ON THE AUTOMORPHISM GROUPS OF HRUSHOVSKI'S PSEUDOPLANES ASSOCIATED TO SMALL RATIONAL NUMBERS

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ABSTRACT. The automorphism groups of Hrushovski's pseudoplanes associated to rational numbers α with $1/3 > \alpha \geq 1/4$ are simple groups.

1. INTRODUCTION

D. Evans, Z. Ghadernezhad, and K. Tent have shown that the automorphisms groups of certain countable structures obtained by using the Hrushovski amalgamation method are simple groups. Among them, there are generic structure of $\mathbf{K}_f$ for certain f with coefficient $1/2$ for the predimension function. They conjectured that the automorphism group of the generic structure of $\mathbf{K}_f$ is a simple group if the coefficient of the predimension function for $\mathbf{K}_f$ is rational.

In this paper, we show that the automorphism groups of Hrushovski's original "pseudoplanes" associated to a predimension function with rational coefficients α with $1/3 > \alpha \geq 1/4$ is a simple group. Similar argument shows the simplicity of the automorphism groups when $1/2 > \alpha > 0$. The statement holds in the case $1 > \alpha > 1/2$ as well, but we need more argument. The Method by D. Evans, Z. Ghadernezhad, and K. Tent works in the case $\alpha = 1/2$. We are going to treat the general case with $1 > \alpha > 0$ in another paper [15].

We essentially use notation and terminology from Baldwin-Shi [3] and Wagner [16]. We also use some terminology from graph theory [4].

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For a set X, $[X]^n$ denotes the set of all subsets of X of size n, and $|X|$ the cardinality of X.

We recall some of the basic notions in graph theory we use in this paper. These appear in [4]. Let G be a graph. $V(G)$ denotes the set of vertices of G. Vertices will be also called *points*. $E(G)$ is the set of edges of G. $E(G)$ is a subset of $[V(G)]^2$. $|G|$ denotes $|V(G)|$ and $e(G)$ denotes $|E(G)|$. The *degree* of a vertex v is the number of edges at v. A vertex of degree 1 is a *leaf*. A vertex of degree 2 or more is a *node*. G is a *path* $x_0x_1\ldots x_k$ if $V(G) = \{x_0,x_1,\ldots,x_k\}$ and $E(G) = \{x_0x_1,x_1x_2,\ldots,x_{k-1}x_k\}$ where the x_i are all distinct. x_0 and x_k are *ends* of G. The number of edges of a path is its *length*. A path of length 0 is a single vertex. G is a *cycle* $x_0x_1\ldots x_{k-1}x_0$ if $k \geq 3$, $V(G) = \{x_0,x_1,\ldots,x_{k-1}\}$ and $E(G) = \{x_0x_1,x_1x_2,\ldots,x_{k-2}x_{k-1},x_{k-1}x_0\}$ where the x_i are all distinct. The number of edges of a cycle is its *length*. A non-empty graph G is *connected* if any two of its vertices are linked by a path in G. A *connected component* of a graph G is a maximal connected subgraph of G. A *forest* is a graph not containing any cycles. A *tree* is a connected forest.

To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E. The language $\{E\}$ will be called *the graph language*.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that X is *connected* in A if X is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of A is a *connected component* of A.

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working.

Let D be a graph and A, B, and C substructures of D. We write $D = B \otimes_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free* *amalgam of B and C over A*. If A is empty, we write $D = B \otimes C$, and D is also called a *free amalgam of B and C*.

Definition 1.1. Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ by $\delta(A) = |A| - e(A)\alpha$.
(2) Let A and B be substructures of a common graph. Put $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition 1.2. Let A and B be graphs with $A \subseteq B$, and suppose A is finite.
$A \leq B$ if whenever $A \subseteq X \subseteq B$ with X finite then $\delta(A) \leq \delta(X)$.
$A < B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta(A) < \delta(X)$.
We say that A is *closed* in B if $A < B$.
$A <^- B$ if whenever $A \subsetneq X \subsetneq B$ with X finite then $\delta(A) < \delta(X)$.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset < A$.
Some facts about $<$ appear in [3, 16, 17]. Some proofs are given in [12].

Fact 1.3. *Let A and B be disjoint substructures of a common graph. Then $\delta(A/B) = \delta(A) + e(A,B)$.*

Fact 1.4. *If $A < B \subseteq D$ and $C \subseteq D$ then $A \cap C < B \cap C$.*

Fact 1.5. *Let $D = B \otimes_A C$.*

(1) $\delta(D/A) = \delta(B/A) + \delta(C/A)$.
(2) *If $A < C$ then $B < D$.*
(3) *If $A < B$ and $A < C$ then $A < D$.*

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) < C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over A* if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 1.6. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K},<)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.
(2) If $A \subseteq M$ then $A \in \mathbf{K}$.
(3) For any $A, B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. There is a smallest B satisfying $A \subseteq B < M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called the *closure* of A in M.

Fact 1.7 ([3, 16, 17])**.** *Let* $(\mathbf{K},<)$ *be an amalgamation class. Then there is a generic structure of* $(\mathbf{K},<)$*. Let* M *be a generic structure of* $(\mathbf{K},<)$*. Then any isomorphism between finite closed substructures of* M *can be extended to an automorphism of* M*.*

Definition 1.8. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K},<)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A < B$ and $A < C$ then $D \in \mathbf{K}$.

By Fact 1.5 (2), we have the following.

Fact 1.9. *Let* $\mathbf{K}$ *be a subclass of* $\mathbf{K}_\alpha$*. If* $(\mathbf{K},<)$ *has the free amalgamation property then it has the amalgamation property.*

Definition 1.10. Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that f is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at x. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f,<)$ has a countably categorical theory [17].

A graph X is *normal to* f if $\delta(X) \geq f(|X|)$. A graph A belongs to $\mathbf{K}_f$ if and only if U is normal to f for any substructure U of A.

2. Theorems by Evans, Ghadernezhad, and Tent

In this section, we fix a generic structure M of $\mathbf{K}_f$. Many of the following definitions and facts are by Evans, Ghadernezhad, and Tent [5].

Definition 2.1. Let $A \subseteq M$. $\mathrm{Aut}(M/A)$ denotes the set of automorphisms of M fixing A pointwise. Let $b \in M$. $\mathrm{orb}(b/A)$ denotes the $\mathrm{Aut}(M/A)$-orbit of b. So, $\mathrm{orb}(b/A) = \{\sigma(b) \mid \sigma \in \mathrm{Aut}(M/A)\}$.

Definition 2.2. Let $A \subseteq M$ be finite. The *dimension* $d(A)$ of A is defined by $d(A) = \delta(\mathrm{cl}(A))$. Let $B \subseteq M$ be also finite. The *relative dimension* $d(A/B)$ is defined by $d(A/B) = d(AB) - d(B)$.

Replace M by any graph G. Then we represent the corresponding dimensions by $d_G(A)$ and $d_G(A/B)$.

Definition 2.3. Suppose $b \in M$ and $A < M$ with A finite. We say that b is *basic* over A if $b \notin A$ and whenever $A \subseteq C < M$ and $d(b/C) < d(b/A)$ then $b \in C$. In this case, $\mathrm{orb}(b/A)$ is called a *basic orbit* over A.

Definition 2.4. We say that M is *monodimensional* if for every finite $A < M$ and basic orbit D over A there is a finite $B < M$ with $M = \mathrm{cl}(BD)$ and $A \subseteq B$.

Definition 2.5. Suppose $A < M$ and $b \in M$ a single element. $b \perp A$ if $\mathrm{cl}(bA) = bA$ and $d(b/A) = d(b)$.

Fact 2.6. *Suppose $A < M$ and $b_1, b_2 \in M$ be single elements. If $b_1 \perp A$ and $b_2 \perp A$ then b_1 and b_2 are conjugate over A in M.*

Proof. Suppose $b_1 \perp A$ and $b_2 \perp A$. We have $\mathrm{cl}(b_1A) = b_1A$ by the definition. So, $\delta(b_1/A) = \delta(b_1) = 1$. This means that there are no edges between b_1 and A. By the same argument, there are no edges between b_2 and A. Hence, b_1A and b_2A are isomorphic over A and also $b_1A < M$ and $b_2A < M$. Therefore, the partial isomorphism between b_1A and b_2A over A can be extended to an automorphism of M by Fact 1.7. $\square$

Fact 2.7. *If $M = \mathrm{cl}(AD)$ for some finite $A < M$ and a basic orbit D over A then M is monodimensional.*

Fact 2.8. *If M is monodimensional then the automorphism group of M is a simple group.*

3. Hrushovski's Boundary Functions

Definition 3.1 ([7])**.** Let α be a positive real number. We define x_n, e_n, k_n, d_n for integers $n \geq 1$ by induction as follows: Put $x_1 = 2$ and $e_1 = 1$. Assume that x_n and e_n are defined. Let r_n be a smallest rational number r such that $r = k/d > \alpha$ with $d \leq e_n$ where k and d are positive integers. Let

k_n/d_n be a reduced fraction with $k_n/d_n = r_n$. Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let $a_0 = (0,0)$, and $a_n = (x_n, x_n - e_n\alpha)$ for $n \geq 1$. Let f_α be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ whose graph on interval $[x_n, x_{n+1}]$ with $n \geq 0$ is a line segment connecting a_n and a_{n+1}. We call f_α a *Hrushovski's boundary function associated to* α.

Fact 3.2 ([7]). *Let f_α be a Hrushovski's boundary function and $\{x_i\}$, $\{e_i\}$, $\{k_i\}$, $\{d_i\}$ sequences in the definition of f_α.*

Suppose C is an extension of B by x points and z edges, $|B| \geq x_n$ and $x/z \geq k_n/d_n$ for some n, and B is normal to f_α. Then C is normal to f_α.

Fact 3.3 ([7]). *Let $D = B \otimes_A C$. If $\delta(A) < \delta(B)$, $\delta(A) < \delta(C)$, and A, B, C are normal to f_α then D is normal to f_α.*

Fact 3.4 ([7]). *Let α be a real number with $0 < \alpha < 1$. Then f_α is strictly increasing and concave, and $(\mathbf{K}_{f_\alpha}, <)$ has the free amalgamation property. Therefore, there is a generic structure of $(\mathbf{K}_{f_\alpha}, <)$. Any one point structure is closed in any structure in $\mathbf{K}_{f_\alpha}$. If α is rational then f_α is unbounded.*

Definition 3.5. Two positive rational numbers β, β' are called a *Farey pair* if there are positive integers h, k, h', k' such that $\beta = h/k$, $\beta' = h'/k'$, and $hk' - h'k = 1$. Note that $\beta > \beta'$ in this case and h/k and h'/k' are reduced fractions. Let $\beta'' = (h+h')/(k+k')$. β'' is called a *mediant* of β and β'.

The following lemma follows from well-known properties of Farey sequences (Farey series) [6]. It is a good exercise to prove it directly.

Lemma 3.6. *Let h/k and h'/k' be reduced fractions which are a Farey pair. Let u, v be positive integers.*

(1) *If $u/v > h'/k'$ with $v < k + k'$ then $u/v \geq h/k$.*
(2) *If $h/k > u/v > h'/k'$ then $v \geq k + k'$.*
(3) *Let h''/k'' be the mediant of h/k and h'/k'. Then h/k and h''/k'' are a Farey pair and h''/k'' and h'/k' are a Farey pair.*
(4) *Suppose β and β' are a Farey pair, and γ and γ' are a Farey pair. If the mediant of β and β' and the mediant of γ and γ' are the same then $\beta = \gamma$ and $\beta' = \gamma'$.*

The following is easy.

Lemma 3.7. (1) *Let $C = A \otimes_p B$ where p is a single vertex and $A, B \in \mathbf{K}_f$. Then $C \in \mathbf{K}_f$.*

(2) *Any finite forests belong to* $\mathbf{K}_f$.

Lemma 3.8. *Suppose* $1/3 > \alpha = c/d \geq 1/4$ *where* c/d *is a reduced fraction. Let* $k \geq 0$ *be an integer satisfying* $(k+2)/(3k+7) > \alpha \geq (k+1)/(3k+4)$.

(1) *The first several terms of the sequences defining* f_α *are given by the following chart:*

x_i	2	3	4	5	$\cdots$	$k+4$	$k+5$
e_i	1	2	4	7	$\cdots$	$3k+4$	$3k+7$
k_i	1	1	1	1	$\cdots$	1	$k+2$
d_i	1	2	3	3	$\cdots$	3	$3k+7$

(2) *Suppose* C *is an extension of* B *by* x *points and* z *edges,* $|B| \geq 4$, $(3/4)|B| \geq z$, $x/z > \alpha$, *and* B *is normal to* f_α. *Then* C *is normal to* f_α.

(3) *Let* $\beta = a/b$ *be a reduce fraction such that* β *and* α *are a Farey pair with* $0 < b < a$. *Let* $\beta_k = (a+kc)/(b+kd)$ *with integers* $k \geq 1$. *Then there are* $i_0 < i_1$ *with* $k_{i_0}/d_{i_0} = \beta$ *and* $k_{i_1}/d_{i_1} = \beta_1$. *If* $i \geq i_1$ *then* $x_{i+2} - x_{i+1} - a \geq 2(x_{i+1} - x_i - a)$ *and* $f_\alpha(x_{i+1}) - f_\alpha(x_i) = 1/d$.

Proof. (1) Starting from $x_1 = 2$, $e_1 = 1$, the value is obvious up to $x_3 = 4$ and $e_3 = 4$. Since $1/3 > \alpha \geq 1/4$, d_3 cannot be 4. Hence, $k_3/d_3 = 1/3$. $1/3$ and $(k+1)/(3k+4)$ are a Farey pairs. So, for any u/v with $u/v > \alpha \geq (k+1)/(3k+4)$ and $v < 3k+7$ we have $u/v \geq 1/3$. Hence, we have $x_i = k+5$ and $e_i = 3k+7$ for some i. Since $(k+2)/(3k+7)$ and $(k+1)/(3k+4)$ are Farey pairs, $(k+2)/(3k+7)$ is the best approximation of α strictly from above with a denominator at most $3k+7$.

(2) Choose i satisfying $x_i \leq |B| < x_{i+1}$. Since $x_3 = 4 \leq |B|$, $4 \leq x_i$. Then $x_i \leq e_i$ and $k_i/d_i \leq 1/3$. Also, we have $d_i \leq e_i$. So, $|B| < x_{i+1} = x_i + k_i \leq e_i + (1/3)e_i = (4/3)e_i$. Hence, $z \leq (3/4)|B| < e_i$. By the choice of k_i/d_i, we have $x/z \geq k_i/d_i$. Since $x_i \leq |B|$, C is normal to f_α by Fact 3.2.

(3) Consider Farey pairs among β_k's and α. Proof is left to the readers. $\square$

4. Special Structures

We associate sequences of vectors to rational numbers. Let S and S' be a sequence of vectors of a same size. SS' denotes the concatenation of S and S'. S^k denotes the concatenation of k copies of S. $\sum S$ denotes the sum of all terms in S. $\langle v_1, v_2, \cdots \rangle$ denotes a sequence with explicit terms.

Definition 4.1. We define a *special sequence for* α with $1/3 \geq \alpha \geq 1/4$ inductively as follows:

$\langle(1,3)\rangle$ is a unique special sequence for $1/3$. $\langle(1,4)\rangle$ is a unique special sequence for $1/4$.

Let β and γ be a Farey pair, and α the mediant of β and γ. Suppose S_1 and S_2 are special sequences for β and γ respectively. Then the concatenation of S_1 and S_2 is a special sequence for α.

Proposition 4.2. *Let β and γ be a Farey pair with $1/3 \geq \beta > \gamma \geq 1/4$ and suppose special sequences S_1 and S_2 exists for β and γ respectively. Let α be a rational number with $\beta > \alpha > \gamma$. Then there exists uniquely a special sequence for α. Moreover, the special sequence for α is a concatenation of copies of S_1 and S_2.*

Proof. Represent these rational numbers in reduced fraction forms, say $\alpha = u/v$, $\beta = x/y$, $\gamma = z/w$. We prove the statement by induction on $v - (y+w)$.

Let $\theta = (x+z)/(y+w)$. θ is the mediant of β and γ. Then $\alpha > \theta$, $\alpha = \theta$, or $\theta > \alpha$. Let S_3 be the concatenation of S_1 and S_2.

Case 1: $\alpha = \theta$. In this case, S_3 is a special sequence for α. Uniqueness follows from Lemma 3.6 (4).

Case 2: $\alpha > \theta$. Then $\beta > \alpha > \theta$, β and θ are a Farey pair. We have $v > y + (y+w) > y+w$. By induction hypothesis, there is a unique special sequence for α which can be obtained by concatenating S_1 and S_3. Since S_3 is a concatenation of S_1 and S_2, the statement holds.

Case 3: $\theta > \alpha$. Similar to Case 2. □

Lemma 4.3. *Let $\alpha = c/d$ be a reduced fraction with $1/3 \geq \alpha \geq 1/4$. Let S be the special sequence for α.*

(1) $\sum S = (c,d)$.
(2) *Let S' be a non-empty proper prefix of S. Let $(u,v) = \sum S$. Then $u/v > \alpha$.*
(3) *Let S' be a non-empty consecutive subsequence of S. Let $(u,v) = \sum S$. Then $u/(v-1) > \alpha$.*

Proof. The statements are true for $\alpha = 1/3,\ 1/4$. Let β and γ be a Farey pair such that α is their mediant. Let $h/k = \beta$ and $h'/k' = \gamma$ be reduced fractions. Let S_1 and S_2 be special sequences for β, γ respectively. By induction hypothesis, S_1 and S_2 satisfies (1) – (3). $S = S_1 S_2$ is the special sequence for α.

(1) is clear. We show (2). Let S' be a non-empty prefix of S, and put $(u,v) = \sum S'$. If S' is a proper prefix of S_1, then $u/v > \beta > \alpha$. Otherwise, we can write $S' = S_1 S_2'$ where S_2' is a non-empty proper prefix of S_2. Put $(u',v') = \sum S_2'$. Then $u'/v' > \gamma$. Together with $h/k = \beta > \gamma$, we have $u/v = (h+u')/(k+v') > \gamma$. Also, we have $v < k+k' = d$. α and γ are a Farey pair. Hence $u/v > \alpha$ by Lemma 3.6 (1).

We can show (3) similarly. □

Lemma 4.4. *Let $\alpha = c/d$ be a reduced fraction with $1/3 \geq \alpha \geq 1/4$, and S the special sequence for α. Let S' be a consecutive subsequence of S^n with $n \geq 1$, and put $(u,v) = \sum S'$. Then $u/(v-1) > \alpha$.*

Proof. We prove by induction on n. The statement holds if $n = 1$ by Lemma 4.3 (3).

Suppose $n > 1$. Then $S^n = S^{n-1}S$. If S' is a subsequence of S^{n-1}, then $u/(v-1) > \alpha$ by the induction hypothesis.

Suppose $S' = S_1' S_2'$ with S_1' is a subsequence of S_1' and S_2' is a non-empty prefix of S. Put $(u_1,v_1) = \sum S_1'$ and $(u_2,v_2) = \sum S_2'$. We have $u_1/(v_1-1) > \alpha$ by induction hypothesis, and $u_2/v_2 > \alpha$ by Lemma 4.3 (2). Since $u = u_1 + u_2$ and $v - 1 = (v_1 - 1) + v_2$, we have $u/(v-1) > \alpha$. □

Definition 4.5. Let S be a special sequence of α with $1/3 \geq \alpha \geq 1/4$. We can assume that $S = \langle (1,f_0),(1,f_1),\ldots,(1,f_{n-1}) \rangle$.

A graph W is called a *twig* for α if $W = LF$, L is a path $p_0 p_1 \ldots p_{n-1}$ where n is a length of S, F is the set of all leaves of W where each leaf in F is adjacent to some p_i, the number of leaves in F adjacent to p_0 is f_0, and the number of leaves in F adjacent to p_i is $f_i - 1$ for each i with $1 \leq i \leq n-1$. L is called the *main path* of W. Note that L consists of the nodes of W. p_0 is called the *left most node* of W.

A graph W is called a *wreath* for α if $W = CF$, C is a cycle $p_0 p_1 \ldots p_{m-1} p_0$ where m is a multiple of n, F is the set of all leaves of W where each leaf in F is adjacent to some p_i, and the number of leaves in F adjacent to p_i is $f_{i \bmod n} - 1$ for each i with $0 \leq i < m$.

A substructure B of W is *full* if a node p of W belongs to B then any leaves q of W adjacent to p belongs to B.

Lemma 4.6. *Let W be a twig or a wreath for α, F the set of leaves of W.*

(1) $\delta_\alpha(W/F) = 0$.

(2) *Let B be a proper substructure of W. If $B - F$ is non-empty then $\delta_\alpha(B/B \cap F) > 0$.*

Proof. Let W be a twig for α. Let S be the special sequence for α, n the length of S, L the main path of W, and F the set of leaves of W. Let B be a substructure of W. Let $B_1, \ldots, B_\ell$ be the list of connected components of B. Then $\delta(B/F) = \sum_{i=1}^{\ell} \delta(B_i/F)$.

So, we can assume that B is connected. Hence, $B \cap L$ is a path in L. Let $B \cap L = p_1 \cdots p_m$ respecting the order of S. Let B' be the full substructure of W with $B' \cap L = B \cap L$. $B' = B$ or B' can be obtained by adding some edges to B. Hence, $\delta(B) \geq \delta(B')$.

Let S' be the subsequence of S corresponding to $p_1 \cdots p_m$. Let $(u, v) = \sum S'$. Then $\delta(B'/F) = u - v\alpha$ if p_1 is the left most node of W, and $\delta(B'/F) = u - (v-1)\alpha$ otherwise. By Lemma 4.3, we have (1) and (2).

If W is a wreath for α, we can argue similarly using S^k for some k instead of S. □

Lemma 4.7. *Let $\beta = a/b$ and $\alpha = c/d$ be two reduced fractions which are a Farey pair. Let W be a twig for β, and F the set of leaves of W.*

(1) $\delta_\alpha(W/F) = 1/d$.
(2) *Let B be a proper substructure of W. If $B - F$ is non-empty then* $\delta_\alpha(B/B \cap F) > 1/d$.
(3) *Suppose W is a closed substructure of a generic structure. Then a node of W is basic over F.*

Proof. (1) W has a nodes and b edges. Since $ad - bc = 1$, we have $a - b(c/d) = 1/d$.

(2) Let x be the number of nodes of W in B and z be the number of edges in B. By Lemma 4.6 (2), we have $x/z > \beta > \alpha$. Since $b < d$, x/z and α cannot be a Farey pair by Fact 3.6 (2). We have $xd - zc > 0$ and $xd - zc \neq 1$. Hence, $x - zc/d > 1/d$.

(3) Clear from the definition of basic elements. □

5. Monodimensionality

Proposition 5.1. *Suppose $1/3 > \alpha = c/d \geq 1/4$ and c, d are positive integers. Let $\beta = a/b$ where $b < d$, $ad - bc = 1$, and a, b are positive integers. Let G be a graph such that $G = B \otimes_F W$ where W is a wreath for α, F is the set of leaves of W, $B = \otimes_A \{B_q \mid q \in F\}$, each B_q is a twig for β, A the set of common leaves of B_q, q is the left most node of B_q, and $F \cap B_q = \{q\}$. Let C be the cycle in W.*

(1) *Let G' be a substructure of G. If $G' \cap C$ is a path in C then G' is normal to f_α.*
(2) *If C is sufficiently large then G belongs to $\mathbf{K}_{f_\alpha}$.*

Proof. (1) Since $1/3$ and $1/4$ are a Farey pair and $1/3 > c/d \geq 1/4$, $d = 4$ or $3+7 \leq d$. Hence, $3 < d$. Since a/b and c/d are a Farey pair and $b < d$, We have $1/3 \geq a/b$ by Lemma 3.6 (1). Let $k \geq 0$ be an integer with $(k+2)/(3k+7) > \alpha \geq (k+1)/(3k+4)$.

Case A: $\alpha = c/d = (k+1)/(3k+4)$.
In this case, $a/b = 1/3$. $\langle(1,3)\rangle^k\langle(1,4)\rangle$ is a special sequence for α.
If $p_0p_1\cdots p_kp_{k+1}$ is a path in C with $e(p_0,F) = 3$, then $e(p_i,F) = 2$ for $i = 1, \ldots, k$, and $e(p_{k+1},F) = 3$.
Since $a/b = 1/3$, $|A| = 3$ and each B_q for $q \in F$ is isomorphic to $K_{1,3}$ (a star with 3 leaves in A and a single node in F). For each $p \in C$, $e(p,F)$ is 2 or 3.
Let $C' = G' \cap C$, $F' = G' \cap F$, $B' = G' \cap B$, and $A' = G' \cap A$. Note that $B' = A'F'$. We can assume $C' = p_1 \cdots p_\ell$ respecting the order of the special sequence for α.

Subcase A1: $e(p_i,F') = 2$ for any $i < \ell$, and $e(p_\ell,F') \geq 2$.
Assume $1 < \ell$. We have $|G'| - |B'| = \ell$. There are 2 edges from p_1 to F'. For i with $1 < i \leq \ell - 1$, there is 1 edge from p_i to p_{i-1}, and 2 edges from p_i to F'. There is 1 edge from p_ℓ to $p_{\ell-1}$, and at most 3 edges from p_ℓ to F'. Hence, we have $e(G') - e(B') \leq 2 + 3(\ell-2) + 4 = 3\ell$. We have $|B'| \geq 4$ because $p_1 \neq p_\ell$ and $e(p_1,F') = 2$ and $e(p_\ell,F') \geq 2$, and by the structure of G. So, G' is normal to f_α by Fact 3.2 and Lemma 3.8.
Assume $\ell = 1$. If $e(p_\ell,F') = 3$, then $|B'| \geq 3$. G' is an extension of B' by 1 point and 3 edges. If $|B'| = 3$ then G' must be a tree. Hence, G' is normal to f_α. If $|B'| \geq 4$, then G' is normal to f_α by Fact 3.2 and Lemma 3.8.

Subcase A2: $e(p_i,F') \geq 2$ for any $i \leq \ell$.
Let F'_i be the set of points in F' adjacent to p_i. Let $G'_i = A'F'_1 \cdots F'_i p_1 \cdots p_i$. We have $G'_\ell = G'$.
If there are no i with $1 \leq i \leq \ell$ and $e(p_i,F') = 3$ then G' is normal to f_α by Subcase A1.
Otherwise, let i_1 be a smallest i satisfying $e(p_i,F') = 3$ and $1 \leq i \leq \ell$. Then G'_{i_1} is normal to f_α by Subcase A1.

Claim 1. *Suppose G'_j is normal to f_α with $e(p_j,F') = 3$. Then G' is normal to f_α.*

We prove the claim by induction on the number of p_i satisfying $e(p_i, F') = 3$ and $j < i \leq \ell$.

Suppose there are no i with $e(p_i, F') = 3$ and $j < i \leq \ell$. Then $e(p_i, F') = 2$ and there is 1 edge from p_i to p_{i-1} for any i with $j < i \leq \ell$. We have $G' = G'_j F' p_{j+1} \cdots p_\ell$. So, $|G'| - |G'_j F'| = \ell - j$, and $e(G') - e(G'_j F') = 3(\ell - j)$. We can assume that $j < \ell$. Hence, $|B'| \geq 4$. Therefore, G' is normal to f_α by Fact 3.2 and Lemma 3.8.

Suppose there is i with $e(p_i, F') = 3$ and $j < i \leq \ell$. Let j' be a smallest i with $e(p_i, F') = 3$ and $j < i \leq \ell$. Since $e(p_j, F') = 3$ and $e(p_{j'}, F') = 3$, we have $j' \geq j + k + 1$.

Suppose $j' = j + k + 1$. Let q_1, q_2, q_3 be points in F' adjacent to p_j. Put $G''_{j'} = G'_j F_{j+1} \cdots F_{j+k} q_1 q_2$. We have $G'_{j'} = G''_{j'} p_{j+1} \cdots p_{j+k} p_{j+k+1} q_3$. Note that $e(p_i, G''_{j'}) = 3$ for $j + 1 \leq i \leq j + k$, $e(p_{j+k+1}, G''_{j'}) = 4$, and $e(q_3, G''_{j'}) = |A'|$. So, $|G'_{j'}| - |G''_{j'}| = k + 2$, and $e(G'_{j'}) - e(G''_{j'}) \leq 3k + 4 + |A'|$. If $|A'| \leq 2$ then $e(G'_{j'}) - e(G''_{j'}) \leq 3(k+2)$. Since $e(p_j, F'_j) = 3$, we have $|G'_j| \geq 4$. Hence, $|G''_{j'}| \geq |G'_j| \geq 4$. Therefore, $G'_{j'}$ is is normal to f_α by Fact 3.2 and Lemma 3.8. If $|A'| = 3$ then $e(G'_{j'}) - e(G''_{j'}) = 3k + 7$. In this case, $|G''_{j'}| = 2k + 2 + |G'_j| \geq 2k + 6 > 2k + 5$. Therefore, $G'_{j'}$ is is normal to f_α by Fact 3.2 and Lemma 3.8.

Suppose $j' > j + k + 1$. Let $j' = j + (k + 2 + k')$ with $k' \geq 0$. Put $G'''_{j'} = G'_j F_{j+1} \cdots F_{j'}$. Put $z = e(G') - e(G'''_{j'})$ and $x = |G'| - |G'''_{j'}|$. Then $x = k + k' + 2$, and $z = 3(k + 1 + k') + 4 = 3(k + k') + 7$. We have $x/z \geq (k+2)/(3k+7)$. On the other hand, $|G'''_{j'}| \geq |G'_j| + 2k + 2k' + 2 + 3 \geq k + 5$. Therefore, G' is normal to f_α by Fact 3.2 and Lemma 3.8.

Subcase A3: $C' = p_1 \cdots p_\ell$ is a path in C.

By induction on the number of i with $e(p_i, F') \leq 1$, we can reduce this subcase to Subcase A2.

If there is no i with $e(p_i, F') = 1$, and $1 \leq i \leq \ell$, then the case is Subcase A2.

Suppose $e(p_i, F') = 1$, and $1 < i < \ell$. Then G' is an extension of $G' - p_i$ with 1 point and 3 edges. If A' is empty then G' is a tree. Hence, G' belongs to $\mathbf{K}_{f_\alpha}$. We can assume A' is non-empty. So, we have $|G' - p_i| \geq 4$. Hence, it is enough to show that $G' - p_i$ is normal to f_α. $G' - p_i$ is a free amalgam over B' of two substructures satisfying the condition of Subcase A3.

There are several other cases to consider, but they can be handled similarly.

We have proved (1) for Case A.

Case B: $(k+2)/(3k+7) > \alpha = c/d > (k+1)/(3k+4)$.

$(k+2)/(3k+7)$ and $(k+1)/(3k+4)$ are a Farey pair. So, we have $3k+7 < d$ by Lemma 3.6 (2). β and α are also a Farey pair. Hence $(k+2)/(3k+7) \geq \beta$ by Lemma 3.6 (1).

Note that $\langle(1,3)\rangle^{k+1}\langle(1,4)\rangle$ and $\langle(1,3)\rangle^{k}\langle(1,4)\rangle$ are special sequences for $(k+2)/(3k+7)$ and $(k+1)/(3k+4)$ respectively. By Proposition 4.2, special sequences for β and α are concatenations of copies of these sequences.

Put $C' = G \cap C$, $F' = G' \cap F$, $A' = G' \cap A$, $B' = G' \cap B$, $B'_q = G' \cap B_q$ for each $q \in F$. Let F'_p be the set of points in F' adjacent to $p \in C'$.

Note that there are three points in A which are adjacent to any points in F. Let $A_1 = \{a_1, a_2, a_3\}$ be the set of such three points.

Subcase B1: $C' = p_1 \cdots p_\ell$, $e(p_i, F') \geq 2$ for each $p_i = 1, \ldots, \ell$, and B'_q has at least 2 nodes (i.e. $|B'_q - A| \geq 2$) for any q in F' adjacent to some point in C'.

We have

$$\begin{aligned} e(C'B') - e(B') &= e(C') + e(C', B') \\ &= (\ell - 1) + \sum_{i=1}^{\ell} e(p_i, F') \\ &= e(p_1, F') + \sum_{i=2}^{\ell} (1 + e(p_i, F')). \end{aligned}$$

Suppose $e(p_i, F') = 2$. Let q and q' be 2 points adjacent to p_i. Then $|B'_q B'_{q'} - A'| \geq 4$. Hence, $(1 + e(p_i, F'))/|B'_q B'_{q'} - A'| \leq 3/4$.

Suppose $e(p_i, F') = 3$. Let q, q' and q'' be 3 points adjacent to p_i. Then $|B'_q B'_{q'} B'_{q''} - A'| \geq 6$. Hence, $(1 + e(p_i, F'))/|B'_q B'_{q'} - A'| \leq 4/6 = 2/3 < 3/4$.

Therefore, $(e(C'B') - e(B'))/|B'| \leq 3/4$. So, $(e(C'B') - e(B')) \leq (3/4)|B'|$. We also have $B' < B'C'$ by Lemma 4.6. Thus, $\ell/(e(C'B') - e(B')) > \alpha$. Hence, $G' = B'C'$ is normal to f_α by Lemma 3.8 (2).

Subcase B2: $C' = p_1 \cdots p_\ell$ and $e(p_i, F') \geq 2$ for each $p_i = 1, \ldots, \ell$.

Note that normality of G' to f_α depends only on $|G'|$ and $e(G')$. Suppose $p_i q$ and $p_j q'$ are edges between C' and F'. Removing these edges and put $p_i q'$ and $p_j q$ as new edges will not change $|G'|$, $e(G')$, $e(p_i, F')$, and $e(p_j, F')$. So, we can assume that if p_i is adjacent to $q \in F'$ with $|B'_q - A'| \geq 2$ and p_j is adjacent to $q' \in F'$ with $|B'_{q'} - A'| = 1$ then $i \leq j$.

Put $G'_i = B'_1 \cdots B'_i p_1 \cdots p_i$.

Let i_0 be a largest $i \leq \ell$ satisfying $e(p_i, F') = 3$ and $|B_q - A'| \geq 2$ for any three q in F' adjacent to p_i.

Suppose such i_0 exists. Then G'_{i_0} is normal to f_α by Subcase B1.

If there is no such i_0, let i_1 be a smallest i satisfying $e(p_i, F') = 3$. Suppose such i_1 exists. Then G'_{i_1} is normal to f_α by the argument in Subcase A2 of Case A. If there are no such i_0 and i_1 then G' is normal to to f_α by the argument in Subcase A1 of Case A.

Now, we can show the following claim as in Subcase A2 of Case A. Note that $q \in F'$ and $|B'_q - A'| = 1$ then B'_q is a star with at most 3 edges.

Claim 2. *Suppose G'_j is normal to f_α with $e(p_j, F') = 3$. Then G' is normal to f_α.*

Therefore, G' is normal to f_α in Subcase B2 also.

Subcase B3: $C' = p_1 \cdots p_\ell$ is a path in C.
Same way as in Subcase A3 in Case A.

(2) Let G' be any substructure of G.

Case 1: $G' \cap C \neq C$.

Let C_1, …, C_m be the connected components of $G' \cap C$. For each i, $B'C_i$ is normal to f_α by (1), and $B' < B'C_i$ also. Hence, G is normal to f_α.

Case 2: $G' \cap C = C$.

Suppose that $G' \cap F$ is a proper substructure of F. By exchanging two edges between C and F if necessary, we can assume that $e(p, F') = 1$ for some $p \in C$ with $e(p, F) = 2$. $G' - p$ is normal to f_α by Case 1. Then G' is an extension of $G' - p$ by 1 point and 3 edges. Since C is sufficiently large, we can assume that $|G' - p| \geq 4$. Hence, G' is normal to f_α.

Now, we can assume that $W \subset G'$.

$B = \otimes_A \{B_q \mid q \in F\}$ is a member of $\mathbf{K}_{f_\alpha}$. We have $|B| = A + a|F|$, and $\delta(B) = |A| + (1/d)|F|$ because $\delta_\alpha(B_q/A) = 1/d$ for each $q \in F$.

Let B' be a substructure of B'. Then $|B'| \leq |B|$. By Lemma 4.6, $\delta(B') \geq |A| + (1/d)|F|$ if $A' = B' \cap A = A$. Suppose $|A'| < |A|$. Then any substructures B'_q of B_q with $B'_q \cap A = A'$ is a proper substructure of B_q. So, we have $\delta_\alpha(B'_q/A') \geq 2/d$ by Lemma 4.7. Hence, $\delta(B') \geq |A'| + (2/d)|F|$. Comparing $|A'| + (2/d)|F|$ and $|A| + (1/d)|F|$, for sufficiently large F, we have $|A'|+(2/d)|F|>|A|+(1/d)|F|$. Therefore, for sufficiently large F, we have $\delta(B')\geq|A|+(1/d)|F|$ for any substructures B' of B with $F\subseteq B'$. On the other hand, $|A|+(1/d)|F|$ is a linear function of $|B|=|A|+a|F|$. Therefore, there is a linear function $f_1(x)$ with positive coefficient such that $f_1(|B|)=\delta(B)$, and $\delta(B')\geq\delta(B)$ for any substructures B' of B with $F\subseteq B'$ if $|B|$ is sufficiently large.

Since, B is normal to f_α, we have $f_1(x)\geq f_\alpha(x)$. By Lemma 3.8 (3), $f_\alpha^{-1}(y)$ behave like exponential function. So, for sufficiently large x, we have $f_1(x)>f_\alpha(2x)$.

Let F be sufficiently large. Suppose $F\subseteq B'\subseteq B$. Then $\delta(B')\geq\delta(B)=f_1(|B|)\geq f_\alpha(2|B|)\geq f_\alpha(2|B'|)$. Since $|C|\leq|F|\leq|B'|$, $\delta(B'C)=\delta(B')\geq f_\alpha(2|B'|)\geq f_\alpha(|B'C|)$. Therefore, $G'=B'C$ is normal to f_α. $\square$

Definition 5.2. Let β and α be a Farey pair with $1/3\geq\beta>\alpha\geq1/4$. Let B be a twig for β, b a node of B and A the leaves of B.

(G,c) is called a *basic tower* for α over A if $A<G$, $c\in G$ and c has distance at least 2 from A, $F\subseteq G$, and B_q for each $q\in F$ such that $d_G(c/F)=0$, $A<B_q<G$, $q\in B_q$ and (B_q,q) is isomorphic to (B,b) over A.

Note that if G is a closed substructure of a generic structure, then the elements in F are basic over A and are pairwise conjugate over A.

Proposition 5.3. (1) *Let G be the structure in Proposition 5.1. Let c be a point in the cycle of W. Then (G,c) is a basic tower for α over A.*

(2) *Let (G,c) be a basic tower for α over A. Let $H=D\otimes_F W$ where W is a wreath for α, F is the set of leaves of W, $D=\otimes_A\{G_q\mid q\in F\}$, each G_q is isomorphic to G over A, and $F\cap G_q=\{q\}$.*

If W is sufficiently large then choosing c' from the main cycle of W, (H,c') is a basic tower for α over A. Moreover, $d_H(c'/A)>d_G(c/A)$.

The proof is easier than that for Proposition 5.1. We can use Lemma 3.8 (2).

Using this proposition many times, we can show that there is a basic tower (G'',c'') over A such that $d_{G''}(c''/A)>1$. This means that $Ac''<G''$. Embed G'' in the generic structure as a closed substructure. Then c'' is in the closure of a basic orbit over A, and $c''\perp A$. As in [14], we can prove the following theorem.

Theorem 5.4. *Let α be a rational number with $1/3 > \alpha \geq 1/4$. Then the generic structure M of $\mathbf{K}_{f_\alpha}$ is monodimensional. Therefore, the automorphism group of M is a simple group.*

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