor}(A)$.

The condtion (2) is stronger than (1). Model theory of differentially closed fields and separably closed fields is needed to fill the gap of assumptions on Γ in fML and fMM. The authors of [BBPa, BBPb, BBPc] managed to proceed their project without appealing to the dichotomy theorem of Zariski geometry.

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