TREE-INDISCERNIBILITY IN SOP_1 AND ANTICHAIN TREES

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Abstract. We study some tree properties and related indiscernibilities. First, we show that there is a tree-indiscernibility which preserves witnesses of SOP_1. Secondly we introduce notions of antichain tree property and show that every SOP_1-NSOP_2 theory (having SOP_1 but not SOP_2) has an antichain tree by using that tree-indiscernibility. And we construct a structure witnessing SOP_1-NSOP_2 in the formula level, *i.e.* there is a formula having SOP_1 but any finite conjunction of it does not have SOP_2. (This work is joint work with JinHoo Ahn at Yonsei University.)

1. Introduction

The notion of SOP_1 and SOP_2 were introduced by Džamonja and Shelah in [1]. It is known that the implication $\mathrm{SOP}_2 \Rightarrow \mathrm{SOP}_1$ holds but it is still unknown whether the converse is true or not. We focus on the problem of equality of SOP_1 and SOP_2, and discuss some related topics.

2. Tree indiscernibility for witnesses of SOP_1

Let us recall a notion of SOP_1 in [1].

Definition 2.1. Let $\varphi(x,y)$ be a formula in T. We say $\varphi(x,y)$ has *1-strong order property* (SOP_1) if there is a tree $\langle a_\eta\rangle_{\eta\in {}^{<\omega}2}$ such that

(1) For all $\eta \in {}^{\omega}2$, $\{\varphi(x, a_{\eta\lceil\alpha}) \mid \alpha < \omega\}$ is consistent, and
(2) For all $\eta,\nu \in {}^{<\omega}2$, $\{\varphi(x, a_{\eta^\frown\langle 1\rangle}), \varphi(x, a_{\eta^\frown\langle 0\rangle^\frown\nu})\}$ is inconsistent.

We say T has SOP_1 if it has a SOP_1 formula. We say T is NSOP_1 if it does not have SOP_1.

In this section, we develop a tree-indiscernibility which can be applied to witnesses of SOP_1. The outline of proof came from [1]. But the proof in [1] omits some important step. We leave sketch of proof here, explain what proof of [1] omits, and how we complement it.

Definition 2.2. For $\bar\eta = \langle \eta_0, ..., \eta_n\rangle$, $\bar\nu = \langle \nu_0, ..., \nu_n\rangle$ ($\eta_i, \nu_i \in {}^{\omega>}2$ for each $i \le n$), we say $\bar\eta \approx_\alpha \bar\nu$ if they satisfies

(i) $\bar\eta$ and $\bar\nu$ are $\wedge$-closed,
(ii) $\eta_i \trianglelefteq \eta_j$ if and only if $\nu_i \trianglelefteq \nu_j$ for all $i, j \le n$,
(iii) $\eta_i{}^\frown d \trianglelefteq \eta_j$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j$ for all $i, j \le n$ and $d \le 1$.

We say $\bar\eta \approx_\beta \bar\nu$ if they satisfy (i), (ii), (iii), and

(iv) $\eta_i{}^\frown\langle 1\rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 1\rangle = \nu_j$ for all $i, j \le n$.

We say $\bar{\eta} \approx_\gamma \bar{\nu}$ if they satisfy (i), (ii), (iii), (iv), and

(v) $\eta_i{}^\frown\langle 0\rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 0\rangle = \nu_j$ for all $i, j \leq n$.

(vi) $\eta_i = \sigma^\frown\langle 0\rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau^\frown\langle 0\rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

(vii) $\eta_i = \sigma^\frown\langle 1\rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau^\frown\langle 1\rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

We say $\langle a_\eta\rangle_{\eta\in{}^{\omega>}2}$ is α-indiscernible (β, γ-indiscernible, resp.) if $\bar{\eta} \approx_\alpha \bar{\nu}$ ($\bar{\eta} \approx_\beta \bar{\nu}$, $\bar{\eta} \approx_\gamma \bar{\nu}$, resp.) implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

Recall the modeling property of α-indiscernibility in [2].

Fact 2.3. [2, Proposition 2.3] *For any $\langle a_\eta\rangle_{\eta\in{}^{\omega>}2}$, there exists $\langle b_\eta\rangle_{\eta\in{}^{\omega>}2}$ such that*

(i) *$\langle b_\eta\rangle_{\eta\in{}^{\omega>}2}$ is α-indiscernible,*

(ii) *for any finite set Δ of $\mathcal{L}$-formulas and $\wedge$-closed $\bar{\eta} = \langle \eta_0, ..., \eta_n\rangle$, there exists $\bar{\nu} \approx_\alpha \bar{\eta}$ such that $\bar{b}_{\bar{\eta}} \equiv_\Delta \bar{a}_{\bar{\nu}}$.*

In order to make the proof shorter we introduce some notation.

Notation 2.4. (i) For each $\eta \in {}^{\omega>}2$, $l(\eta)$ denotes the domain of η.

(ii) For each $\eta \in {}^{\omega>}2$ with $l(\eta) > 0$, η^- denotes $\eta\lceil_{l(\eta)-1}$ and $t(\eta)$ denotes $\eta(l(\eta) - 1)$.

(iii) For $\bar{\eta} = \langle \eta_0, ..., \eta_n\rangle$, $\mathrm{cl}(\bar{\eta})$ denotes $\langle \eta_0 \wedge \eta_0, ..., \eta_0 \wedge \eta_n\rangle^\frown ...^\frown\langle \eta_n \wedge \eta_0, ..., \eta_n \wedge \eta_n\rangle$.

(iv) η and ν are said to be incomparable (denoted by $\eta \perp \nu$) if $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$.

Note that $\eta = \eta^{-\frown} t(\eta)$ for all η with $l(\eta) > 0$. The following remarks will also be useful.

Remark 2.5. Suppose $\langle \eta_0, ..., \eta_n\rangle \approx_\gamma \langle \nu_0, ..., \nu_n\rangle$. Then it follows that

(i) $\eta_i \wedge \eta_j = \eta_k$ if and only if $\nu_i \wedge \nu_j = \nu_k$ for all $i.j.k \leq n$,

(ii) $\eta_i^- \trianglelefteq \eta_j^-$ if and only if $\nu_i^- \trianglelefteq \nu_j^-$ for all $i, j \leq n$,

(iii) for all $i, j \leq n$, if $\eta_i \perp \eta_j$ then $\eta_i^- \wedge \eta_j^- = \eta_i \wedge \eta_j$,

(iv) $\eta_i{}^\frown\langle d\rangle \trianglelefteq \eta_j^-$ if and only if $\nu_i{}^\frown\langle d\rangle \trianglelefteq \nu_j^-$ for all $i, j \leq n$ and $d \leq 1$,

(v) $\eta_i^{-\frown}\langle d\rangle \trianglelefteq \eta_j^-$ if and only if $\nu_i^{-\frown}\langle d\rangle \trianglelefteq \nu_j^-$ for all $i, j \leq n$ and $d \leq 1$,

Lemma 2.6. *Suppose $\varphi(x, \bar{y})$ witnesses* SOP$_1$*. Then there exists a γ-indiscernible tree $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ which witnesses* SOP$_1$ *with φ.*

Sketch of Proof. Suppose $\varphi(x, \bar{y})$ witnesses SOP$_1$ with $\langle a_\eta\rangle_{\eta\in{}^{\omega>}2}$. For each $\eta \in {}^{\omega>}2$, put $b_\eta = a_{\eta^\frown\langle 0\rangle}{}^\frown a_{\eta^\frown\langle 1\rangle}$. By Fact 2.3, there exists an α-indiscernible $\langle c_\eta\rangle_{\eta\in{}^{\omega>}2}$ such that for any $\bar{\eta}$ and finite subset Δ of $\mathcal{L}$-formulas, $\bar{\nu} \approx_\alpha \bar{\eta}$ and $\bar{b}_{\bar{\nu}} \equiv_\Delta \bar{c}_{\bar{\eta}}$ for some $\bar{\nu}$. Note that c_η is of the form $c^0_\eta{}^\frown c^1_\eta$ where $|c^0_\eta| = |c^1_\eta| = |\bar{y}|$ for each $\eta \in {}^{\omega>}2$. For each $\eta \in {}^{\omega>}2$ with $l(\eta) \geq 1$, we define d'_η by

$$d'_\eta = \begin{cases} c^0_{\eta^-} & \text{if } t(\eta) = 0 \\ c^1_{\eta^-} & \text{if } t(\eta) = 1 \end{cases}$$

and put $d_\eta = d'_{\langle 0\rangle^\frown\eta}$ for each $\eta \in {}^{\omega>}2$. We show that φ witnesses SOP$_1$ with $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ and $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ is γ-indiscernible. Then $\langle \varphi(x, y), \langle d_\eta\rangle_{\eta\in{}^{\omega>}2}\rangle$ witnesses SOP$_1$. One can show this by using the fact that $\langle c_\eta\rangle_{\eta\in{}^{\omega>}2}$ is based on $\langle b_\eta\rangle_{\eta\in{}^{\omega>}2}$.

To show that $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ is γ-indiscernible, suppose that $\langle \eta_0,...,\eta_n\rangle \approx_\gamma \langle \nu_0,...,\nu_n\rangle$. For each $i\le n$, let $\sigma_i=\langle 0\rangle^\frown\eta_i$ and $\tau_i=\langle 0\rangle^\frown\nu_i$. By definition of $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$, it is enough to show that $d'_{\sigma_0}...d'_{\sigma_n}\equiv d'_{\tau_0}...d'_{\tau_n}$. Clearly $\langle\sigma_0,...,\sigma_n\rangle\approx_\gamma\langle\tau_0,...,.\tau_n\rangle$. It's not difficult, but after a rather laborious calculation, one can show that

$$\mathrm{cl}(\langle\sigma_0^-,...,\sigma_n^-\rangle)\approx_\alpha \mathrm{cl}(\langle\tau_0^-,...,\tau_n^-\rangle).$$

By α-indiscernibility of $\langle c_\eta\rangle_{\eta\in{}^{\omega>}2}$, we have $\bar{c}_{\mathrm{cl}(\langle\sigma_0^-,...,\sigma_n^-\rangle)}\equiv\bar{c}_{\mathrm{cl}(\langle\tau_0^-,...,\tau_n^-\rangle)}$. In particular, we have $c_{\sigma_0^-}...c_{\sigma_n^-}\equiv c_{\tau_0^-}...c_{\tau_n^-}$. By definition of $\langle d'_\eta\rangle_{\eta\in{}^{\omega>}2}$,

$$d'_{\sigma_0^-\frown\langle 0\rangle}d'_{\sigma_0^-\frown\langle 1\rangle}...d'_{\sigma_n^-\frown\langle 0\rangle}d'_{\sigma_n^-\frown\langle 1\rangle}\equiv d'_{\tau_0^-\frown\langle 0\rangle}d'_{\tau_0^-\frown\langle 1\rangle}...d'_{\tau_n^-\frown\langle 0\rangle}d'_{\tau_n^-\frown\langle 1\rangle}.$$

Note that in general, if $m_{\xi_0}...m_{\xi_k}\equiv n_{\zeta_0}...n_{\zeta_k}$ and $i_0<...<i_e\le k$, then $m_{\xi_{i_0}}...m_{\xi_{i_e}}\equiv n_{\zeta_{i_0}}...n_{\zeta_{i_e}}$ Since we assume $\langle\eta_0,...,\eta_n\rangle\approx_\gamma\langle\nu_0,...,\nu_n\rangle$, we have $t(\sigma_i)=t(\tau_i)$ for each $i\le n$. Thus

$$d'_{\sigma_0}...d'_{\sigma_n}\equiv d'_{\tau_0}...d'_{\tau_n}$$

as desired. This shows that $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ is γ-indiscernible, and completes the proof. □

Note that even if $i_0<...<i_e\le k$, $j_0<...<j_e\le k$ and $m_{\xi_0}...m_{\xi_k}\equiv n_{\zeta_0}...n_{\zeta_k}$, it is not sure that $m_{\xi_{i_0}}...m_{\xi_{i_e}}\equiv n_{\zeta_{j_0}}...n_{\zeta_{j_e}}$. So if we want to say $d'_{\sigma_0}...d'_{\sigma_n}\equiv d'_{\tau_0}...d'_{\tau_n}$ from the fact that

$$d'_{\sigma_0^-\frown\langle 0\rangle}d'_{\sigma_0^-\frown\langle 1\rangle}...d'_{\sigma_n^-\frown\langle 0\rangle}d'_{\sigma_n^-\frown\langle 1\rangle}\equiv d'_{\tau_0^-\frown\langle 0\rangle}d'_{\tau_0^-\frown\langle 1\rangle}...d'_{\tau_n^-\frown\langle 0\rangle}d'_{\tau_n^-\frown\langle 1\rangle}$$

in the last paragraph of proof of Lemma 2.6, it must be guaranteed that $t(\sigma_i)=t(\tau_i)$ for each $i\le n$. This is why we introduce $\approx_\gamma$ and find a γ-indiscernible witness of SOP_1 first, not directly find β-indiscernible one as in [1]. The proof in [1] uses the similar argument in this note, tries to show directly that there exists a β-indiscernible tree witnessing SOP_1 without using γ-indiscernibility. So, by the problem mentioned above, the proof ends incomplete.

Theorem 2.7. *If $\varphi(x,y)$ witnesses SOP_1, then there exists a β-indiscernible tree $\langle e_\eta\rangle_{\eta\in{}^{\omega>}2}$ which witnesses SOP_1 with φ.*

Proof. By Lemma 2.6, there exists a γ-indiscernible tree $\langle d_\eta\rangle_{\eta\in{}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h:{}^{\omega>}2\to{}^{\omega>}2$ by

$$h(\eta)=\begin{cases}\langle\rangle & \text{if}\quad \eta=\langle\rangle\\ h(\eta^-)^\frown\langle 01\rangle & \text{if}\quad t(\eta)=0\\ h(\eta^-)^\frown\langle 1\rangle & \text{if}\quad t(\eta)=1,\end{cases}$$

and put $e_\eta=d_{h(\eta)}$ for each $\eta\in{}^{\omega>}2$. Then $\langle e_\eta\rangle_{\eta\in{}^{\omega>}2}$ is β-indiscernible, and φ witnesses SOP_1 with $\langle e_\eta\rangle_{\eta\in{}^{\omega>}2}$. □

3. Antichain tree property

In this section, we introduce a notion of tree property which is called *antichain tree property (ATP)* and explain how to construct an antichain tree in a SOP_1-NSOP_2 theory. Simply the concept of antichain trees is opposite to the concept of SOP_2 in the following sense.

Definition 3.1. (i) A subset X of ${}^{\omega>}2$ is called an antichain if it is pairwisely incomparable (*i.e.* for all $\eta,\nu\in X$, $\eta\ntrianglelefteq\nu$ and $\nu\ntrianglelefteq\eta$. We denote it $\eta\perp\nu$).

(ii) A tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is called an *antichain tree* if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x,a_\eta) \mid \eta \in X\}$ is consistent if and only if X is pairwisely incomparable.

(iii) We say φ has *antichain tree property (ATP)* if φ forms an antichain tree with some $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$, T has ATP if it has an ATP formula, and T is *NATP (non-ATP)* if it does not have ATP.

And the definition of SOP_2 can be written as follows. Notice the difference between (ii) of Definition 3.1 above and Definition 3.2 below.

Definition 3.2. We say $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ witnesses SOP_2 if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x,a_\eta) : \eta \in X\}$ is consistent if and only if X is pairwisely 'comparable'.

In this sense we can consider ATP to have the opposite nature of SOP_2.

If an antichain tree $\langle \varphi, \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is given, we can find a witness of SOP_1 and a witness of TP_2 by restricting the parameter set $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ as follows.

Proposition 3.3. *If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is an antichain tree, then $\varphi(x,y)$ witnesses* SOP_1.

Proof. By compantnss, it is enough to show that for each $n \in \omega$, there exists $h_n : {}^{n\geq}2 \to {}^{\omega>}2$ such that

(i) $\{\varphi(x, b_{\eta\lceil i}) : i \leq n\}$ is consistent for all $\eta \in {}^{n}2$,

(ii) $\{\varphi(x, b_{\eta^\frown\langle 0\rangle^\frown\nu}), \varphi(x, b_{\eta^\frown\langle 1\rangle})\}$ is inconsistent for all $\eta, \nu \in {}^{n>}2$ with $\eta{}^\frown\langle 0\rangle{}^\frown\nu$, $\eta{}^\frown\langle 1\rangle \in {}^{n\geq}2$,

where $b_\eta = a_{h_n(\eta)}$ for each $\eta \in^{n\geq} 2$. We use induction. Define $h_0 : {}^{0\geq}2 \to {}^{\omega>}2$ by $h_0(\langle\rangle) = \langle 1\rangle$, $h_0(\langle 0\rangle) = \langle 011\rangle$, and $h_0(\langle 1\rangle) = \langle 0\rangle$. For $n \in \omega$, assume such h_n exists. Define $h_{n+1} : {}^{n+1\geq}2 \to {}^{\omega>}2$ by

$$h_{n+1}(\eta) = \begin{cases} \langle 1\rangle & \text{if } \eta = \langle\rangle \\ \langle 011\rangle^\frown h_n(\nu) & \text{if } \eta = \langle 0\rangle^\frown\nu \text{ for some } \nu \in {}^{n\geq}2 \\ \langle 0\rangle^\frown h_n(\nu) & \text{if } \eta = \langle 1\rangle^\frown\nu \text{ for some } \nu \in {}^{n\geq}2. \end{cases}$$

It is easy to show that $\langle \varphi, \langle b_\eta \rangle_{\eta \in {}^{n\geq}2} \rangle$ witnesses SOP_1 for each $n \in \omega$ where $b_\eta = a_{h_n(\eta)}$. □

Definition 3.4. We say a formula $\varphi(x,y)$ has TP_2 if there exists an array $\langle a_{i,j} \rangle_{i,j\in\omega}$ such that $\{\varphi(x,a_{i,j_0}), \varphi(x,a_{i,j_1}\}$ is inconsistent for all $i, j_0, j_1 \in \omega$ with $j_0 \neq j_1$, and $\{\varphi(x, a_{i,f(i)})\}_{i\in\omega}$ is consistent for all $f : \omega \to \omega$. We say a theory T has TP_2 if there exists a formula having TP_2 modulo T.

Proposition 3.5. *If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is an antichain tree, then $\varphi(x,y)$ witnesses* TP_2.

Proof. For each $n \in \omega$, choose any antichain $\{\eta_0, ..., \eta_{n-1}\}$ in ${}^{\omega>}2$. Define $h_n : n \times n \to {}^{\omega>}2$ by

$$h_n(i,j) = \eta_i{}^\frown\langle 0\rangle^j.$$

Then $\{\varphi(x, a_{h_n(i,f(i))})\}_{i<n}$ is consistent for all $f : n \to n$ and $\{\varphi(x, a_{h_n(i,j)})\}_{j<n}$ is 2-inconsistent for all $i < n$. By compactness, there exists $h : \omega \times \omega \to {}^{\omega>}2$ such that $\langle \varphi, \langle b_{i,j} \rangle_{i,j<\omega} \rangle$ witnesses TP_2 where $b_{i,j} = a_{h(i,j)}$. □

Now we show Theorem 3.7 which claims that an antichain tree exists in any SOP_1-$NSOP_2$ theory. Before we begin the construction, we need a lemma.

Lemma 3.6. *For any $c : {}^{\omega_1>}2 \to \omega$, one can find $g : {}^{\omega>}2 \to {}^{\omega_1>}2$ and $i \in \omega$ such that*

(i) $g(\eta)^\frown\langle l\rangle \trianglelefteq g(\eta^\frown\langle l\rangle)$ *for all* $\eta \in {}^{\omega>}2$ *and* $l \leq 1$,
(ii) $c(g(\eta)) = i$ *for all* $\eta \in {}^{\omega_1>}2$.

Theorem 3.7. *Suppose there exists $\varphi(x, y)$ which witnesses* SOP$_1$ *and there is no $n \in \omega$ such that $\bigwedge_{i=0}^{n} \varphi(x, y_i)$ witnesses* SOP$_2$*. Then there exists $\langle b_\eta\rangle_{\eta\in{}^{\omega>}2}$ such that $\langle \varphi(x, y), \langle b_\eta\rangle_{\eta\in{}^{\omega>}2}\rangle$ forms an antichain tree.*

Sketch of Proof. By Theorem 2.7 and compactness, there exists an β-indiscernible $\langle a_\eta\rangle_{\eta\in{}^{\omega_1>}2}$ which witnesses SOP$_1$ with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega>}2$ by

$$h(\eta) = \begin{cases} \langle 1\rangle & \text{if } \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 001\rangle & \text{if } t(\eta) = 0 \\ h(\eta^-)^\frown\langle 011\rangle & \text{if } t(\eta) = 1. \end{cases}$$

For each $i, k \in \omega$ and $\eta, \xi \in {}^{\omega_1>}2$, put

$$\begin{aligned} L_i &= \{h(\nu') : l(\nu') = i\}, & L_i(\eta) &= \{\eta^\frown\nu : \nu \in L_i\} \\ 1_\xi &= \{\xi^\frown\langle 1^d\rangle : d \in \omega\}, & 1_\xi(\eta) &= \{\eta^\frown\nu : \nu \in 1_\xi\} \\ 1_\xi^k &= \{\xi^\frown\langle 1^0\rangle, ..., \xi^\frown\langle 1^k\rangle\}, & 1_\xi^k(\eta) &= \{\eta^\frown\nu : \nu \in 1_\xi^k\} \\ M_i &= L_i \cup 1_{h(\langle 0^i\rangle)}, & M_i(\eta) &= \{\eta^\frown\nu : \nu \in M_i\} \\ M_i^k &= L_i \cup 1^k_{h(\langle 0^i\rangle)}, & M_i^k(\eta) &= \{\eta^\frown\nu : \nu \in M_i^k\} \\ m_i^k &= h(\langle 0^i\rangle)^\frown\langle 1^k\rangle, & m_i^k(\eta) &= \eta^\frown m_i^k. \end{aligned}$$

For each $X \subseteq {}^{\omega_1>}2$, let Φ_X denote $\{\varphi(x, a_\eta) : \eta \in X\}$.

Then one can show that here exists $\eta \in {}^{\omega_1>}2$ such that $\Phi_{M_i(\eta)}$ is consistent for all $i \in \omega$. By β-indiscernibility, we may assume $\eta = \langle\rangle$. For each $\eta \in {}^{\omega>}2$, put $b_\eta = a_{h(\eta)}$. Then $\langle\varphi(x, b_\eta)\rangle_{\eta\in{}^{\omega>}2}$ is an antichain tree. □

Corollary 3.8. *If* T *is* SOP$_1$ *and* NSOP$_2$*, then* T *has ATP. The witness of ATP can be selected to be strong indiscernible.*

Proof. If a theory has SOP$_1$ and does not have SOP$_2$, then the theory has a formula which witnesses SOP$_1$ and any finite conjunction of the formula does not witness SOP$_2$. So we can apply Theorem 3.7. The theory has a witness of ATP. Furthermore, we can obtain a strong indiscernible witness of ATP by using compactness and the modeling property in [3]. □

As we observed in the beginning of this section, one can find witnesses of SOP$_1$ and TP$_2$ from a witness of an antichain tree by restricting the set of parameters. But we can not use the same method for finding a witness of SOP$_2$.

Remark 3.9. The following are true.

(i) Suppose $\langle\varphi(x, y), \langle a_\eta\rangle_{\eta\in{}^{\omega>}2}\rangle$ is an antichain tree. Then there is no $h : {}^{2\geq}2 \to {}^{\omega>}2$ such that $\langle\varphi(x, y), \langle b_\eta\rangle_{\eta\in{}^{2\geq}2}\rangle$ satisfies the conditions of SOP$_2$, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2\geq}2$.
(ii) Suppose $\langle\varphi(x, y), \langle a_\eta\rangle_{\eta\in{}^{\omega>}2}\rangle$ witnesses SOP$_2$. Then there is no $h : {}^{2\geq}2 \to {}^{\omega>}2$ such that $\langle\varphi(x, y), \langle b_\eta\rangle_{\eta\in{}^{2\geq}2}\rangle$ forms an antichain tree with height 2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2\geq}2$.

Proof. (i) To get a contradiction, suppose there exists such h. Then $h(\langle 00\rangle)$, $h(\langle 01\rangle)$, $h(\langle 10\rangle)$, and $h(\langle 11\rangle)$ are pairwisely comparable in ${}^{\omega>}2$, so they are linearly ordered. We may assume $h(\langle 00\rangle)$ is the smallest. Since $h(\langle 0\rangle)$ and $h(\langle 00\rangle)$ are incomparable, $h(\langle 0\rangle)$ and $h(\langle 11\rangle)$ are incomparable. Thus $\{\varphi(x, b_{\langle 0\rangle}), \varphi(x, b_{\langle 11\rangle})\}$ is consistent. This is a contradiction.
(ii) To get a contradiction, suppose there exists such h. Then $h(\langle 00\rangle)$, $h(\langle 01\rangle)$, $h(\langle 10\rangle)$, and $h(\langle 11\rangle)$ are pairwisely comparable in ${}^{\omega>}2$, so they are linearly ordered. We may assume $h(\langle 00\rangle)$ is the smallest. Since $h(\langle 0\rangle)$ and $h(\langle 00\rangle)$ are incomparable, $h(\langle 0\rangle)$ and $h(\langle 11\rangle)$ are incomparable. Thus $\{\varphi(x, b_{\langle 0\rangle}), \varphi(x, b_{\langle 11\rangle})\}$ is inconsistent. This is a contradiction. □

But it does not mean the existence of an antichain tree prevents the theory having a witness of SOP_2. We will see in Section 4 that there exists an example of a structure whose theory has a formula $\varphi(x, y)$ which forms an antichain tree (so it witnesses SOP_1) and $\bigwedge_{i<n} \varphi(x, y_i)$ do not witness SOP_2 for all $n \in \omega$. But our example has SOP_2.

We end this section with the following remarks. They discuss the possibility of that the concept of ATP can be helpful for solving the problem of equality of SOP_1 and SOP_2.

Remark 3.10. If the existence of an antichain tree always implies the existence of a witness of SOP_2, then $\mathrm{SOP}_1 = \mathrm{SOP}_2$ by Corollary 3.8.

Remark 3.11. If there exists a NSOP_2 theory having an antichain tree, then $\mathrm{SOP}_1 \supsetneq \mathrm{SOP}_2$ by Proposition 3.3.

4. An example of antichain tree

In the last section, we showed the existence of an antichain tree in SOP_1-NSOP_2 context. It is natural to ask if an antichain tree exists without classification theoretical hypothesis. We construct a structure of relational language whose theory has a formula $\varphi(x, y)$ which forms an antichain tree and $\bigwedge_{i<n} \varphi(x, y_i)$ do not witness SOP_2 for all $n \in \omega$. Note that φ also witnesses SOP_1 by Proposition 3.3.

We begin the construction with language $\mathcal{L} = \{R\}$ where R is a binary relation symbol. For each $n \in \omega$, let $\alpha_n \in \omega$ be the number of all maximal antichains in ${}^{n>}2$, and β_n be the set of all maximal antichains in ${}^{n>}2$. We can choose a bijection from α_n to β_n for each $n \in \omega$, say μ_n. For each $n \in \omega$, let A_n and B_n be finite sets such that $|A_n| = \alpha_n$ and $|B_n| = |{}^{n>}2|$. We denote their elements by

$$A_n = \{a^n_l : l < \alpha_n\},$$
$$B_n = \{b^n_\eta : \eta \in {}^{n>}2\}.$$

And let N_n be the disjoint union of A_n and B_n for each $n \in \omega$.

For each $n \in \omega$, let $\mathcal{C}_n$ be an $\mathcal{L}$-structure such that $\mathcal{C}_n = \langle C_n; R^{\mathcal{C}_n}\rangle$, where $R^{\mathcal{C}_n} = \{\langle a^n_l, b^n_\eta\rangle \in A_n \times B_n : \eta \in \mu_n(l)\}$. For each $n \in \omega$, let ι_n be a map from $\alpha_n \cup {}^{n>}2$ to $\alpha_{n+1} \cup {}^{n+1>}2$ which maps $c \mapsto c$ for all $c \in \alpha_n \cup {}^{n>}2$, and define $\iota^*_n : C_n \to C_{n+1}$ by $a^n_l \mapsto a^{n+1}_{\iota_n(l)}$ and $b^n_\eta \mapsto b^{n+1}_{\iota_n(\eta)}$. Then ι^*_n is an embedding. So we can regard $\mathcal{C}_n$ as a substructure of $\mathcal{C}_{n+1}$ with respect to ι^*_n. Let $\mathcal{C}$ be $\bigcup_{n<\omega} \mathcal{C}_n$, A and B denote $\bigcup_{n<\omega} A_n$ and $\bigcup_{n<\omega} B_n$ respectively.

Then we have the following observations.

Proposition 4.1. *$R(x, y)$ forms an antichain tree in* $\mathrm{Th}(\mathcal{C})$.

Proposition 4.2. *$\bigwedge_{i<n} R(x, y_i)$ does not witness* SOP_2 *for all $n \in \omega$.*

But $\mathrm{Th}(\mathcal{C})$ has a witness of SOP_2. Let $\varphi(x, y) = \neg\exists w(R(w, x) \wedge R(w, y)) \wedge \exists z(x \neq z \neq y \neq x \wedge \exists w(R(w, x) \wedge R(w, z)) \wedge \neg\exists w(R(w, y) \wedge R(w, z)))$. Then φ says "y is a predecessor of x in the set of parameters." ($i.e.$, $y \lhd x$) So, $\langle \varphi(x, y), \langle b_\eta \rangle_{\eta \in \omega^{>}2} \rangle$ witnesses SOP_2, where $b_\eta = b^n_\eta$ for some $n \in \omega$. b_η is well-defined by the constructions of $\mathcal{C}$.

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